Problem 1

What is the difference between a minimum spanning set for a vector space and a basis for a vector space. Provide an example highlighting this difference.

Ans: The minimum spanning set does not have to be part of the vector space $V$, while a basis has to be an element of the vector set. For example, consider the vector space

$$V = \{x | x = \alpha \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \alpha \in \mathbb{R}\}$$

which is a straight line through the origin.

A minimal spanning set for $V$ are the two vectors:

$$v^1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad v^2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Take away any of the two vectors and the remaining vector does not span $V$. However they are not a basis as neither $v^1$ nor $v^2$ are an element of $V$. 
Problem 2

Simon & Blume question 11.2 (page 243). Explain your answer.

Ans: There are several ways to test for linear dependence of the vectors \( \mathbf{v}^1, \mathbf{v}^2, \ldots, \mathbf{v}^n \). The definition of linear independence was

Show that the linear equation system \([\mathbf{v}^1 \ \mathbf{v}^2 \ \cdots \ \mathbf{v}^n] \mathbf{x} = \mathbf{0}\) has the only solution \( \mathbf{x} = \mathbf{0} \). (Where \( \mathbf{x} \) is a \((n \times 1)\) column vector).

a) (i) Given: \( \mathbf{v}^1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \mathbf{v}^2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \)

(ii) Let’s consider the linear equation system \([\mathbf{v}^1 \ \mathbf{v}^2] \mathbf{x} = \mathbf{0}\)

\[
\begin{pmatrix}
2 & 1 \\
1 & 2
\end{pmatrix}
\]

Add (-2) times the second row to the first row:

\[
\begin{pmatrix}
0 & -3 \\
1 & 2
\end{pmatrix}
\]

Add (-2) times the first row to the second:

\[
\begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix}
\]

Hence we know from the first row that \( x_1 = 0 \) and from the second row that \( x_2 = 0 \)

(iii) Since 0 is the only solution to the linear equation system in (ii), we know that \( \mathbf{v}^1 \) and \( \mathbf{v}^2 \) are linearly independent.

b) (i) Given: \( \mathbf{v}^1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \mathbf{v}^2 = \begin{pmatrix} -4 \\ -2 \end{pmatrix} \)

(ii) Let’s consider the linear equation system \([\mathbf{v}^1 \ \mathbf{v}^2] \mathbf{x} = \mathbf{0}\)

\[
\begin{pmatrix}
2 & -4 \\
1 & -2
\end{pmatrix}
\]
Add (-2) times the second row to the first row:

\[ \begin{pmatrix} 0 & 0 & : & 0 \\ 1 & -2 & : & 0 \end{pmatrix} \]

Clearly, \( x_1 = 2 \) and \( x_2 = 1 \) is a nonzero solution to the system

(iii) Since the linear equation system in (ii) has a nonzero solution, we know that \( \mathbf{v}^1 \) and \( \mathbf{v}^2 \) are linearly dependent.

\[ \begin{align*}
\text{c)} & \quad \text{(i) Given: } \mathbf{v}^1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \mathbf{v}^2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \\
\text{(ii) Let's consider the linear equation system } [\mathbf{v}^1 \mathbf{v}^2] \mathbf{x} = \mathbf{0} \\
& \quad \begin{pmatrix} 1 & 0 & : & 0 \\ 1 & 1 & : & 0 \\ 0 & 1 & : & 0 \end{pmatrix}
\end{align*} \]

We know from the first row that \( x_1 = 0 \) and from the third row that \( x_2 = 0 \)

(iii) Since 0 is the only solution to the linear equation system in (ii), we know that \( \mathbf{v}^1 \) and \( \mathbf{v}^2 \) are linearly independent.

\[ \begin{align*}
\text{d)} & \quad \text{(i) Given: } \mathbf{v}^1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \mathbf{v}^2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \mathbf{v}^3 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \\
\text{(ii) Let's consider the linear equation system } [\mathbf{v}^1 \mathbf{v}^2 \mathbf{v}^3] \mathbf{x} = \mathbf{0} \\
& \quad \begin{pmatrix} 1 & 0 & 1 & : & 0 \\ 1 & 1 & 0 & : & 0 \\ 0 & 1 & 1 & : & 0 \end{pmatrix}
\end{align*} \]

Add (-1) times the first row to the third row:

\[ \begin{pmatrix} 1 & 0 & 1 & : & 0 \\ 1 & 1 & 0 & : & 0 \\ -1 & 1 & 0 & : & 0 \end{pmatrix} \]

Add the third row to the second row:

\[ \begin{pmatrix} 1 & 0 & 1 & : & 0 \\ 0 & 2 & 0 & : & 0 \\ -1 & 1 & 0 & : & 0 \end{pmatrix} \]

Divide the second row by 2

\[ \begin{pmatrix} 1 & 0 & 1 & : & 0 \\ 0 & 1 & 0 & : & 0 \\ -1 & 1 & 0 & : & 0 \end{pmatrix} \]
Add (-1) times the second row to the third row:

\[
\begin{pmatrix}
1 & 0 & 1 & : & 0 \\
0 & 1 & 0 & : & 0 \\
-1 & 0 & 0 & : & 0 \\
\end{pmatrix}
\]

Add the third row to the first row:

\[
\begin{pmatrix}
0 & 0 & 1 & : & 0 \\
0 & 1 & 0 & : & 0 \\
-1 & 0 & 0 & : & 0 \\
\end{pmatrix}
\]

We know from the first row that \( x_3 = 0 \), from the second row that \( x_2 = 0 \), and from the third row that \( x_1 = 0 \).

(iii) Since 0 is the only solution to the linear equation system in (ii), we know that \( \mathbf{v}^1 \), \( \mathbf{v}^2 \) and \( \mathbf{v}^3 \) are linearly independent.
Problem 3

Simon & Blume question 11.9 (page 246).

Ans: This question asks you to do a basis transformation, i.e., write a vector \( \mathbf{b} \) as a linear combination of vectors.

a) (i) Given: \( \mathbf{v}^1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \mathbf{v}^2 = \begin{pmatrix} 1 \\ 4 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 2 \\ 2 \end{pmatrix} \),

(ii) Let’s consider the linear equation system
\[
\begin{pmatrix} \mathbf{v}^1 & \mathbf{v}^2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}
\]

\[
\begin{pmatrix}
1 & 1 : 2 \\
2 & 4 : 2
\end{pmatrix}
\]

Add (-2) times the first row to the second row:
\[
\Leftrightarrow \begin{pmatrix}
1 & 1 : 2 \\
0 & 2 : -2
\end{pmatrix}
\]

Divide the second row by 2:
\[
\Leftrightarrow \begin{pmatrix}
1 & 1 : 2 \\
0 & 1 : -1
\end{pmatrix}
\]

Add (-1) times the second row to the first:
\[
\Leftrightarrow \begin{pmatrix}
1 & 0 : 3 \\
0 & 1 : -2
\end{pmatrix}
\]

(iii) We know from (ii) that \( \mathbf{b} \) can be written as \( \mathbf{b} = 3 \mathbf{v}^1 - \mathbf{v}^2 \).

b) (i) Given: \( \mathbf{v}^1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \mathbf{v}^2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \mathbf{v}^3 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \),

(ii) consider the equation system
\[
\begin{pmatrix} \mathbf{v}^1 & \mathbf{v}^2 & \mathbf{v}^3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}
\]

\[
\begin{pmatrix}
1 & 1 & 0 : 1 \\
1 & 0 & 1 : 2 \\
0 & 1 & 1 : 3
\end{pmatrix}
\]
Add (-1) times the first row to the second row:

\[
\begin{pmatrix}
1 & 1 & 0 & : & 1 \\
0 & -1 & 1 & : & 1 \\
0 & 1 & 1 & : & 3
\end{pmatrix}
\]

Add the second row to the third row:

\[
\begin{pmatrix}
1 & 1 & 0 & : & 1 \\
0 & -1 & 1 & : & 1 \\
0 & 0 & 2 & : & 4
\end{pmatrix}
\]

Divide the third row by 2

\[
\begin{pmatrix}
1 & 1 & 0 & : & 1 \\
0 & -1 & 1 & : & 1 \\
0 & 0 & 1 & : & 2
\end{pmatrix}
\]

Add (-1) times the third row to the second row:

\[
\begin{pmatrix}
1 & 1 & 0 & : & 1 \\
0 & -1 & 0 & : & -1 \\
0 & 0 & 1 & : & 2
\end{pmatrix}
\]

Multiply the third row by (-1)

\[
\begin{pmatrix}
1 & 1 & 0 & : & 1 \\
0 & 1 & 0 & : & 1 \\
0 & 0 & 1 & : & 2
\end{pmatrix}
\]

Add (-1) times the second row to the first row:

\[
\begin{pmatrix}
1 & 0 & 0 & : & 0 \\
0 & 1 & 0 & : & 1 \\
0 & 0 & 1 & : & 2
\end{pmatrix}
\]

(iii) We know from (ii) that \( \mathbf{b} \) can be written as \( \mathbf{v}^2 + 2 \mathbf{v}^3 \)
Problem 4


Ans: This question asks you to determine whether a set of vectors is a basis or not. From Simon & Blume Theorem 11.7 (page 248) you know that a basis of $\mathbb{R}^n$ contains exactly $n$ vectors and from Theorem 11.8 you know that they form a basis iff the $n$ vectors are linearly independent. There are again several ways to do this.

(i) Following problem 2, show that 3 vectors are linearly independent.
(ii) Calculate the determinant of the 3 vectors in part b)-d). If it is nonzero, we know that the 3 vectors are linearly independent and therefore form a basis

Since we did method (i) in problem 3, let’s do method (ii) here.

a) Two vectors cannot span $\mathbb{R}^3$ and we therefore know that the vectors can’t be a basis. (In general, a set of $n$ vectors can span at most a space of dimension $n$).

b) \[ \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 1 & 1 \end{vmatrix} \]

Develop after the third column:
\[ \det = 1 \begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} + 1 \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} = 1 \cdot 2 + 1 \cdot (1 - 1) = 0 \]

Since the determinant is zero, the three vectors are linearly dependent and thus can’t be a basis.

c) \[ \begin{vmatrix} 6 & 5 & 4 \\ 3 & 2 & 1 \\ 9 & 8 & 7 \end{vmatrix} \]

Develop after the third column:
\[ \det = 4 \begin{vmatrix} 5 & 4 \\ 8 & 7 \end{vmatrix} - 1 \begin{vmatrix} 6 & 5 \\ 9 & 8 \end{vmatrix} + 7 \begin{vmatrix} 6 & 5 \\ 3 & 2 \end{vmatrix} = 4(40 - 28) - (48 - 45) + 7(12 - 15) = 24 - 3 - 21 = 0 \]

Since the determinant is zero, the three vectors are linearly dependent and thus can’t be a basis.

d) \[ \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 1 & 0 \end{vmatrix} \]

Develop after the third column:
\[ \det = 1 \begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} = 1 - 2 = -1 \]

Since the determinant is nonzero, the three vectors are linearly independent and span $\mathbb{R}^3$ and hence are a basis.

e) 4 vectors can not be a basis for $\mathbb{R}^3$ as they have to be linearly dependent. (In general, if your vector space has dimension $n$ and you have more than $n$ vectors, they have to be linearly dependent).
Problem 5

Consider the following conjecture: Let $V$ be a vector space spanned by the set $\{v_1, v_2, \ldots v_n\}$. The set $\{v_1, v_2, \ldots v_n\}$ is a minimal spanning set for $V$ iff the vectors $v_1, v_2, \ldots v_n$ are linearly independent vectors. Is the conjecture true? Prove your answer.

Ans: The conjecture is false because of the “$\Leftarrow$”: Consider:

$$v^1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, v^2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, v^3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

And let $V = \{\alpha \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \alpha \in \mathbb{R}\}$ (A line in the plane where $x_3 = 0$).

Clearly, $v^1, v^2, \text{ and } v^3$ are linearly independent, they span $V$ (As they span $\mathbb{R}^3$ and hence any subset of it) but they are not a minimal spanning set (as you can drop $v^3$ and still span $V$).
Problem 6

Let $U$ and $W$ be vector subspaces of the vectorspace $V$. Define the space

$$ U + W = \{ x | x = u + w, u \in U, w \in W \} $$

Show that $U + W$ is a vector subspace of $V$.

**Ans:** Recall that you have to check three conditions to show that $X$ is a vector subspace of the vector space $V$:

(i) $X$ has to be nonempty
(ii) $\forall x_1, x_2 \in X : x_1 + x_2 \in X$
(iii) $\forall x_1 \in X, \alpha \in \mathbb{R}^+: \alpha x_1 \in X$

Now let’s prove the claim that if $U, W$ are vector subspaces then so is $U + W$.

(i) Since $0$ is an element of any vector subspace it is an element of $U$ and $W$. But therefore

$$ 0 = \underbrace{0}_{\in U} + \underbrace{0}_{\in W} $$

and hence $0$ is an element of $U + W$. Consequently, $U + W$ is nonempty.

(ii) (1) $\forall x_1, x_2 \in U + W : \exists u_1, u_2 \in U$ and $\exists w_1, w_2 \in W$ such that

$$ x_1 = u_1 + w_1 \text{ and } x_2 = u_2 + w_2. $$

(2) Since $U$ is a vector space we know that $u_1 + u_2 \in U$

(3) Since $W$ is a vector space we know that $w_1 + w_2 \in W$

(4) From (2) and (3) we therfore know that

$$ x_1 + x_2 = u_1 + w_1 + u_2 + w_2 \quad (\text{By (1)}) $$

is an element of $U + W$

(iii) (1) $\forall x_1 \in U + W : \exists u_1 \in U$ and $\exists w_1 \in W$ such that

$$ x_1 = u_1 + w_1 $$

(2) Since $U$ is a vector space we know that $\forall \alpha \in \mathbb{R}^+: \alpha u_1 \in U$

(3) Since $W$ is a vector space we know that $\forall \alpha \in \mathbb{R}^+: \alpha w_1 \in W$

(4) From (2) and (3) we therfore know that

$$ \alpha x_1 = \alpha (u_1 + w_1) = \underbrace{\alpha u_1}_{\in U} + \underbrace{\alpha w_1}_{\in W} \quad (\text{By (1)}) $$

is an element of $U + W$
Problem 7

a) Let A be a symmetric (nxn) matrix with one or more negative eigenvalues. What can you say about the determinant of the matrix and its rank?

b) Let A be symmetric (2x2) matrix. Is it true that A is indefinite iff its determinant is negative. Explain your answer (No formal proof necessary).

Ans:

a) Let A be a symmetric (nxn) matrix with one or more negative eigenvalues. What can you say about the determinant of the matrix and its rank?

The determinant of a symmetric matrix is the product of its eigenvalues. Therefore, the determinant is

(i) negative if it has an odd number of negative eigenvalues (and all eigenvalues are different from zero).

(ii) positive if it has an even number of negative eigenvalues (and all eigenvalues are different from zero).

(iii) zero if it has at least one eigenvalue that is zero.

The number of negative eigenvalues doesn’t say anything about the rank of a matrix. From (iii) above you know that the rank is of full rank iff the matrix has only nonzero eigenvalues.

b) Let A be symmetric (2x2) matrix. Is it true that A is indefinite iff its determinant is negative?

We know that every symmetric (2x2) matrix has exactly 2 real eigenvalues. Let’s consider all possible cases:

(i) Both eigenvalues are positive ⇒ the matrix is positive definite

(ii) Both eigenvalues are negative ⇒ the matrix is negative definite

(iii) One eigenvalue is positive, the other is zero ⇒ the matrix is positive semi-definite

(iv) One eigenvalue is negative, the other is zero ⇒ the matrix is negative semi-definite

(v) Both eigenvalues are zero ⇒ the matrix is both positive semi-definite and negative semi-definite

(vi) One eigenvalue is positive, the other one is negative ⇒ the matrix is indefinite

Hence, A is indefinite iff its determinant is negative. (The product of its two eigenvalues is negative).