ARE211 FINAL EXAM

DECEMBER 12, 2003

This is the final exam for ARE211. As announced earlier, this is an open-book exam. Try to allocate your 180 minutes in this exam wisely, and keep in mind that leaving any questions unanswered is not a good strategy. Make sure that you do all the easy questions, and easy parts of hard questions, before you move onto the hard questions.

Problem 1 *(15 points)*

(1) In Euclidean space, under the Pythagorean metric, assume that a sequence (x_n) satisfies

$$|x_n - x_{n+1}| \leq \alpha |x_n - x_{n-1}|$$

for each $n = 2, 3, ...$ for some fixed $0 < \alpha < 1$.
Show that (x_n) is a convergent sequence.

(2) Let (X, d) be a complete metric space. A function $f : X \to X$ is called a contraction if there exists some $0 < \alpha < 1$ such that $\forall x, y \in X$,

$$d(f(x), f(y)) \leq \alpha d(x, y)$$

α is called a contraction constant.
Show that for every contraction f on a complete metric space (X, d), there exists a unique point $x \in X$ such that $f(x) = x$. *(Such a point $x \in X$ is called a fixed point.)*
(Hint: To prove this, construct a sequence that has the property defined in (1). An understanding of completeness will also be helpful.)

Problem 2 *(20 points)*

In this problem, all scalars are assumed to be real. We define a projection matrix to be a square matrix P such that

$$P^2 = P^T = P$$

(1) **Show** that every eigenvalue of a projection matrix is either 1 or 0.

(2) **Prove** that if Z is an $n \times r$ matrix such that $Z^T Z = I_r$, then ZZ^T is a projection matrix.

(3) Let P be an $n \times n$ projection matrix such that $P \neq \mathbf{0}$, **show** that there is an integer r and an $n \times r$ matrix Z with the following properties: $1 \leq r \leq n$, $Z^T Z = I_r$, and $ZZ^T = P$.
(Hint: Use the result of (1) and consider the following theorem:
If A is a real symmetric matrix, then
(a) all the eigenvalues of A are real numbers;
(b) A is diagonalizable —- there exist a diagonal matrix D and an invertible matrix S, both with entirely real entries, such that $S^{-1}AS = D$;
(c) the matrix S of (b) can be chosen so that $S^T = S^{-1}$.)

Problem 3 *(10 points)*

The Taylor series expansion of $f(a+h)$: is given by

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \frac{h^3}{3!}f^{(3)}(a) + \cdots ,$$

where the right hand side is to be interpreted as a convergent series. The special case of the Taylor series expansion when $a = 0$ is called the Maclaurin expansion of f.

(1) For $x \in \mathbb{R}$, **find** the Maclaurin expansion for the exponential function $f(x) = e^x$.

(2) Use the Maclaurin expansion for e^x to **find** $\lim_{x\to\infty} xe^{-x}$.*(You don't need to prove the limit.)*

Problem 4 *(20 points)*

(1) Use **first order conditions** to find all the critical points of $f(x, y) = x + y^2$ subject to the constraint $2x^2 + y^2 = 1$.

(2) Use the **second order conditions** to classify the critical points you have identified in (1), i.e., to distinguish between the following four categories: (a) local max; (b) local min; (c) global max on the constraint set; (d) global min on the constraint set.

Problem 5 *(20 points)*

Consider the following problem:

$$\max_{x_1,x_2} f(x_1, x_2) = x_1^2 + x_1 + 4x_2^2$$
$$s.t.\ 2x_1 + 2x_2 \leq 1, x_1 \geq 0, x_2 \geq 0$$

(1) **Solve** this maximization problem using either the Lagrangian or the Kuhn-Tucker method.

(2) Apply the **Envelope Theorem** to estimate the solution to the following problem (which is identical except for the coefficient on x_2^2 in the objective function).

$$\max_{x_1,x_2} f(x_1, x_2) = x_1^2 + x_1 + 4.1x_2^2$$
$$s.t.\ 2x_1 + 2x_2 \leq 1, x_1 \geq 0, x_2 \geq 0$$

Problem 6 *(15 points)*

Consider a supply-demand model for two goods A and B, the markets for which are interrelated in the following way: the supply of each good depends only on its only price, i.e.$S_A = S_A(p_A), S_B = S_B(p_B)$, but the demand for each good depends on both prices and on income. We assume that both goods are normal.

Let the demand function for good $i (i = A, B)$ be $D_i(p_A, p_B, y)$, where p_i is the price of good i, and y is income as before; let the supply function be $S_i(p_i)$. We assume that

$$\frac{\partial D_A}{\partial p_A} < 0 < \frac{dS_A}{dp_A}, \frac{\partial D_A}{\partial y} > 0, \frac{\partial D_B}{\partial p_B} < 0 < \frac{dS_B}{dp_B}, \frac{\partial D_B}{\partial y} > 0$$

For the moment, we make NO assumptions about the cross-price effects, i.e. the signs of $\frac{\partial D_A}{\partial p_B}$ and $\frac{\partial D_B}{\partial p_A}$.

Defining the excess demand functions

$$\begin{aligned} ED_A(p_A, p_B, y) &= D_A(p_A, p_B, y) - S_A(p_A) \\ ED_B(p_A, p_B, y) &= D_B(p_A, p_B, y) - S_B(p_B) \end{aligned}$$

(1) **Give** the equilibrium condtions for this market using excess demand functions.

(2) Assume that, for a given value of y, there is a unique pair of equilibrium prices (p_A^*, p_B^*) which satisfy the equilibrium conditions given above. Now please give a **comparative statics analysis** of the impact of a change in income on equilibrium prices, i.e. the sign of $\frac{dp_A^*}{dy}$ and $\frac{dp_B^*}{dy}$. Explain how the results depend on the signs of $\frac{\partial D_A}{\partial p_B}$ and $\frac{\partial D_B}{\partial p_A}$, and also the signs of $\frac{\partial D_A}{\partial y}$ and $\frac{\partial D_B}{\partial y}$.