This is the final exam for ARE211. As announced earlier, this is an open-book exam. Try to allocate your 180 minutes in this exam wisely, and keep in mind that leaving any questions unanswered is not a good strategy. Make sure that you do all the easy questions, and easy parts of hard questions, before you move onto the hard questions.

Problem 1  (15 points)

(1) In Euclidean space, under the Pythagorean metric, assume that a sequence \((x_n)\) satisfies

\[ |x_n - x_{n+1}| \leq \alpha |x_n - x_{n-1}| \]

for each \(n = 2, 3, \ldots\) for some fixed \(0 < \alpha < 1\).

Show that \((x_n)\) is a convergent sequence.

Solution: Let \(c = |x_2 - x_1| \). An easy inductive argument shows that for each \(n\) we have

\[ |x_{n+p} - x_n| \leq \sum_{i=1}^{p} |x_{n+i} - x_{n+i-1}| \leq c \sum_{i=1}^{p} \alpha^{n+i-2} \leq \frac{c}{1-\alpha} \alpha^{n-1} \]

holds for all \(n\) and all \(p\). Since \(\lim \alpha^n = 0\), it follows that \((x_n)\) is a Cauchy sequence, and hence a convergent sequence, under this complete metric space.

(2) Let \((X, d)\) be a complete metric space. A function \(f : X \rightarrow X\) is called a contraction if there exists some \(0 < \alpha < 1\) such that

\[ d(f(x), f(y)) \leq \alpha d(x, y) \]

\(\alpha\) is called a contraction constant.

Show that for every contraction \(f\) on a complete metric space \((X, d)\), there exists a unique point \(x \in X\) such that \(f(x) = x\). (Such a point \(x \in X\) is called a fixed point.)

(Hint: To prove this, construct a sequence that has the property defined in (1). An understanding of completeness will also be helpful.)

Solution: Note first that if \(f(x) = x\) and \(f(y) = y\) hold, then the inequality \(d(f(x), f(y)) \leq \alpha d(x, y)\) easily implies that \(d(x, y) = 0\), and so \(x = y\). That is, \(f\) has at most one fixed point. To see that \(f\) has a fixed point, choose some \(a \in X\), and then define the sequence \((x_n)\) inductively by

\[ x_1 = a \text{ and } x_{n+1} = f(x_n) \text{ for } n = 1, 2, \ldots \]

From our condition, it follows that

\[ d(x_{n+1}, x_n) = d(f(x_n), f(x_{n-1})) \leq \alpha d(x_n, x_{n-1}) \]

holds for \(n = 2, 3, \ldots\). Thus, as in part (1), we have shown that \((x_n)\) is a convergent sequence. Let \(x = \lim x_n\). Now by observing that \(f\) is (uniformly) continuous, we obtain that

\[ x = \lim_{n \to \infty} x_{n+1} = \lim_{n \to \infty} f(x_n) = f(x) \]

and so \(x\) is a unique fixed point for \(f\).
Problem 2 (20 points)

In this problem, all scalars are assumed to be real. We define a projection matrix to be a square matrix $P$ such that

$$P^2 = P^T = P$$

(1) **Show** that every eigenvalue of a projection matrix is either 1 or 0.

**Solution:** Let $\lambda$ be an eigenvalue of a projection matrix $P$, and let $x$ be a corresponding eigenvector: $x \neq 0$, and $Px = \lambda x$. Since $P = P^2$, $Px = P(Px)$, hence $\lambda x = \lambda (\lambda x) = \lambda^2 x$. Since $x \neq 0$, it follows that $\lambda = \lambda^2$, so $\lambda$ is either 1 or 0.

(2) **Prove** that if $Z$ is an $n \times r$ matrix such that $Z^T Z = I_r$, then $ZZ^T$ is a projection matrix.

**Solution:** Let $Z$ be an $n \times r$ matrix such that $Z^T Z = I_r$. Then $ZZ^T$ is $n \times n$ and

$$(ZZ^T)^T = ZZ^T, (ZZ^T)^2 = ZZ^T ZZ^T = ZI_r Z^T = ZZ^T$$

Hence $ZZ^T$ is a projection matrix.

(3) Let $P$ be an $n \times n$ projection matrix such that $P \neq 0$, **show** that there is an integer $r$ and an $n \times r$ matrix $Z$ with the following properties: $1 \leq r \leq n$, $Z^T Z = I_r$, and $ZZ^T = P$.

(Hint: Use the result of (1) and consider the following theorem:

If $A$ is a real symmetric matrix, then

(a) all the eigenvalues of $A$ are real numbers;
(b) $A$ is diagonalizable --- there exist a diagonal matrix $D$ and an invertible matrix $S$, both with entirely real entries, such that $S^{-1} AS = D$;
(c) the matrix $S$ of (b) can be chosen so that $S^T = S^{-1}$.)

**Solution:** By (1), the characteristic polynomial of $P$ is $(\lambda - 1)^r \lambda^{n-r}$ for some $r$. Let $D$ be the diagonal matrix whose first $r$ diagonal entries are equal to 1 and whose remaining diagonal entries are all zero. Hence, by the Theorem above, there exists an orthogonal matrix, $SST = I$, such that $P = SDS^T$. Partition $S$ as $[Z \ Y]$, where $Z$ consists of the first $r$ columns. Then the equation $SS^T = I$ may be written

$$\begin{bmatrix} Z^T & Z^T Y \\ Y^T & Y^T Y \end{bmatrix} = \begin{bmatrix} I_r & 0 \\ 0 & I_{n-r} \end{bmatrix}$$

In particular, $Z^T Z = I_r$. Also, the equation $P = SDS^T$ may be written

$$P = [Z \ Y] \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} Z^T \\ Y^T \end{bmatrix}$$

Hence

$$P = [Z \ Y] \begin{bmatrix} Z^T \\ 0 \end{bmatrix} = ZZ^T$$

as required.
Problem 3 (10 points)

The Taylor series expansion of \( f(a + h) \) is given by

\[
f(a + h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \frac{h^3}{3!}f^{(3)}(a) + \cdots,
\]

where the right hand side is to be interpreted as a convergent series. The special case of the Taylor series expansion when \( a = 0 \) is called the Maclaurin expansion of \( f \).

(1) For \( x \in \mathbb{R} \), find the Maclaurin expansion for the exponential function \( f(x) = e^x \).

**Solution:** \( f(x) = f'(x) = \cdots = f^{(n)}(x) = e^x \), whence \( f(0) = f'(0) = \cdots = f^{(n)}(0) = 1 \). The Maclaurin expansion is therefore

\[
e^x = \sum_{n=0}^{\infty} \frac{1}{n!}x^n = 1 + x + \frac{1}{2!}x^2 + \cdots + \frac{1}{n!}x^n + \cdots \quad (x \in \mathbb{R})
\]

(2) Use the Maclaurin expansion for \( e^x \) to find \( \lim_{x \to \infty} xe^{-x} \). (You don’t need to prove the limit.)

**Solution:** From the series for \( e^x \),

\[
\frac{e^x}{x} = \frac{1}{x} + 1 + \frac{x}{2!} + \frac{x^2}{3!} + \cdots
\]

As \( x \to \infty \), \( \frac{1}{x} + 1 \to 1 \), and the terms \( \frac{1}{x} \), \( \frac{x}{2!} \), \( \frac{x^2}{3!} \), \( \cdots \) all \( \to \infty \); hence \( \frac{e^x}{x} \to \infty \), so \( \lim_{x \to \infty} xe^{-x} = 0 \).
Problem 4 (20 points)

(1) Use **first order conditions** to find all the critical points of \( f(x, y) = x + y^2 \) subject to the constraint \( 2x^2 + y^2 = 1 \).

**Solution:** \( x^* \) is a critical point of \( f \), if \( f'(x^*) = 0 \). This condition requires that \( x^* \) not be an endpoint of the interval under consideration. The graph of \( f \) on the constraint is plotted below:

![Graph of f(x,y) and constraint](image)

(2) Use the **second order conditions** to classify the critical points you have identified in (1), i.e., to distinguish between the following four categories: (a) local max; (b) local min; (c) global max on the constraint set; (d) global min on the constraint set.

**Solution:** This part has exactly the same spirit as the last question in problem set 9. Minimum is \((-\frac{1}{\sqrt{2}}, 0)\); Maximum is \((\frac{1}{4}, \pm \frac{\sqrt{2}}{8})\); Local minimum at \((\frac{1}{\sqrt{2}}, 0)\).
Problem 5  (20 points)

Consider the following problem:

\[
\begin{align*}
\max_{x_1, x_2} & \quad f(x_1, x_2) = x_1^2 + x_1 + 4x_2^2 \\
\text{s.t.} & \quad 2x_1 + 2x_2 \leq 1, x_1 \geq 0, x_2 \geq 0
\end{align*}
\]

(1) **Solve** this maximization problem using either the Lagrangian or the Kuhn-Tucker method.

**Solution:** See Example 18.13 in Simon and Blume.

(2) Apply the **Envelope Theorem** to estimate the solution to the following problem (which is identical except for the coefficient on \(x_2^2\) in the objective function).

\[
\begin{align*}
\max_{x_1, x_2} & \quad f(x_1, x_2) = x_1^2 + x_1 + 4.1x_2^2 \\
\text{s.t.} & \quad 2x_1 + 2x_2 \leq 1, x_1 \geq 0, x_2 \geq 0
\end{align*}
\]

**Solution:** This part has exactly the same spirit as the first question in problem set 10.

\[
\begin{align*}
\max_{x_1, x_2} & \quad f(x_1, x_2) = x_1^2 + x_1 + ax_2^2 \\
\text{s.t.} & \quad 2x_1 + 2x_2 \leq 1, x_1 \geq 0, x_2 \geq 0
\end{align*}
\]

\[
L = x_1^2 + x_1 + ax_2^2 - \lambda_1(2x_1 + 2x_2 - 1) + \lambda_2 x_1 + \lambda_3 x_2
\]

For \(a = 4\), \(x_1^* = 0\), \(x_2^* = 0.5\), \(\lambda_1^* = 2\), \(\lambda_2^* = 3\), \(\lambda_3^* = 0\), and \(f^* = 1\). At these values,

\[
\frac{\partial L}{\partial a} = x_2^* = 0.5^2 = 0.25
\]

so

\[
f^*(4.1) \approx f^*(4) + \frac{\partial L}{\partial a} \Delta a = 1 + 0.25(0.1) = 1.025
\]
Problem 6 (15 points)

Consider a supply-demand model for two goods $A$ and $B$, the markets for which are interrelated in the following way: the supply of each good depends only on its only price, i.e. $S_A = S_A(p_A), S_B = S_B(p_B)$, but the demand for each good depends on both prices and on income. We assume that both goods are normal.

Let the demand function for good $i (i = A, B)$ be $D_i(p_A, p_B, y)$, where $p_i$ is the price of good $i$, and $y$ is income as before; let the supply function be $S_i(p_i)$. We assume that $\frac{\partial D_A}{\partial p_A} < 0 < \frac{\partial S_A}{\partial p_A}, \frac{\partial D_A}{\partial y} > 0, \frac{\partial D_B}{\partial p_B} < 0 < \frac{\partial S_B}{\partial p_B}, \frac{\partial D_B}{\partial y} > 0$

For the moment, we make NO assumptions about the cross-price effects, i.e. the signs of $\frac{\partial D_A}{\partial p_B}$ and $\frac{\partial D_B}{\partial p_A}$.

Defining the excess demand functions

$$ED_A(p_A, p_B, y) = D_A(p_A, p_B, y) - S_A(p_A)$$
$$ED_B(p_A, p_B, y) = D_B(p_A, p_B, y) - S_B(p_B)$$

(1) **Give** the equilibrium conditions for this market using excess demand functions.

**Solution:** $ED_A(p^*_A, p^*_B, y) = 0$ and $ED_B(p^*_A, p^*_B, y) = 0$

(2) Assume that, for a given value of $y$, there is a unique pair of equilibrium prices $(p^*_A, p^*_B)$ which satisfy the equilibrium conditions given above. Now please give a **comparative statics analysis** of the impact of a change in income on equilibrium prices, i.e. the sign of $\frac{dp^*_A}{dy}$ and $\frac{dp^*_B}{dy}$. Explain how the results depend on the signs of $\frac{\partial D_A}{\partial p_B}$ and $\frac{\partial D_B}{\partial p_A}$, and also the signs of $\frac{\partial D_A}{\partial y}$ and $\frac{\partial D_B}{\partial y}$.

**Solution:** To apply comparative statics, let us consider the Jacobian matrix

$$J = \begin{bmatrix} \frac{\partial EDA}{\partial p_A} & \frac{\partial EDA}{\partial p_B} \\ \frac{\partial EDA}{\partial p_A} & \frac{\partial EDA}{\partial p_B} \end{bmatrix}$$

The implicit function theorem tells us that if $J$ is invertible at the given equilibrium, there is a unique local solution for $p_A$ and $p_B$ in terms of $y$, which may be differentiated as follows:

$$\begin{bmatrix} \frac{dp^*_A}{dy} \\ \frac{dp^*_B}{dy} \end{bmatrix} = -J^{-1} \begin{bmatrix} \frac{\partial EDA}{\partial p_A} \\ \frac{\partial EDA}{\partial p_B} \end{bmatrix}$$

Let $\Delta = \det J$. Now assume that $\Delta > 0$, which implies that cross-price effects on demand are not so big as to swamp the own-price effects on supply and demand. Given that $\Delta > 0$, the matrix $J$ is invertible. We have

$$J^{-1} = \frac{1}{\Delta} \begin{bmatrix} \frac{\partial EDA}{\partial p_A} & -\frac{\partial EDA}{\partial p_B} \\ -\frac{\partial EDA}{\partial p_A} & \frac{\partial EDA}{\partial p_B} \end{bmatrix}$$
\[
\begin{align*}
\left[ \frac{dp_A}{dy} \right]_{y_B} &= -\frac{1}{\Delta} \left[ \frac{\partial E_{D_B}}{\partial p_A} \frac{\partial E_{D_B}}{\partial y} - \frac{\partial E_{D_A}}{\partial p_B} \frac{\partial E_{D_A}}{\partial y} \right] \\
&= -\frac{1}{\Delta} \left[ \frac{(-)(+)}{\partial p_A} \frac{(+)(+)}{\partial y} - \frac{(-)(+)}{\partial p_A} \frac{(+)(+)}{\partial y} \right]
\end{align*}
\]

For example, we therefore have the following information about the sign of \( \frac{dp_A}{dy} \): if \( \frac{\partial p_A}{\partial p_B} > 0 \), then \( \frac{dp_A}{dy} > 0 \); if \( \frac{\partial p_A}{\partial p_B} < 0 \), then \( \frac{dp_A}{dy} \) could be negative if \( \frac{\partial p_B}{\partial y} \) were large enough relative to \( \frac{\partial p_A}{\partial y} \). The similar analysis applies to \( \frac{dp_A}{dy} \). Next, conduct the similar analysis assuming \( \Delta < 0 \).