Fall, 2003 **ARE202A**

FINAL EXAM
DECEMBER 8 2003

Tackle first the question you think is the easier one. It's always a good strategy to make attempts at all parts of the question, because then you always have a chance at partial credit. If you omit a part, then you lose that chance! **Don't attempt either bonus part till you've done what you can on the non-bonus parts of both questions.**

Problem 1. (60 points)

Consider a consumer with Cobb-Douglas preferences:

$$U = A.Q^{\alpha}L^{\beta}$$

with $A > 0$, $\alpha = 1/3$ and such that $\beta = 1/2$, and where Q is a composite consumer good and L is leisure (not labor). The consumer maximizes his utility subject to non-negativity constraints ($Q \geq 0$, $L \geq 0$), a time constraint $L \leq T$ (where T is the total time available) and subject to his budget constraint:

$$pQ \leq (1-\tau)w(T-L)+Y$$

where p is the price of the composite commodity, τ is the tax rate on wage income ($\tau \in [0,1)$), w is the wage rate per unit of labor time, and Y is other non-taxed income. We assume that the consumer's other income represents a small share of his maximum wage budget, i.e., that $Y < \frac{w(1-\tau)T}{2}$.

(a) Write the utility maximization problem in the usual form.

$$\begin{aligned} \underset{Q,L}{\text{Max}} \quad U(Q,L) &= AQ^{1/3}L^{1/2} \\ \text{subject to} \quad pQ+(1-\tau)wL &\leq (1-\tau)wT+Y \\ -Q &\leq 0 \\ -L &\leq 0 \\ L &\leq T \end{aligned}$$

(b) Draw the feasible set.

The graph is shown on Figure 1.

(c) On a graph, identify geometrically a segment representing the set of points that could potentially satisfy the Mantra (i.e., be candidates for solution), given what you know about the utility function. (Hint: look at the picture you've drawn for (b))

See Figure 2. The gradient of U is defined as:

$$\bigtriangledown(U) = (U/3Q \quad U/2L)$$

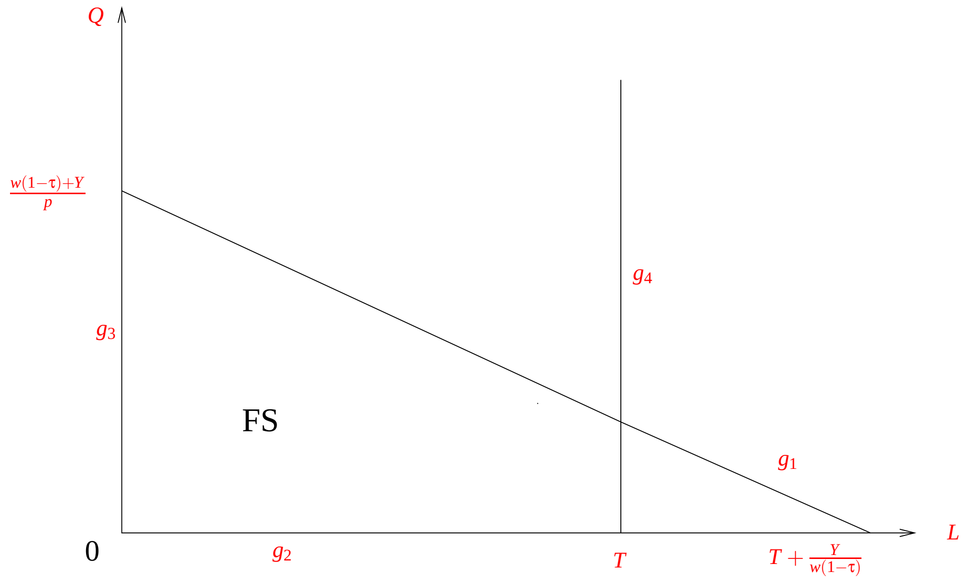

FIGURE 1. Feasible set

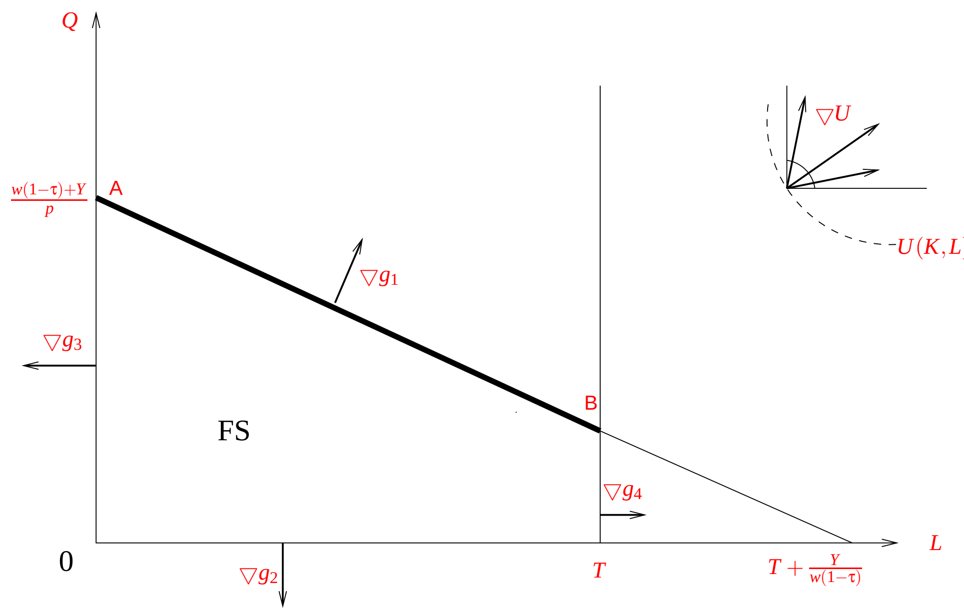

FIGURE 2. Potential solutions according to the Mantra: the segment [AB]

(for $Q > 0$ and $L > 0$), so both components of the gradient of U are strictly positive. Thus the gradient vector of U will point towards the North-East. The gradients of the constraints are the following:

$$\nabla g_1 = (p \quad w(1-\tau))$$
$$\nabla g_2 = (-1 \quad 0)$$
$$\nabla g_3 = (0 \quad -1)$$
$$\nabla g_4 = (0 \quad 1)$$

So we see that the gradient of U will never be in the positive cone of constraints (2), (3) or (4) by themselves. In fact the gradient of U may only be in the positive cone of constraint (1) or in the positive cone spanned by the gradients of constraints (1) and (3) or (1) and (4). This means that without further computations, the set of points candidate for solution according to the Mantra is the segment of points noted [AB] in Figure 2.

(d) In your answer to (c), there should be exactly two points at which two constraints are satisfied with equality. Use the Mantra to check arithmetically (by finding nonnegative coefficients that enable you to write the gradient of U as a nonnegative linear combination of the gradients of the two constraints) whether or not the KKT conditions can be satisfied at either of these points.

In Figure 2, these points are noted A and B. At point A, because the constraints (1) and (3) are satisfied with equality, $L=0$ and $Q=\frac{w(1-\tau)T+Y}{P}$. At this point $\bigtriangledown U=(0 \quad \infty)$ which is not well defined.In fact, because this gradient is not finite, we cannot find a finite linear combination of the gradients of g_1 and g_3. So strictly speaking we cannot apply the Mantra. At point B, because the constraints (1) and (4) are satisfied with equality, $L=T$ and $Q=\frac{Y}{P}$. At this point $\bigtriangledown U=((1/3)(\frac{Y}{P})^{-2/3}T^{1/2} \quad (1/2)(\frac{Y}{P})^{1/3}T^{-1/2})$ which is strictly positive. We just need to see if we can write this gradient as a positive linear combination of the gradients of g_1 and g_4. Let $(\phi_1,\phi_2)\in\mathbb{R}^2$. We solve the following system:

$$\phi_1 * p+\phi_2 * 0=\frac{1}{3}(\frac{Y}{P})^{-2/3}T^{1/2}$$
$$\phi_1 * w(1-\tau)+\phi_2 * 1=\frac{1}{2}(\frac{Y}{P})^{1/3}T^{-1/2}$$

This system leads to the following result:

$$\phi_1=\frac{1}{3p}(\frac{Y}{P})^{-2/3}T^{1/2}$$
$$\phi_2=\frac{1}{p}(\frac{Y}{P})^{-2/3}T^{-1/2}[\frac{Y}{2}-\frac{w}{3}(1-\tau)T]$$

so $\phi_1>0$ and $\phi_2=\frac{1}{p}(\frac{Y}{P})^{-2/3}T^{-1/2}\frac{1}{2}[Y-\frac{2w}{3}(1-\tau)T]$; since $Y<\frac{w(1-\tau)T}{2}$, we can write $Y-\frac{2}{3}*w(1-\tau)T\leq Y-\frac{w(1-\tau)T}{2}<0$ so $\phi_2<0$ which means that the gradient of U is not in the nonnegative cone formed by the gradients of g_1 and g_4, i.e., point B does not satisfy the Mantra either.

(e) Check whether the Constraint Qualification holds at each of the points you have identified in (c) (You should consider three cases.)

First the CQ conditions: we will show that the Jacobian of the gradients is full rank (a sufficient condition for the CQ to hold). For the set of points identified in (c) we see that only up to two constraints may be satisfied with equality: g_1 and g_3 or g_1 and g_4. The gradients are the following:

$$\bigtriangledown g_1=(p \quad w(1-\tau))$$
$$\bigtriangledown g_3=(0 \quad -1)$$
$$\bigtriangledown g_4=(0 \quad 1)$$

so as long as $p>0$ and $w(1-\tau)>0$ the gradient of g_1 will not be proportional to the gradient of g_3 or of g_4, which means that the two couples of gradients are linearly independent. Thus, in any case on this set the Jacobian of the constraint will be full rank. Note also that in the case where only

g_1 is satisfied with equality the only gradient of the constraint is linearly independent because it is nonzero. Hence, the CQ are satisfied.

(f) Check whether the necessary conditions of the KKT are also going to be sufficient. In this problem, you do *not* need to check a bordered Hessian condition, you just need to check a Hessian condition. Explain why.

The feasible set is clearly convex. We check the Hessian of the utility function.

$$\bigtriangledown(U) = (U/3Q \quad U/2L)$$

defined for $Q > 0$ and $L > 0$.

$$H_u(Q,L) = \begin{pmatrix} (-2/9)U/Q^2 & U/6(QL) \\ U/6(QL) & (-1/4)U/L^2 \end{pmatrix}$$

The first leading principal minor is strictly negative. The second is:

$$(-2/9)U/Q^2 * (-1/4)U/L^2 - (U/6(QL))^2 = U^2/(Q^2L^2)(2/36 - 1/36) > 0$$

So the Hessian of $U(\cdot)$ is negative definite, meaning that $U(\cdot)$ is strictly concave and thus strictly quasi-concave. Thus the necessary conditions of the KKT are also going to be sufficient. (You don't need to check the Bordered Hessian because the function is concave.)

(g) *Without doing any further computations*, what can you conclude about the solution, if any, to this problem? If a solution does exist, what system of equations will deliver it? Explain what theorem(s) you are invoking in order to conclude what you are concluding, and explain why it is appropriate to invoke it/them.

The feasible set is clearly compact and convex, and the objective function is strictly concave and hence strictly quasi-concave. Hence the conditions for existence of a unique solution are satisfied. Since the only possible solutions lie on the bold line in Figure 2, and we've ruled out both A and B, the solution must lie in the interior of the line segement joining A and B.

Because the constraint qualification is satisfied along this line segment, the KKT conditions are necessary for a solution. Because the gradient of the function is non-vanishing along it, because the objective is strictly quasi-concave and the feasible set is convex, the KKT conditions are also sufficient. Accordingly you have to solve for the KKT conditions.

(h) **Bonus part:** Write the Lagrangian and the first-order conditions to the problem defined in (a), and use the results of (g) to compute the solution to this problem.

We can define the Lagrangian as:

$$L(Q,L,\lambda,\mu,\nu,\rho) = AQ^{1/3}L^{1/2} - \lambda(pQ + w(1-\tau)L - w(1-\tau)T - Y) + \mu Q + \nu L - \rho(L-T)$$

The first-order necessary and sufficient conditions are:

$$\frac{\partial L}{\partial Q} = \frac{1}{3}AQ^{-2/3}L^{1/2} - \lambda p + \mu = 0 \tag{1}$$

$$\frac{\partial L}{\partial L} = \beta AQ^{1/3}L^{-1/2} - \lambda w(1-\tau) + \nu - \rho = 0 \tag{2}$$

$$\lambda \geq 0 \quad \mu \geq 0 \quad \nu \geq 0 \quad \rho \geq 0 \tag{3}$$

$$\lambda(pQ + w(1-\tau)L - w(1-\tau)T - Y) = 0 \tag{4}$$

$$\mu Q = 0 \tag{5}$$

$$\nu L = 0 \tag{6}$$

$$\rho(L - T) = 0 \tag{7}$$

$$\frac{\partial L}{\partial \lambda} = -(pQ + (1-\tau)wL - (1-\tau)wT - Y) \geq 0 \tag{8}$$

$$\frac{\partial L}{\partial \mu} = Q \geq 0 \tag{9}$$

$$\frac{\partial L}{\partial \nu} = L \geq 0 \tag{10}$$

$$\frac{\partial L}{\partial \rho} = -(L - T) \geq 0 \tag{11}$$

From our conclusions in (c) and (d), we know that only constraint g_1 is satisfied with equality. so all the other are not, meaning that $\mu = 0$, $\nu = 0$, and $\rho = 0$, and $Q > 0$, $L > 0$. This simplifies the two first FOCs to:

$$\frac{\partial L}{\partial Q} = \frac{A}{3}Q^{-2/3}L^{1/2} - \lambda p = 0$$
$$\frac{\partial L}{\partial L} = \frac{A}{2}Q^{1/3}L^{-1/2} - \lambda w(1-\tau) = 0$$

Suppose $\lambda = 0$ then we would get $\alpha AQ^{-2/3}L^{1/2} = 0$ but we know that $A > 0$,$Q > 0$, and $L > 0$ so this is not possible. Thus $\lambda > 0$, which means that the constraint satisfied with equality is binding. First we rearrange the two FOCs and we write the equality constraint:

$$\lambda = \frac{AQ^{-2/3}L^{1/2}}{3p}$$
$$\lambda = \frac{AQ^{1/3}L^{-1/2}}{2w(1-\tau)}$$
$$pQ + w(1-\tau)L = w(1-\tau)T + Y$$

By setting the left hand side of the two first equations to be equal we get: $L = \frac{3pQ}{2w(1-\tau)}$. Then replacing this value into the equality constraint :

$$\begin{aligned} Q^{\star} &= \frac{w(1-\tau)T + Y}{p(1+\frac{3}{2})} \\ &= \frac{2[w(1-\tau)T + Y]}{5p} \end{aligned}$$

$$\begin{aligned} L^{\star} &= \frac{T}{1+\frac{2}{3}} + \frac{Y}{w(1-\tau)(1+\frac{2}{3})} \\ &= \frac{3T}{5} + \frac{3Y}{5w(1-\tau)} \end{aligned}$$

In addition: $\lambda^\star = \frac{A(Q^\star)^{-2/3}(L^\star)^{1/2}}{3p}$, and $\mu^\star = \nu^\star = \rho^\star = 0$.

Problem 2. (40 points).

Proof of the Stolper-Samuelson Theorem: *"An increase in the price of a good will raise the real returns to the factor used intensively in that good and lower the real return to the other factor."*

Consider the following two-sector, two-input model (2*2): Good 1 is produced at a relative price p and good 2 is the *numeraire*, sold at a price of 1. Prices are determined on the world market and so are exogenous to the economy. The two goods are produced with labor L at a wage w and capital K with a rental price r. The economy's aggregate endowment of both inputs is fixed. We assume that sector 1 is labor intensive, and sector 2 is capital intensive, so that when inputs are used optimally, $\frac{K_1}{L_1} < \frac{K_2}{L_2}$.

Because of free entry, the firms earn zero profits in equilibrium, thus:

$$C^1(w,r) = p$$
$$C^2(w,r) = 1$$

where C^i denotes the *unit cost function* for good i, formally defined as

$$C^i(w,r) = \min_{L,K} \{wL_i + rK_i \text{ s.t. } q^i(L_i,K_i) \geq 1\}$$

where $q^i(\cdot)$ is the production function, meaning that it is defined as the minimum cost to produce one unit of good i. Suppose that the minimum is obtained with the strictly positive quantities of input $L_i^\star$ and $K_i^\star$.

(a) Write the equilibrium conditions for this system as the level set of some $f : \mathbb{R}^3 \to \mathbb{R}^2$. What are the exogenous and the endogenous variables? (Hint: Neither L nor K should appear in your answer.)

Let $f(p;w,r) = \begin{bmatrix} C^1(w,r) & - & p \\ C^2(w,r) & - & 1 \end{bmatrix}$. The equilibrium of the system is now represented as the level set corresponding to zero of the function f.

(b) Write down the unit cost functions $C^i(\cdot)$ in terms of $L_i^\star$ and $K_i^\star$.

Using the notations defined for the optimal choice variables, we can write $C^i(w,r) = wL_i^\star + rK_i^\star$.

(c) Show that the derivative of $C^i(\cdot)$ with respect to wages is equal to $L_i^\star$.

C^i is the maximized objective function, and the wage will not enter the production constraint, so by the Envelope Theorem:

$$\frac{dC^i(w,r)}{dw} = \frac{\partial C^i(w,r)}{\partial w} = L_i^\star$$

(d) Similarly, what is the derivative of $C^i(\cdot)$ with respect to rental prices? (Write your result in terms of $L_i^\star$ and $K_i^\star$)

By the exact same reasoning as in (c), $\frac{dC^i}{dr} = K_i^\star$

(e) *Assume* for the moment that the condition for using the implicit function theorem is satisfied. Using your answers to (c) and (d), write down the differential of the Jacobian of the implicit function relating the wage and rental rates to the price of good 1. (Hint: in terms of the notation used in lectures, your answer should be of the form $\mathbf{dx} = Jg(\alpha)\mathbf{d}\alpha$.)

$$dC^1(w,r) = \frac{\partial C^1(w,r)}{\partial w}dw + \frac{\partial C^1(w,r)}{\partial r}dr = dp \tag{12}$$

$$dC^2(w,r) = \frac{\partial C^2(w,r)}{\partial w}.dw + \frac{\partial C^2(w,r)}{\partial r}dr = 0 \tag{13}$$

Using the results from (c) and (d), we get:

$$L_1^\star dw + K_1^\star dr = dp \tag{14}$$

$$L_2^\star dw + L_2^\star dr = 0 \tag{15}$$

or in matrix form:

$$\begin{bmatrix} dw \\ dr \end{bmatrix} = \begin{bmatrix} L_1^\star & K_1^\star \\ L_2^\star & K_2^\star \end{bmatrix}^{-1} \begin{bmatrix} dp \\ 0 \end{bmatrix}$$

(f) Now check to see if the condition for the implicit function theorem is satisfied

We check the sufficient condition of the implicit function theorem, using the differential system derived in (e):

$$\begin{vmatrix} L_1^\star & K_1^\star \\ L_2^\star & K_2^\star \end{vmatrix} = L_1^\star K_2^\star - L_2^\star K_1^\star$$

Because sector 1 is labor intensive, we know that: $\frac{K_1^\star}{L_1^\star} < \frac{K_2^\star}{L_2^\star}$ so that $L_2^\star K_1^\star < L_1^\star K_2^\star$ which means that $L_1^\star K_2^\star - L_2^\star K_1^\star > 0$. Thus we know that we can define w and r as implicit functions of p. We will note them $w(p)$ and $r(p)$.

(g) Show that a small increase in the price of good 1 leads to an increase in the wage.

We use the first-order approximation: $w(p+dp) = w(p) + \frac{dw}{dp}dp$. To compute $\frac{dw}{dp}$ we use the implicit function theorem and apply Cramer's rule:

$$\frac{dw}{dp} = \begin{vmatrix} 1 & K_1^\star \\ 0 & K_2^\star \end{vmatrix} * (L_1^\star K_2^\star - L_2^\star K_1^\star)^{-1} \tag{16}$$

$$= \frac{K_2^\star}{L_1^\star K_2^\star - L_2^\star K_1^\star} \tag{17}$$

As $K_2^\star > 0$ and the denominator is also strictly positive, we find that $\frac{dw}{dp} > 0$ so that an increase in the price of good 1 leads to an increase in the wage.

(h) Show that a small increase in the price of good 1 leads to a decrease in rental prices.

Similarly: $r(p+dp) = r(p) + \frac{dr}{dp}dp$, then we apply Cramer's rule:

$$\frac{dr}{dp} = \begin{vmatrix} L_1^\star & 1 \\ L_2^\star & 0 \end{vmatrix} * (L_1^\star K_2^\star - L_2^\star K_1^\star)^{-1} \tag{18}$$

$$= \frac{-L_2^\star}{L_1^\star K_2^\star - L_2^\star K_1^\star} \tag{19}$$

As $-L_2^\star < 0$ and the denominator is strictly positive, we find that $\frac{dr}{dp} < 0$ so that an increase in the price of good 1 leads to a decrease in the rental price.

(i) What is the effect of a small increase in the price p on the real rental price (r/p)?

From our results in (f), if p increases then r decreases then necessarily r/p decreases.

(j) **Bonus part:** What is the the effect of an increase in the price p on the real wage (w/p)? (Hint 1: use the zero-profit conditions, Hint 2: take it for granted that $L_1^\star K_2^\star - L_2^\star K_1^\star - (L_1^\star/r) < 0$.)

This is a little more tricky. Obviously we cannot simply conclude that we will find a decrease in the real wage as in (h). First, using the Chain Rule: $\frac{d(w(p)/p)}{dp} = \frac{1}{p}(\frac{dw(p)}{dp} - \frac{w(p)}{p})$. This means that we need to compare $\frac{dw}{dp}$ and $\frac{w(p)}{p}$. We have already derived $\frac{dw}{dp}$ in (e), so we just need to derive $\frac{w(p)}{p}$. Use the first zero-profit condition that we can rewrite it as:

$$wL_1^\star + rK_1^\star = p$$
$$wL_2^\star + rK_2^\star = 1$$

The second equation leads to: $w = \frac{1-rK_2^\star}{L_2^\star}$ and plugging this expression into the first equation leads to:

$$p = rK_1^\star + \frac{L_1^\star}{L_2^\star}(1 - rK_2^\star)$$

Now we get:

$$\begin{aligned}\frac{w(p)}{p} &= \frac{(1-rK_2^\star)/L_2^\star}{rK_1^\star + (L_1^\star(1-rK_2^\star)/L_2^\star)} \\ &= \frac{1-rK_2^\star}{rL_2^\star K_1^\star + L_1^\star - rL_1^\star K_2^\star} \\ &= \frac{1/r - K_2^\star}{(L_2^\star K_1^\star - L_1^\star K_2^\star) + L_1^\star/r} \\ &= \frac{(K_2^\star) - 1/r}{(L_1^\star K_2^\star - L_2^\star K_1^\star) - (L_1^\star/r)}\end{aligned}$$

Using Hint 2 so changing the sense of inequality when multiplying by the denominator of $\frac{w(p)}{p}$, we get:

$$\begin{aligned}\frac{dw}{dp} > \frac{w}{p} \quad &\Leftrightarrow \frac{K_2^\star}{L_1^\star K_2^\star - L_2^\star K_1^\star} > \frac{(K_2^\star) - 1/r}{(L_1^\star K_2^\star - L_2^\star K_1^\star) - (L_1^\star/r)} \\ &\Leftrightarrow K_2^\star(L_1^\star K_2^\star - L_2^\star K_1^\star) - \frac{L_1^\star K_2^\star}{r} < (K_2^\star - \frac{1}{r})(L_1^\star K_2^\star - L_2^\star K_1^\star) \\ &\Leftrightarrow -L_1^\star K_2^\star < L_2^\star K_1^\star - L_1^\star K_2^\star \\ &\Leftrightarrow 0 < L_2^\star K_1^\star\end{aligned}$$

Which implies that $\frac{d(w(p)/p)}{dp} > 0$ so the real wage increases with a raise in price.

You have just proved the Stolper-Samuelson Theorem !