Fall, 2003 **ARE202A**

FINAL EXAM
DECEMBER 8 2003

Tackle first the question you think is the easier one. It's always a good strategy to make attempts at all parts of the question, because then you always have a chance at partial credit. If you omit a part, then you lose that chance! **Don't attempt either bonus part till you've done what you can on the non-bonus parts of both questions.**

Problem 1. (60 points)

Consider a consumer with Cobb-Douglas preferences:

$$U = A.Q^{\alpha}L^{\beta}$$

with $A > 0$, $\alpha = 1/3$ and such that $\beta = 1/2$, and where Q is a composite consumer good and L is leisure (not labor). The consumer maximizes his utility subject to non-negativity constraints ($Q \geq 0$, $L \geq 0$), a time constraint $L \leq T$ (where T is the total time available) and subject to his budget constraint:

$$pQ \leq (1-\tau)w(T-L)+Y$$

where p is the price of the composite commodity, τ is the tax rate on wage income ($\tau \in [0,1)$), w is the wage rate per unit of labor time, and Y is other non-taxed income. We assume that the consumer's other income represents a small share of his maximum wage budget, i.e., that $Y < \frac{w(1-\tau)T}{2}$.

(a) Write the utility maximization problem in the usual form.

(b) Draw the feasible set.

(c) On a graph, identify geometrically a segment representing the set of points that could potentially satisfy the Mantra (i.e., be candidates for solution), given what you know about the utility function. (Hint: look at the picture you've drawn for (b))

(d) In your answer to (c), there should be exactly two points at which two constraints are satisfied with equality. Use the Mantra to check arithmetically (by finding nonnegative coefficients that enable you to write the gradient of U as a nonnegative linear combination of the gradients of the two constraints) whether or not the KKT conditions can be satisfied at either of these points.

(e) Check whether the Constraint Qualification holds at each of the points you have identified in (c) (You should consider three cases.)

(f) Check whether the necessary conditions of the KKT are also going to be sufficient. In this problem, you do *not* need to check a bordered Hessian condition, you just need to check a Hessian condition. Explain why.

(g) *Without doing any further computations*, what can you conclude about the solution, if any, to this problem? If a solution does exist, what system of equations will deliver it? Explain what theorem(s) you are invoking in order to conclude what you are concluding, and explain why it is appropriate to invoke it/them.

(h) **Bonus part:** Write the Lagrangian and the first-order conditions to the problem defined in (a), and use the results of (g) to compute the solution to this problem.

Problem 2. (40 points).

Proof of the Stolper-Samuelson Theorem: *"An increase in the price of a good will raise the real returns to the factor used intensively in that good and lower the real return to the other factor."*

Consider the following two-sector, two-input model (2*2): Good 1 is produced at a relative price p and good 2 is the *numeraire*, sold at a price of 1. Prices are determined on the world market and so are exogenous to the economy. The two goods are produced with labor L at a wage w and capital K with a rental price r. The economy's aggregate endowment of both inputs is fixed. We assume that sector 1 is labor intensive, and sector 2 is capital intensive, so that when inputs are used optimally, $\frac{K_1}{L_1} < \frac{K_2}{L_2}$.

Because of free entry, the firms earn zero profits in equilibrium, thus:

$$C^1(w,r) = p$$
$$C^2(w,r) = 1$$

where C^i denotes the *unit cost function* for good i, formally defined as

$$C^i(w,r) = \min_{L,K} \{wL_i + rK_i \text{ s.t. } q^i(L_i,K_i) \geq 1\}$$

where $q^i(\cdot)$ is the production function, meaning that it is defined as the minimum cost to produce one unit of good i. Suppose that the minimum is obtained with the strictly positive quantities of input $L_i^\star$ and $K_i^\star$.

(a) Write the equilibrium conditions for this system as the level set of some $f : \mathbb{R}^3 \to \mathbb{R}^2$. What are the exogenous and the endogenous variables? (Hint: Neither L nor K should appear in your answer.)

(b) Write down the unit cost functions $C^i(\cdot)$ in terms of $L_i^\star$ and $K_i^\star$.

(c) Show that the derivative of $C^i(\cdot)$ with respect to wages is equal to $L_i^\star$.

(d) Similarly, what is the derivative of $C^i(\cdot)$ with respect to rental prices? (Write your result in terms of $L_i^\star$ and $K_i^\star$)

(e) *Assume* for the moment that the condition for using the implicit function theorem is satisfied. Using your answers to (c) and (d), write down the differential of the Jacobian of the implicit function relating the wage and rental rates to the price of good 1. (Hint: in terms of the notation used in lectures, your answer should be of the form $\mathbf{dx} = Jg(\alpha)\mathbf{d}\alpha$.)

(f) Now check to see if the condition for the implicit function theorem is satisfied

(g) Show that a small increase in the price of good 1 leads to an increase in the wage.

(h) Show that a small increase in the price of good 1 leads to a decrease in rental prices.

(i) What is the effect of a small increase in the price p on the real rental price (r/p)?

(j) **Bonus part:** What is the the effect of an increase in the price p on the real wage (w/p)? (Hint 1: use the zero-profit conditions, Hint 2: take it for granted that $L_1^\star K_2^\star - L_2^\star K_1^\star - (L_1^\star/r) < 0$.)

You have just proved the Stolper-Samuelson Theorem !