Problem 1
Please use the Pythagorean metric in the following problem. (The Pythagorian metric is another name for the $d^2$ metric, defined in class as $d^2(x, y) = \sqrt{\sum_{i=1}^{n}(x_i - y_i)^2}$.)

a) Consider the sequence $x_n = 2 + \frac{(-1)^n}{n}$ defined on $\mathbb{R}$. Prove (i) that the sequence is a convergent sequence using the definition of a convergent sequence and show (ii) that the sequence is a Cauchy sequence using the definition of a Cauchy sequence.

b) Now consider the sequence $x_n = 2 + \frac{(-1)^n}{n}$ defined on $S = \mathbb{R} \setminus \{2\}$. Using your proof from part a) argue that it is still a Cauchy sequence in $S$. Prove that it is not a convergent sequence in $S$.

(Note: The set $A \setminus B$ is defined as: $A \setminus B = \{x | x \in A, x \notin B\}$).

Problem 2

a) Prove that a sequence $x_n$ in $X$ converges in the discrete metric if and only if there exists $\bar{x} \in X$ and a $N \in \mathbb{N}$ such that for all $n > N$, $x_n = \bar{x}$.

b) In class we showed that every Cauchy sequence in $\mathbb{R}$ with respect to the Pythagorean metric is also a convergent sequence in $\mathbb{R}$ with respect to the Pythagorean metric. Show that every Cauchy sequence in $\mathbb{R}$ with respect to the discrete metric is also a convergent sequence in $\mathbb{R}$ with respect to the discrete metric.

c) Problem 1 showed you that a Cauchy sequence that is defined on a strict subset of $\mathbb{R}$ does not have to converge in that subset. Again only considering the discrete metric, can we say that every Cauchy sequence defined on a subset $S \subset \mathbb{R}$ is also a convergent sequence in that subset. If yes, show why. If not, give a counter-example.

Problem 3
Show that every convergent sequence (in an arbitrary universe $X$ with respect to any metric defined on $X \times X$) is a Cauchy sequence under the same metric. (Hint: This proof is very short. Use the general definition of a metric).
Problem 4
For each of the following, draw and describe the $\epsilon$-ball $B_d(x, \epsilon; X)$ for some $\epsilon > 0$ around the point $x$ in the specified metric $d(x,y)$ and universe $X$. (In part (d) you might not be able to draw it, so just sketch parts of it). Give a brief explanation of why the $\epsilon$-ball looks the way it does. For (d) and (e), consider two cases, one where $\epsilon < 1$ and another where $\epsilon > 1$.

<table>
<thead>
<tr>
<th>Part</th>
<th>$x$</th>
<th>$d(x,y)$</th>
<th>$X$</th>
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<tbody>
<tr>
<td>(a)</td>
<td>3</td>
<td>$</td>
<td>x-y</td>
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<tr>
<td>(b)</td>
<td>(2,1)</td>
<td>$\max{</td>
<td>x_1-y_1</td>
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<td>(c)</td>
<td>(1,2)</td>
<td>$\sum_{i=1}^2</td>
<td>x_i-y_i</td>
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<tr>
<td>(d)</td>
<td>2</td>
<td>discrete metric</td>
<td>Rationals $\mathbb{Q}$</td>
</tr>
<tr>
<td>(e)</td>
<td>(2,2)</td>
<td>Pythagorean metric</td>
<td>$\mathbb{Z} \times \mathbb{Z}$ where $\mathbb{Z}$ are the integers</td>
</tr>
</tbody>
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Problem 5
Show that each subset $S$ of an arbitrary universe $X$ is an open set in $X$ under the discrete metric.

Problem 6
Fix $a, b, c \in \mathbb{R}$ with $a < b < c$. Consider the following two subsets of $\mathbb{R}^2$:

$$S_h = \{ x \in \mathbb{R}^2 | a < x_1 < b ; x_2 = c \}$$

$$S_v = \{ x \in \mathbb{R}^2 | x_1 = a ; b < x_2 < c \}$$

Loosely speaking, $S_h$ is a line segment parallel to the horizontal axis and $S_v$ is a line segment parallel to the vertical axis.

- You know from Problem 5 that all $S_h$ and $S_v$ are open sets under the discrete metric.
  a) Show that neither $S_h$ nor $S_v$ are open sets under the Pythagorean metric.
  b) Is it possible that all $S_h$ are open sets and all $S_v$ are not open sets under the same metric.
    If yes, then look far and wide and give an example of such a metric. If not, prove why it isn’t possible.
Since we want to keep problem sets shorter, there are two more optional problems below. We do *strongly* recommend that you do them. I will grade them and record that you did them, but the points will not be a part of your grade unless you are a “border-line” case when it comes time to calculate the final grades.

**Optional Problem 1**
Prove the following: Given $S \subset \mathbb{R}$, $b \in \mathbb{R}$ is a greatest lower bound (infimum) of $S$ iff $b$ is a lower bound for $S$ and $\forall \, \epsilon > 0, \exists \, s \in S$, such that $s - b < \epsilon$. A very similar proof is in the notes; please try this first on your own without referring to that proof.

**Optional Problem 2**
Recall the definition of point-wise convergence we gave in class. A sequence of functions $f_n$ converges point-wise to a function $f$ on a set $X$ in the metric $d$ if $\forall \, x \in X$, given $\epsilon > 0 \exists \, N(x, \epsilon) \in \mathbb{N}$ such that $n > N$ implies $d(f_n(x), f(x)) < \epsilon$.

This definition implies that $N$ depends on the epsilon *and* on the $x$. As we discussed in section, it may not be possible to find an $N$ that works for *every* $x$ simultaneously. If you succeed in finding such an $N$, you have uniform convergence.

We define uniform convergence as: A sequence of functions $f_n$ converges uniformly to a function $f$ on the set $X$ in the metric $d$ if $\forall \, \epsilon > 0 \exists \, N(\epsilon) \in \mathbb{N}$ s.t. $\forall \, x \in X$, $n > N$ implies $d(f_n(x), f(x)) < \epsilon$.

For each of the following state whether they are true or not. If they are correct, prove them. If they are false, give a counter-example.

a) For a given universe $X$ and a metric $d$, every sequence of functions that converges uniformly also converges point-wise.

b) For a given universe $X$ and the Pythagorian metric, every point-wise convergent sequence of functions also converges uniformly.

c) For a given universe $X$ and the discrete metric, every point-wise convergent sequence of functions also converges uniformly.