ARE211 FINAL EXAM

DECEMBER 12, 2003

This is the final exam for ARE211. As announced earlier, this is an open-book exam. Try to allocate your 180 minutes in this exam wisely, and keep in mind that leaving any questions unanswered is not a good strategy. Make sure that you do all the easy questions, and easy parts of hard questions, before you move onto the hard questions.

Problem 1 *(15 points)*

(1) In Euclidean space, under the Pythagorean metric, assume that a sequence (x_n) satisfies

$$|x_n - x_{n+1}| \leq \alpha |x_n - x_{n-1}|$$

for each $n = 2, 3, ...$ for some fixed $0 < \alpha < 1$.
Show that (x_n) is a convergent sequence.

Solution: *Let $c = |x_2 - x_1|$.An easy inductive argument shows that for each n we have $|x_{n+1} - x_n| \leq c\alpha^{n-1}$. Thus,*

$$|x_{n+p} - x_n| \leq \sum_{i=1}^{p} |x_{n+i} - x_{n+i-1}| \leq c \sum_{i=1}^{p} \alpha^{n+i-2} \leq \frac{c}{1-\alpha}\alpha^{n-1}$$

holds for all n and all p. Since $\lim \alpha^n = 0$, it follows that (x_n) is a Cauchy sequence, and hence a convergent sequence, under this complete metric space.

(2) Let (X, d) be a complete metric space. A function $f : X \to X$ is called a contraction if there exists some $0 < \alpha < 1$ such that $\forall x, y \in X$,

$$d(f(x), f(y)) \leq \alpha d(x, y)$$

α is called a contraction constant.
Show that for every contraction f on a complete metric space (X, d), there exists a unique point $x \in X$ such that $f(x) = x$. *(Such a point $x \in X$ is called a fixed point.)*
(Hint: To prove this, construct a sequence that has the property defined in (1). An understanding of completeness will also be helpful.)

Solution: *Note first that if $f(x) = x$ and $f(y) = y$ hold, then the inequality $d(f(x), f(y)) \leq \alpha d(x, y)$ easily implies that $d(x, y) = 0$, and so $x = y$. That is, f has at most one fixed point. To see that f has a fixed point, choose some $a \in X$, and then define the sequence (x_n) inductively by*

$$x_1 = a \text{ and } x_{n+1} = f(x_n) \text{ for } n = 1, 2, ...$$

From our condition, it follows that

$$d(x_{n+1}, x_n) = d(f(x_n), f(x_{n-1})) \leq \alpha d(x_n, x_{n-1})$$

holds for $n = 2, 3, ...$Thus, as in part (1), we have shown that (x_n) is a convergent sequence. Let $x = \lim x_n$. Now by observing that f is (uniformly) continuous, we obtain that

$$x = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} f(x_n) = f(x)$$

and so x is a unique fixed point for f.

Problem 2 *(20 points)*

In this problem, all scalars are assumed to be real. We define a projection matrix to be a square matrix P such that

$$P^2 = P^T = P$$

(1) **Show** that every eigenvalue of a projection matrix is either 1 or 0.

Solution: *Let λ be an eigenvalue of a projection matrix P, and let x be a corresponding eigenvector: $x \neq 0$, and $Px = \lambda x$. Since $P = P^2$, $Px = P(Px)$, hence $\lambda x = P(\lambda x) = \lambda Px = \lambda^2 x$.Since $x \neq 0$, it follows that $\lambda = \lambda^2$, so λ is either 1 or 0.*

(2) **Prove** that if Z is an $n \times r$ matrix such that $Z^T Z = I_r$, then ZZ^T is a projection matrix.

Solution: *Let Z be an $n \times r$ matrix such that $Z^T Z = I_r$.Then ZZ^T is $n \times n$ and*

$$(ZZ^T)^T = ZZ^T,\ (ZZ^T)^2 = ZZ^T ZZ^T = ZI_r Z^T = ZZ^T$$

Hence ZZ^T is a projection matrix.

(3) Let P be an $n \times n$ projection matrix such that $P \neq \mathbf{0}$, **show** that there is an integer r and an $n \times r$ matrix Z with the following properties: $1 \leq r \leq n$, $Z^T Z = I_r$, and $ZZ^T = P$.
(Hint: Use the result of (1) and consider the following theorem:
If A is a real symmetric matrix, then
(a) all the eigenvalues of A are real numbers;
(b) A is diagonalizable —- there exist a diagonal matrix D and an invertible matrix S, both with entirely real entries, such that $S^{-1}AS = D$;
(c) the matrix S of (b) can be chosen so that $S^T = S^{-1}$.)

Solution: *By (1), the characteristic polynomial of P is $(\lambda - 1)^r \lambda^{n-r}$ for some r. Let D be the diagonal matrix whose first r diagonal entries are equal to 1 and whose remaining diagonal entries are all zero. Hence, by the Theorem above, there exists an orthogonal matrix, $SS^T = I$, such that $P = SDS^T$. Partition S as $[Z\ Y]$, where Z consists of the first r columns. Then the equation $SS^T = I_n$ may be written*

$$\begin{bmatrix} Z^T Z & Z^T Y \\ Y^T Z & Y^T Y \end{bmatrix} = \begin{bmatrix} I_r & 0 \\ 0 & I_{n-r} \end{bmatrix}$$

In particular, $Z^T Z = I_r$.Also, the equation $P = SDS^T$ may be written

$$P = [Z\ Y] \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} Z^T \\ Y^T \end{bmatrix}$$

Hence

$$P = [Z\ Y] \begin{bmatrix} Z^T \\ 0 \end{bmatrix} = ZZ^T$$

as required.

Problem 3 *(10 points)*

The Taylor series expansion of $f(a+h)$: is given by

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \frac{h^3}{3!}f^{(3)}(a) + \cdots,$$

where the right hand side is to be interpreted as a convergent series. The special case of the Taylor series expansion when $a = 0$ is called the Maclaurin expansion of f.

(1) For $x \in \mathbb{R}$, **find** the Maclaurin expansion for the exponential function $f(x) = e^x$.

Solution: $f(x) = f'(x) = \cdots = f^{(n)}(x) = e^x$,*whence* $f(0) = f'(0) = \cdots = f^{(n)}(0) = 1$.*The Maclaurin expansion is therefore*

$$e^x = \sum_{n=0}^{\infty} \frac{1}{n!}x^n = 1 + x + \frac{1}{2!}x^2 + \cdots + \frac{1}{n!}x^n + \cdots \quad (x \in R)$$

(2) Use the Maclaurin expansion for e^x to **find** $\lim_{x\to\infty} xe^{-x}$.*(You don't need to prove the limit.)*

Solution: *From the series for* e^x,

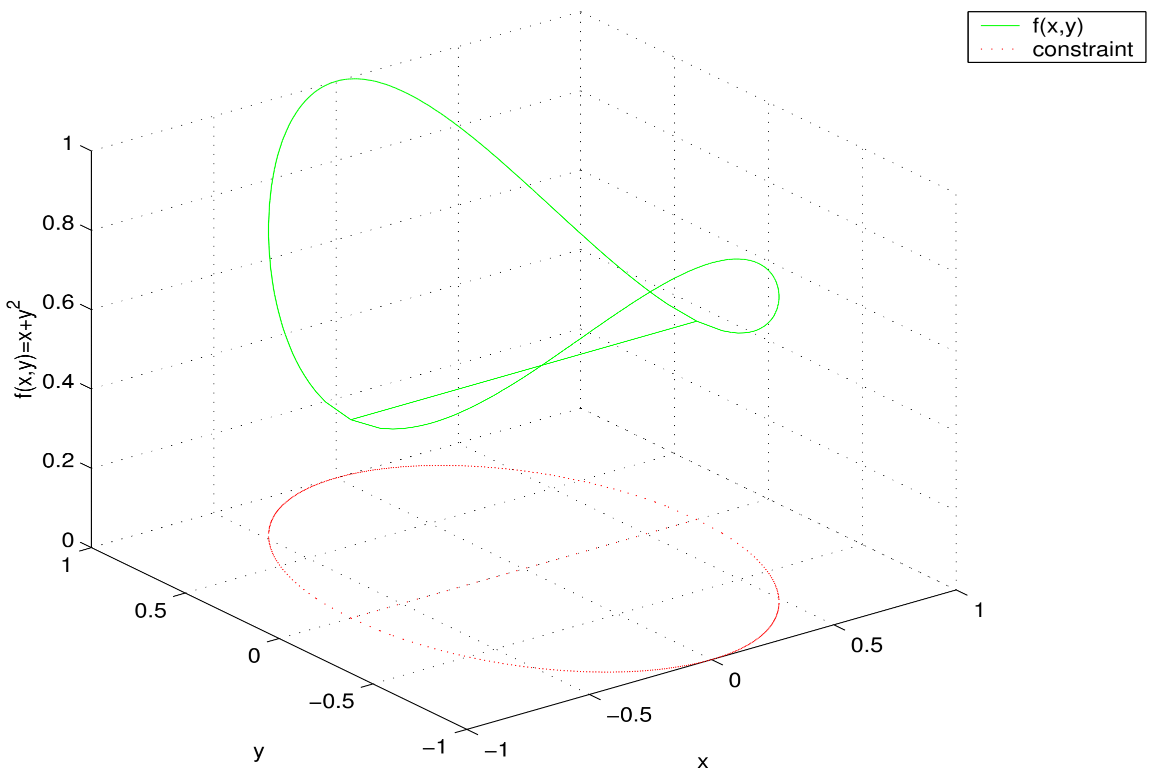
$$\frac{e^x}{x} = \frac{1}{x} + 1 + \frac{x}{2!} + \frac{x^2}{3!} + \cdots$$

As $x \to \infty$, $\frac{1}{x} + 1 \to 1$, *and the terms* $\frac{x}{2!}, \frac{x^2}{3!}, \ldots$ *all* $\to \infty$; *hence* $\frac{e^x}{x} \to \infty$, *so* $\lim_{x\to\infty} xe^{-x} = 0$.

Problem 4 *(20 points)*

(1) Use **first order conditions** to find all the critical points of $f(x, y) = x + y^2$ subject to the constraint $2x^2 + y^2 = 1$.

Solution: *x^* is a critical point of f, if $f'(x^*) = 0$. This condition requires that x^* not be an endpoint of the interval under consideration.* The graph of f on the constraint is plotted below:



(2) Use the **second order conditions** to classify the critical points you have identified in (1), i.e., to distinguish between the following four categories: (a) local max; (b) local min; (c) global max on the constraint set; (d) global min on the constraint set.

Solution: *This part has exactly the same spirit as the last question in problem set 9. Minimum is $(-\frac{1}{\sqrt{2}}, 0)$; Maximum is $(\frac{1}{4}, \pm\frac{\sqrt{7}}{8})$; Local minimum at $(\frac{1}{\sqrt{2}}, 0)$.*

Problem 5 *(20 points)*

Consider the following problem:

$$\max_{x_1,x_2} f(x_1,x_2) = x_1^2 + x_1 + 4x_2^2$$
$$s.t.\ 2x_1 + 2x_2 \le 1, x_1 \ge 0, x_2 \ge 0$$

(1) **Solve** this maximization problem using either the Lagrangian or the Kuhn-Tucker method.

Solution: *See Example 18.13 in Simon and Blume.*

(2) Apply the **Envelope Theorem** to estimate the solution to the following problem (which is identical except for the coefficient on x_2^2 in the objective function).

$$\max_{x_1,x_2} f(x_1,x_2) = x_1^2 + x_1 + 4.1x_2^2$$
$$s.t.\ 2x_1 + 2x_2 \le 1, x_1 \ge 0, x_2 \ge 0$$

Solution: *This part has exactly the same spirit as the first question in problem set 10.*

$$\max_{x_1,x_2} f(x_1,x_2) = x_1^2 + x_1 + ax_2^2$$
$$s.t.\ 2x_1 + 2x_2 \le 1, x_1 \ge 0, x_2 \ge 0$$

$$L = x_1^2 + x_1 + ax_2^2 - \lambda_1(2x_1 + 2x_2 - 1) + \lambda_2 x_1 + \lambda_3 x_2$$

For $a = 4$, $x_1^* = 0, x_2^* = 0.5, \lambda_1^* = 2, \lambda_2^* = 3, \lambda_3^* = 0$, *and* $f^* = 1$. *At these values,*

$$\frac{\partial L}{\partial a} = x_2^{*2} = 0.5^2 = 0.25$$

so

$$f^*(4.1) \approx f^*(4) + \frac{\partial L}{\partial a}\Delta a = 1 + 0.25(0.1) = 1.025$$

Problem 6 *(15 points)*

Consider a supply-demand model for two goods A and B, the markets for which are interrelated in the following way: the supply of each good depends only on its only price, i.e.$S_A = S_A(p_A), S_B = S_B(p_B)$, but the demand for each good depends on both prices and on income. We assume that both goods are normal.

Let the demand function for good $i (i = A, B)$ be $D_i(p_A, p_B, y)$, where p_i is the price of good i, and y is income as before; let the supply function be $S_i(p_i)$. We assume that

$$\frac{\partial D_A}{\partial p_A} < 0 < \frac{dS_A}{dp_A}, \frac{\partial D_A}{\partial y} > 0, \frac{\partial D_B}{\partial p_B} < 0 < \frac{dS_B}{dp_B}, \frac{\partial D_B}{\partial y} > 0$$

For the moment, we make NO assumptions about the cross-price effects, i.e. the signs of $\frac{\partial D_A}{\partial p_B}$ and $\frac{\partial D_B}{\partial p_A}$.

Defining the excess demand functions

$$\begin{aligned} ED_A(p_A, p_B, y) &= D_A(p_A, p_B, y) - S_A(p_A) \\ ED_B(p_A, p_B, y) &= D_B(p_A, p_B, y) - S_B(p_B) \end{aligned}$$

(1) **Give** the equilibrium condtions for this market using excess demand functions.

Solution: $ED_A(p_A^*, p_B^*, y) = 0$ *and* $ED_B(p_A^*, p_B^*, y) = 0$

(2) Assume that, for a given value of y, there is a unique pair of equilibrium prices (p_A^*, p_B^*) which satisfy the equilibrium conditions given above. Now please give a **comparative statics analysis** of the impact of a change in income on equilibrium prices, i.e. the sign of $\frac{dp_A^*}{dy}$ and $\frac{dp_B^*}{dy}$. Explain how the results depend on the signs of $\frac{\partial D_A}{\partial p_B}$ and $\frac{\partial D_B}{\partial p_A}$, and also the signs of $\frac{\partial D_A}{\partial y}$ and $\frac{\partial D_B}{\partial y}$.

Solution: *To apply comparative statics, let us consider the Jacobian matrix*

$$J = \begin{bmatrix} \frac{\partial ED_A}{\partial p_A} & \frac{\partial ED_A}{\partial p_B} \\ \frac{\partial ED_B}{\partial p_A} & \frac{\partial ED_B}{\partial p_B} \end{bmatrix}$$

The implicit function theorem tells us that if J is invertible at the given equilibrium, there is a unique local solution for p_A and p_B in terms of y, which may be differentiated as follows:

$$\begin{bmatrix} \frac{dp_A^*}{dy} \\ \frac{dp_B^*}{dy} \end{bmatrix} = -J^{-1} \begin{bmatrix} \frac{\partial ED_A}{\partial y} \\ \frac{\partial ED_B}{\partial y} \end{bmatrix}$$

Let $\Delta = \det J$. Now assume that $\Delta > 0$, which implies that cross-price effects on demand are not so big as to swamp the own-price effects on supply and demand. Given that $\Delta > 0$, the matrix J is invertible. We have

$$J^{-1} = \frac{1}{\Delta} \begin{bmatrix} \frac{\partial ED_B}{\partial p_B} & -\frac{\partial ED_A}{\partial p_B} \\ -\frac{\partial ED_B}{\partial p_A} & \frac{\partial ED_A}{\partial p_A} \end{bmatrix}$$

It follows that

$$\begin{bmatrix} \frac{dp_A^*}{dy} \\ \frac{dp_B^*}{dy} \end{bmatrix} = -\frac{1}{\Delta}\begin{bmatrix} \frac{\partial ED_B}{\partial p_B} & -\frac{\partial ED_A}{\partial p_B} \\ -\frac{\partial ED_B}{\partial p_A} & \frac{\partial ED_A}{\partial p_A} \end{bmatrix}\begin{bmatrix} \frac{\partial ED_A}{\partial y} \\ \frac{\partial ED_B}{\partial y} \end{bmatrix}$$

$$= -\frac{1}{\Delta}\begin{bmatrix} \overset{(-)}{\frac{\partial ED_B}{\partial p_B}}\overset{(+)}{\frac{\partial D_A}{\partial y}} - \frac{\partial D_A}{\partial p_B}\overset{(+)}{\frac{\partial D_B}{\partial y}} \\ -\frac{\partial D_B}{\partial p_A}\overset{(+)}{\frac{\partial D_A}{\partial y}} + \overset{(-)}{\frac{\partial ED_A}{\partial p_A}}\overset{(+)}{\frac{\partial D_B}{\partial y}} \end{bmatrix}$$

For example, we therefore have the following information about the sign of $\frac{dp_A}{dy}$*: if* $\frac{\partial D_A}{\partial p_B} > 0$,*then* $\frac{dp_A}{dy} > 0$*; if* $\frac{\partial D_A}{\partial p_B} < 0$*, then* $\frac{dp_A}{dy}$ *could be negative if* $\frac{\partial D_B}{\partial y}$ *were large enough relative to* $\frac{\partial D_A}{\partial y}$*. The similiar analysis applies to* $\frac{dp_A}{dy}$*. Next, conduct the similiar analysis assuming* $\Delta < 0$*.*