Problem 1

Let \( A = \begin{pmatrix} 3 & 3 & -1 & 0 \\ -2 & -3 & 1 & 0 \\ -6 & -2 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \) and \( B = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & -6 \end{pmatrix} \)

a) Show that \( A \) and \( B \) are nonsingular.

b) Calculate \( A^{-1}, B^{-1}, (AB)^{-1}, \) and \( (A^T)^{-1}. \)

c) Solve the linear equation system \( Ax = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix} \) and \( (AB)x = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \)

Ans:

a) Show that \( A \) and \( B \) are nonsingular. A matrix is nonsingular if its determinant is nonzero.

\[
\begin{align*}
\det(A) &= \begin{vmatrix} 3 & 3 & -1 & 0 \\ -2 & -3 & 1 & 0 \\ -6 & -2 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} \\
&= \begin{vmatrix} 3 & 3 & -1 \\ -2 & -3 & 1 \\ -6 & -2 & 1 \end{vmatrix} + \begin{vmatrix} 3 & 3 \\ -6 & -2 \end{vmatrix} + \begin{vmatrix} 3 & 3 \\ -2 & -3 \end{vmatrix} \\
&= (-4 - 18) - (-6 + 18) + (-9 + 6) = 14 - 12 - 3 = -1 \neq 0
\end{align*}
\]

Note that for a diagonal matrix the determinant is simply the product of the diagonal elements:

\[
\det(B) = \begin{vmatrix} -1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & -6 \end{vmatrix} = 36 \neq 0
\]

b) Let’s first calculate the inverse of \( A \)

\[
\begin{pmatrix} 3 & 3 & -1 & 0 \\ -2 & -3 & 1 & 0 \\ -6 & -2 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
\]

Add the \( 3^{rd} \) row to the \( 1^{st} \) row and \((-1)\) \( 3^{rd} \) row to the \( 2^{nd} \) row:

\[
\begin{pmatrix} -3 & 1 & 0 & 0 \\ 4 & -1 & 0 & 0 \\ -6 & -2 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}
\]

Add the \( 2^{nd} \) row to the \( 1^{st} \) row:
\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
4 & -1 & 0 & 0 \\
-6 & -2 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 1 & 0 & 0 \\
0 & 1 & -1 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]

Add 6*1\(^{st}\) row to the 3\(^{rd}\) row and (-4)*1\(^{st}\) row to the 2\(^{nd}\) row:

\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & -2 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 1 & 0 & 0 \\
-4 & -3 & -1 & 0 \\
6 & 6 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]

Divide the 2\(^{nd}\) row by (-1) and add 2*2\(^{nd}\) row to the 3\(^{rd}\) row:

\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 1 & 0 & 0 \\
4 & 3 & 1 & 0 \\
14 & 12 & 3 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]

Hence, \(A^{-1}\) = \[
\begin{pmatrix}
1 & 1 & 0 & 0 \\
4 & 3 & 1 & 0 \\
14 & 12 & 3 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]

Since \(B\) is a digonal matrix, the inverse is simply a digonal matrix where each diagonal element is the inverse of the corresponding diagonal element of \(B\).

Hence, \(B^{-1}\) = \[
\begin{pmatrix}
-1 & 0 & 0 & 0 \\
0 & \frac{1}{2} & 0 & 0 \\
0 & 0 & \frac{1}{3} & 0 \\
0 & 0 & 0 & -\frac{1}{6}
\end{pmatrix}
\]

Using the fact that \((AB)^{-1} = B^{-1}A^{-1}\) we know:

\[
(AB)^{-1} = \begin{pmatrix}
-1 & 0 & 0 & 0 \\
0 & \frac{1}{2} & 0 & 0 \\
0 & 0 & \frac{1}{3} & 0 \\
0 & 0 & 0 & -\frac{1}{6}
\end{pmatrix}
\begin{pmatrix}
1 & 1 & 0 & 0 \\
4 & 3 & 1 & 0 \\
14 & 12 & 3 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
= \begin{pmatrix}
-1 & -1 & 0 & 0 \\
2 & \frac{3}{2} & 1 & \frac{1}{2} \\
\frac{14}{3} & 4 & 1 & 0 \\
0 & 0 & 0 & -\frac{1}{6}
\end{pmatrix}
\]

Using the fact that \((A^T)^{-1} = (A^{-1})^T\) we know:

\[
(A^T)^{-1} = \begin{pmatrix}
1 & 4 & 14 & 0 \\
1 & 3 & 12 & 0 \\
0 & 1 & 3 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]

c) Solving \(Ax = b\) we know: \(x = A^{-1}b\)

\[
x = \begin{pmatrix}
1 & 1 & 0 & 0 \\
4 & 3 & 1 & 0 \\
14 & 12 & 3 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
1 \\
-1 \\
0 \\
0
\end{pmatrix}
= \begin{pmatrix}
0 \\
1 \\
2 \\
0
\end{pmatrix}
\]

Solving \((AB)x = b\) we know: \(x = (AB)^{-1}b\)

\[
x = \begin{pmatrix}
-1 & -1 & 0 & 0 \\
2 & \frac{3}{2} & 1 & \frac{1}{2} \\
\frac{14}{3} & 4 & 1 & 0 \\
0 & 0 & 0 & -\frac{1}{6}
\end{pmatrix}
\begin{pmatrix}
1 \\
0 \\
0 \\
0
\end{pmatrix}
= \begin{pmatrix}
-2 \\
\frac{26}{3} \\
\frac{3}{3} \\
0
\end{pmatrix}
\]
Problem 2

**Ans:** In this problem you were asked to solve linear equation systems of the form $Ax = b$ using Cramer’s rule.

**a)** $A = \begin{pmatrix} 5 & 1 \\ 2 & -1 \\ 3 & 1 \\ 4 & -1 \end{pmatrix}$ and $b = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$

\[
\begin{align*}
x_1 &= \frac{5 \cdot 4 - 1 \cdot 2}{5 \cdot 1 - 4 \cdot 2} = \frac{20 - 2}{5 - 8} = \frac{18}{-3} = -6 \\
x_2 &= \frac{5 \cdot 3 - 1 \cdot 4}{5 \cdot 1 - 4 \cdot 2} = \frac{15 - 4}{5 - 8} = \frac{11}{-3} = \frac{-11}{3} \\
\end{align*}
\]

**b)** $B = \begin{pmatrix} 2 & -3 & 0 \\ 4 & -6 & 1 \\ 1 & 10 & 0 \end{pmatrix}$ and $b = \begin{pmatrix} 2 \\ 7 \\ 1 \end{pmatrix}$

Always develop the determinant after the third column:

\[
\begin{align*}
x_1 &= \frac{2 \cdot 1 \cdot 1 - 4 \cdot 1 \cdot 0 - 1 \cdot (-3) \cdot 0}{2 \cdot 1 \cdot 0 - 4 \cdot (-3) \cdot 0 - 1 \cdot 1 \cdot 0} = \frac{2 + 3}{2 + 4} = \frac{5}{6} \\
x_2 &= \frac{2 \cdot 1 \cdot 1 - 4 \cdot 1 \cdot 1 - 0 \cdot 1 \cdot 0}{2 \cdot 1 \cdot 0 - 4 \cdot (-3) \cdot 0 - 1 \cdot 1 \cdot 0} = \frac{2 - 4}{2 + 4} = \frac{-2}{6} = -\frac{1}{3} \\
x_3 &= \frac{2 \cdot 1 \cdot 1 - 4 \cdot 1 \cdot 1 - 0 \cdot 1 \cdot 0}{2 \cdot 1 \cdot 0 - 4 \cdot (-3) \cdot 0 - 1 \cdot 1 \cdot 0} = \frac{2}{2 + 4} = \frac{1}{3} \\
\end{align*}
\]

Problem 3
Show that if the matrix $A$ is nonsingular and symmetric, then the matrix $A^{-1}$ is also symmetric. (You can use as a fact that the left- and right inverse of the matrix $A$ are the same and that the inverse is unique).

**Ans:** $A^{-1}A = I$ (Definition of the inverse)

$= I^T$ (Identity matrix is symmetric)
\[= (\mathbf{A} \mathbf{A}^{-1})^T \quad \text{(Left inverse equals right inverse)}\]
\[= (\mathbf{A}^{-1})^T \mathbf{A}^T \quad \text{(see problem 1a)}\]
\[= (\mathbf{A}^{-1})^T \mathbf{A} \quad \text{(A is symmetric)}\]

Since the inverse is unique we therefore know that: \( \mathbf{A}^{-1} = (\mathbf{A}^{-1})^T \), i.e., \( \mathbf{A}^{-1} \) is symmetric.

**Problem 4**

Given an example of a 2x2 matrix \( \mathbf{A} \) such that the function \( f(\mathbf{x}) = \mathbf{A}\mathbf{x} \)

a) maps the unit circle to itself.

b) maps the unit circle to a line in \( \mathbb{R}^2 \).

c) maps the unit circle to a single point.

d) over several iterations, maps the unit circle to an ellipse the size of Miami.

e) rotates every vector by 45 degrees (counter-clockwise) and stretches it by a factor of 2.

For each of the above, provide a sketch (or series of sketches) and an explanation.

**Ans:** Given an example of a 2x2 matrix \( \mathbf{A} \) such that the function \( f(\mathbf{x}) = \mathbf{A}\mathbf{x} \)

a) **maps the unit circle to itself:**

   The only symmetric matrix that maps the unit circle to itself is the identity matrix (there are also non-symmetric matrices which do so)

   \[ \mathbf{A}_a = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \]

b) **maps the unit circle to a line in \( \mathbb{R}^2 \).**

   Any matrix with rank 1 will do so, for example:

   \[ \mathbf{A}_b = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \]

c) **maps the unit circle to a single point.**

   The only matrix that maps the unit circle to a point is the zero matrix:

   \[ \mathbf{A}_c = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \]

d) **over several iterations, maps the unit circle to an ellipse the size of Miami.**

   Any matrix with two eigenvalues bigger than one in modulus will do so. Since I am lazy, I take the example from part e).

   e) **rotates every vector by 45 degrees (counter-clockwise) and stretches it by a factor of 2.**

   Following the example in the lecture notes, the matrix is:

   \[ \mathbf{A}_e = \begin{pmatrix} \sqrt{2} & -\sqrt{2} \\ \sqrt{2} & \sqrt{2} \end{pmatrix} \]

The images are displayed on the next page. In the first graph of figure 1, the vectors \( \mathbf{v}_1 \) and \( \mathbf{v}_2 \) are the eigenvectors. In the second graph, the matrix has no eigenvectors, but for illustration purposes, the unit vectors \( \mathbf{v}_1 \) and \( \mathbf{v}_2 \) and their images are displayed.
Figure 1. Images of the unit circle under $A_a, A_b,$ and $A_c$ (top graph) and under $A_e$ (bottom graph).
Problem 5

Let $(M^n)$ be a sequence of $2 \times 2$ symmetric matrices. Assume that

(i) there exists $\alpha \in (-1,0)$ such that for all $n$, $\det(M^n) = \alpha$.

(ii) For each $n$, let $\lambda^n_1 > 0$ be the eigenvalue corresponding to $v^1$ for the matrix $M^n$. Assume that $(\lambda^n_1)$ is a strictly decreasing sequence with $\lim_n \lambda^n_1 = 0$.

(iv) For each $n$, denote by $\lambda^n_2$ the eigenvalue corresponding to $v^2$ for the matrix $M^n$. Assume that $|\lambda^n_2| = |\lambda^n_1|$, i.e., the eigenvectors have the same absolute value for $M^1$.

Questions:

(1) Sketch the image of the unit circle $C$ under $M^n$ for three values of $n$ including $n = 1$, say, for example $n = 1, 2, 4$. Your graph should reflect what you know about $\alpha$. Include a sketch of the unit circle on your figure, for reference. Also, indicate the image of the eigenvectors on your graph. Label everything on your graph clearly so there's no ambiguity about what is what.

Ans:

(2) What can you say about the definiteness of each $M^n$?

Ans: Since $\alpha < 0$, there must be one positive and one negative eigenvalue. So each matrix is indefinite.

(3) Does $(\lambda^n_2)$ contain a convergent subsequence? Justify your answer.

Ans: We know that for each $n$, $\det(M^n) = \prod_{i=1}^{2} \lambda^n_i = \alpha$. Hence $\lambda^n_2 = \alpha / \lambda^n_1$. Since $(\lambda^n_1) \to 0$, $(\lambda^n_2)$ increases without bound. Hence it cannot contain a convergent subsequence.

(4) Let $P^n_+ = \{x \in C : x'M^n x > 0\}$ and $P^n_- = \{x \in C : x'M^n x < 0\}$. What is:

(i) $\cup_{n=1}^{\infty} P^n_+$

(ii) $\cap_{n=1}^{\infty} P^n_+$

(iii) $\cup_{n=1}^{\infty} P^n_-$

(iv) $\cap_{n=1}^{\infty} P^n_-$

You may prefer to draw a picture to illustrate some or all of these.

Ans: Answers not in order:

(ii) Since $(\lambda^n_1)$ is positive and shrinking to zero, the sets $P^n_+$ decrease with $n$. The only vector that belongs to $P^n_+$ for all $n$ is $v^1$. Thus $\cap_{n=1}^{\infty} P^n_+ = \{v^1, -v^1\}$

(iii) Eventually, all vectors get pulled toward $v^2$, the eigenvector with the negative eigenvalue that increases without bound. Hence Thus $\cup_{n=1}^{\infty} P^n_- = C \{v^1, -v^1\}$
Problem 6

(1) Given $N \in \mathbb{N}$, $N > 2$, we say that a nonempty set $W$ is an $N$-vector space if $\{v_1, ..., v_N\} \subset W$ and $\alpha \in \mathbb{R}^N$ implies $\sum_{i=1}^{N} \alpha_i v_i \in W$. Show that for any $N \in \mathbb{N}$, a set $W$ is an $N$-vector space iff it is a vector space.

**Ans:** If $N \leq 2$, there’s nothing to prove, so assume $N > 2$. The proof in one direction is completely trivial. Suppose $W$ is an $N$-vector space. Now consider $\{v^1, v^2\} \subset W$ and $\alpha \in \mathbb{R}^2$. Extend $\alpha$ to $\mathbb{R}^n$ by adding zeros and let $\{v^3, ..., v^N\}$ be arbitrarily chosen. We have $\sum_{i=1}^{2} \alpha_i v^i = \sum_{i=1}^{N} \alpha_i v^i \in W$, proving that $W$ is a vector space.

Now suppose $W$ is an vector space. Trivially, we know that $W$ is also a $2$-vector space. Now assume that for some $n \geq 2$, we’ve proved that $W$ is an $n$-vector space. (We have done so for $n = 2$.) We’ll prove that $W$ is also an $(n+1)$-vector space. Arbitrarily pick $\{v^1, ..., v^{n+1}\} \subset W$ and $\alpha \in \mathbb{R}^{n+1}$. Let $w = \sum_{i=1}^{n} \alpha_i v^i$ and note that by assumption $w \in W$. Since $W$ is a vector space $\sum_{i=1}^{n+1} \alpha_i v^i = w + \alpha_{n+1} v^{n+1} \in W$. Therefore, $W$ is an $(n+1)$-vector space.

(2) The remaining parts of this question relate to the following construction. Fix $\theta \in \mathbb{R}^5$ and a set $K \subset \mathbb{N}$. Let

$$X(\theta, K) = \left\{ \text{sequences in } \mathbb{R}^5 \text{ s.t.} \begin{cases} x_n = \theta & \text{for all } n \in K \\ x_{n,2} = x_{n,3} & \text{for all } n \in K^C \end{cases} \right\}$$

where $x_{n,j}$ denotes the $j$'th component of the $n$'th element of the sequence. What is the largest collection of $\theta$'s in $\mathbb{R}^5$ and largest collection of sets $K$’s for which $X(\theta, K)$ is a finite dimensional vector space. To get full marks for this question, you must prove that for the pair of collections that you have identified,

(a) whenever $(\theta, K)$ belongs to this pair of collections, then $X(\theta, K)$ is a finite dimensional vector space,

(b) whenever $(\theta, K)$ does not belong to this pair of collections, then $X(\theta, K)$ is not a finite dimensional vector space.

**Ans:** Let $\Theta$ consist of all co-finite subsets of $\mathbb{N}$. A subset of $\mathbb{N}$ is co-finite if its complement in $\mathbb{N}$, denoted $K^C$, is a finite set. Let $\theta = 0$. Pick an arbitrary co-finite subset $K \subset \mathbb{N}$, pick sequences $x, y \in X(0, K)$, $\alpha, \beta \in \mathbb{R}$ and let $z$ denote the sequence $\alpha x + \beta y$. For all $n \in K$,

$$z_n = \alpha x_n + \beta y_n = \alpha 0 + \beta 0 = 0,$$

while for all $n \in K^C$, $z_{n,2} = z_{n,3}$. Therefore, $z \in X(0, K)$, establishing that $X(0, K)$ is a vector space.

The only way to prove that it’s finite dimensional is to provide a basis for the space. For $k \in K^C$ and $i = 1, 2, 4, 5$, let $y^{k,i}$ denote the sequence defined by, for $n \in \mathbb{N}$ and $j = 1, ..., 5$
\[ y_{n,j}^{k,i} = \begin{cases} 1 & \text{if } n = k \& i = 2 \& j = 2,3 \\ 1 & \text{if } n = k \& i \neq 2 \& j = i \\ 0 & \text{otherwise} \end{cases} \]

We will establish that this is a basis in the answer to the next part. For now note simply that the number of sequences we have defined is 4 times the number of elements in \( K^C \) which is a finite number. Hence \( X(0, K) \) is finite dimensional.

On the other hand,

(a) for \( \theta \neq 0 \), pick \( x, y \in X(\theta, K) \) and let \( z \) denote the sequence \( x + y \). For all \( n \in K \), \( z_n = x_n + y_n = \theta + \theta = 2 \theta \neq \theta \). Therefore, \( z \notin X(\theta, K) \), so that \( X(\theta, K) \) is not a vector space.

(b) If \( K \) is not co-finite, then the number of elements in a basis for \( X(\theta, K) \) is infinite, so that \( X(\theta, K) \) is not finite dimensional.

(3) Fix a set \( K \subset \mathbb{N} \) and \( \theta \in \mathbb{R}^5 \) such that \( X(\theta, K) \) is a finite-dimensional vector space. Find a basis for \( X(\theta, K) \). Do this abstractly, not for a specific \( K \) and \( \theta \). That is, you should give one answer that “works” for all \( K \) and all \( \theta \) such that \( X(\theta, K) \) is a vector space.

Demonstrate that it is a basis. Hint: it is quite possible that you have already partially or fully completed part c) in your answer to part b). If you have, simply refer to your previous answer; don’t repeat work you’ve already done.

**Ans:** A basis was provided in the answer to the previous part. Call it \( Y \). We now just have to check that it is indeed a basis. To verify this, we need to check that: (a) \( Y \) is a subset of \( X(0, K) \); (b) \( Y \) spans \( X(0, K) \); (c) any proper subset of \( Y \) will not span \( X(0, K) \).

Clearly, each element of \( Y \) belongs to \( X(0, K) \). Moreover for an arbitrarily chosen \( x \in X(0, K) \), it is clearly the case that \( x = \sum_{j=1,2,4,5} \sum_{n \in K^C} x_{n,j} y^{n,j} \). Hence \( Y \) spans \( X(0, K) \).

Finally, suppose that \( y^{k,i} \) were omitted from \( Y \), for some \( k \in K^C \) and \( i = 1, 2, 4, 5 \). Since for all remaining \( y \in Y \), \( y_{k,i} = 0 \), \( y^{k,i} \) cannot be written as a linear combination of the remaining elements of \( Y \). Hence \( Y \) is a set of basis vectors for \( X(0, K) \).

(4) Given a set \( K \subset \mathbb{N} \) and \( \theta \in \mathbb{R}^5 \) such that \( X(\theta, K) \) is a finite dimensional vector space, what is the dimension of \( X(\theta, K) \)?

**Ans:** The dimension of \( X(\theta, K) \) is the number of elements of any basis for \( X(\theta, K) \). As noted already, the dimension of \( X(\theta, K) \) is \( 4 \times \# K^C \).

(5) Given a set \( K \subset \mathbb{N} \) and \( \theta \in \mathbb{R}^5 \) such that \( X(\theta, K) \) is a finite dimensional vector space, find a minimal spanning set for \( X(\theta, K) \) that is not a basis. Again, do this abstractly. Demonstrate that it spans, is minimal, but that it isn’t a basis.

**Ans:** For \( k \in K^C \), and \( i = 1, \ldots, 5 \), let \( z^{k,i} \) denote the sequence defined by, for \( n \in \mathbb{N} \) and \( j = 1, \ldots, 5 \),

\[ z^{k,i}_{n,j} = \begin{cases} 1 & \text{if } n = k \& j = i \\ 0 & \text{otherwise} \end{cases} \]

We now verify that \( Z = \{ z^{k,i} \}_{k \in K^C, i = 1, \ldots, 5} \) is a minimal spanning set for \( X(0, K) \) but not a basis. Clearly, no element of this set belongs to \( X(0, K) \), since for \( n \in K^C \), \( z^{k,i}_{n,1} \neq z^{k,i}_{n,3} \). It spans the set however since for an arbitrarily chosen \( x \in X(0, K) \), it is clearly the case that \( x = \sum_{j=1,2,4,5} \sum_{n \in K^C} x_{n,j} y^{n,j} \). To show that \( Z \) is a minimal spanning set, pick arbitrarily \( k \in K^C \), and \( i = 1, \ldots, 5 \) and omit the sequence \( z = z^{k,i} \) from \( Z \). Now consider the “corresponding” member of the basis set \( y \in Y \), defined above, where
\[ y = \begin{cases} \gamma^{k,2} & \text{if } i = 3 \\ \gamma^{k,i} & \text{otherwise} \end{cases} \]. By construction, \( y_{k,i} = 1 \), but for all \( z' \in Z \) except for \( z \), \( z'_{k,i} = 0 \). Therefore, \( y \) cannot be written as a linear combination of the members of \( Z \) if \( z \) is excluded.
Problem 7

Fix $n \in \mathbb{N}$ and a vector $v^0 \in \mathbb{R}^n$, two natural numbers $J > 1$ and $K > 1$, and a nonempty set $Q \subset \{1, \ldots, JK\}$. Now let $\mathbb{M}$ denote the set of all $n \times JK$ matrices $M$ such that $M = [x^{1, \ldots, x^{JK}}]$ where

$$x^m = \begin{cases} v^0 & \text{if } m \in Q \\ v^k & \text{if } m \notin Q \text{ and } m = k + jK, \text{ for } j \in \{0, \ldots, J\} \text{ and } k \in \{1, \ldots, K\} \end{cases}$$

for some $n \times K$ matrix $V = [v^1, \ldots, v^K]$. (Note that each distinct element of $\mathbb{M}$ is defined by a different vector $V$.)

(1) Think of an example of an element of $\mathbb{M}$, for $K = 3$, $J = 4$ a set $Q$ that has at least 3 elements and a $n \times K$ matrix $V$. (Keep it simple!!) Write down $Q$, $n$, $v^0$ and $V$ Then write down your matrix $M$.

Ans: $n = 1$; $Q = \{2, 3, 4\}$; $v^0 = 4$, $V = \{1, 2, 3\}$, $M = [1, 4, 4, 4, 2, 3, 1, 2, 3, 1, 2, 3]$.

(2) Identify conditions under which $\mathbb{M}$ is a vector space.

Ans: The condition is that $v^0 = 0$.

(3) Demonstrate that if the conditions you identified in part B) are satisfied, $\mathbb{M}$ is indeed a vector space.

Ans: Consider $M^1, M^2 \in \mathbb{M}$ and $\alpha \in \mathbb{R}^2$. Let $M^3 = \alpha_1 M^1 + \alpha_2 M^2$ where for $i = 1, 2, 3$, $M^i = [x^{1,i}, \ldots, x^{JK,i}]$ and

$$x^{i,3} = \begin{cases} 0 & \text{if } m \in Q \\ \alpha_1 x^{1,k} + \alpha_2 x^{2,k} & \text{if } m \notin Q \text{ and } m = jk, \text{ for } j \in \{1, \ldots, J\} \text{ and } k \in \{1, \ldots, K\} \end{cases}$$

Hence $M^3 \in \mathbb{M}$.

(4) Demonstrate that if the conditions you identified in part B) are not satisfied, $\mathbb{M}$ is not a vector space.

Ans: Suppose that $n = 1$ and $v^0 = 1$. Let $Q = \{1\}$ and let $J = K = 2$. Consider the matrix $M = [1, 1, 1, 1]$ and note that $2M = [2, 2, 2, 2] \notin \mathbb{M}$.

(5) Assume now that your conditions guaranteeing that $\mathbb{M}$ is a vector space are satisfied. Write down the dimension of $\mathbb{M}$.

Ans: The dimension of $\mathbb{M}$ is $n\ell$, where $\ell = \max[K, JK - \#Q]$. For example, if $Q = \{1, 2, 3\}$ and $K = 4$, then for $J = 1$, then $M \in \mathbb{M}$ implies $M$ consists of one $n$-vector plus 3 zero vectors. Hence, $\ell = 1$ and the dimension of $\mathbb{M}$ is $n$. If $J = 5$, then $M \in \mathbb{M}$ implies $M$ consists of 5 $n$-vectors plus 3 zero vectors. But the fourth and the eighth vectors are the same. Hence $\ell = 4$ and the dimension of $\mathbb{M}$ is $4n$.

(6) Continuing to assume that your conditions guaranteeing that $\mathbb{M}$ is a vector space are satisfied, write down a basis for $\mathbb{M}$.
Ans:
For $i = 1, \ldots, n$, let $e^i$ denote the $i$'th canonical vector, i.e., the vector whose $i$'th component is 1 and all other components are 0. Now for $i = 1, \ldots, n$ and $k = 1, \ldots, K$, let $M_{i,k}$ denote the matrix whose $m$'th column is $\begin{cases} 0 & \text{if } m \in Q \\ e^i & \text{if } m \notin Q \text{ and } m = jk, \text{ for } j \in \{1, \ldots, J\} \\ 0 & \text{otherwise} \end{cases}$. The family of matrices $\{M_{i,k}\}_{i=1,\ldots,n \atop k=1,\ldots,K}$ is a basis for $\mathbb{M}$. 