1. Analysis (cont)

1.8. Topology of $\mathbb{R}^n$ (cont)

1.8.5. Closure of a Set

Definition: Let $A \subset X$. The closure of $A$ in $X$, denoted $\text{cl}(A)$ or $\bar{A}$ in $X$ is the intersection of all closed sets containing $A$.

Theorem: For $A \subset X$, $A$ is closed in $X$ iff $A = \text{cl}(A)$ in $X$.

1.8.6. Boundary of a Set. A point $x$ is a boundary point of a set $A \subset X$ if there are points arbitrarily close to $x$ that are in $A$ and if there are points arbitrarily close to $x$ that are in $X$ but not in $A$.
Example: The set \( A = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1 \} \). Its boundary is the circle \( \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1 \} \).

That is, the boundary is the border between \( A \) and \( X \setminus A \).

Definition: The set of boundary points of \( A \) in \( X \), denoted \( \text{bd}(A) \), is the set \( \text{cl}(A) \cap \text{cl}(X \setminus A) \).

Theorem: Let \( A \subset X \). A point \( x \in \text{bd}(A) \) iff \( \forall \epsilon > 0, \exists y, z \in B(x, \epsilon) \) such that \( y \in A \) and \( z \in X \setminus A \).

Example: The set \( \{1, 2, 3, 4, 5\} \) has no boundary points when viewed as a subset of the integers; on the other hand, when viewed as a subset of \( \mathbb{R} \), every element of the set is a boundary point.

Theorem: A set \( A \subset X \) is closed in \( X \) iff \( A \) contains all of its boundary points.

Note the difference between a boundary point and an accumulation point.

Take the set \( A = \{0\} \subset \mathbb{R} \). \( 0 \) is a boundary point of \( A \) but not an accumulation point. On the other hand, every element of the interval \( A = (0, 1) \subset \mathbb{R} \) is an accumulation point of \( A \), but \( A \) contains none of the boundary points of \( A \).

Though boundary points and accumulation points are resoundingly different in general, there is a close connection between the two concepts. Indeed, many textbooks use one of the following two definitions of a closed set:

1. A set is closed if it contains all of its accumulation points.
2. A set is closed if it contains all of its boundary points.

These two definitions would suggest that the two terms are synonyms for each other, but of course they aren’t. The following theorem explains the relationship.

Theorem: Given a set \( A \subset X \), a point \( x \in X \) that does not belong to \( A \) is a boundary point of \( A \) in \( X \) iff it is an accumulation point of \( A \) in \( X \).

Proof: “Only If.” Suppose \( x \in \text{bd}(A) \), but \( x \notin A \). Fix \( \epsilon > 0 \). By definition of a boundary point, \( \exists y \in B(x, \epsilon) \cap A \). Since \( x \notin A \), \( y \neq x \). We’ve established, then, that \( \forall \epsilon > 0, \exists x \neq y \in B(x, \epsilon) \).

Hence \( x \) is an accumulation point of \( A \). The “If” part is parallel. Please try to do it as an exercise.
1.8.7. Compact Sets. Importance of compact sets: continuous functions defined on compact sets always attain their extrema.

Definition: Given a set $S \subset X$, an open covering of $S$ is a collection $C$ of open sets in $X$ such that $S \subset \bigcup\{O : O \in C\}$.

This is a rather abstract notion because the set $C$ can be arbitrarily large, i.e., uncountably infinite.

Definition: A set $X$ is said to be compact if every open covering $C$ of $X$ has a finite open subcover, i.e., if there exists a finite collection of sets $\{O_1, \ldots, O_n\} \subset C$ such that $X \subset \bigcup_{i=1}^n O_i$.

(Note: In some definitions, e.g., Wikipedia, you'll see equality signs where I have containment signs. The reason is that these definitions apply to spaces not sets. Don’t worry about the distinction, it’s not important for us, just wanted to reassure you that I’m not being inconsistent with Wikipedia (and many other sources)!

Example: Consider the closed set $S = [0, 1]$, and for $n \in \mathbb{N}$, let $C = \{B(x, 1/n) : x \in S\}$. Obviously this uncountably infinite set covers $S$. But there are many finite subsets of $C$ that also cover $S$, in particular, the set $\{O_0, \ldots, O_{2n}\} \subset C$, where for $i = 0, \ldots, 2n$, $O_i = B(i/2n, 1/n)$.

Example: On the other hand, consider the open set $S = (0, 1)$, and its open cover $C = \{B(x, x/2) : x \in S\}$. Note that this set indeed covers $S$: pick $x \in S$, then $x \in B(x, x/2) \in C$. Now consider any $n \in \mathbb{N}$ and any finite set $X = \{x_1, \ldots, x_n\} \subset S$, that is $x_i \in (0, 1)$, for all $x_i \in X$. Now define $\{O_1, \ldots, O_n\} = \{B(x_i, x_i/2) : x_i \in X\}$. We’ll show that this subset doesn’t cover $S$. Let $x = \min\{x_i \in X\}$. Now $x/4 > 0$ so that $x/4 \in S$. However, $x/4 < x - x/2 = x/2 \leq x_i/2$, for all $x_i \in X$. Since any finite subset of $C$ is of the form we just considered, we’ve proved that there is no finite subset of $C$ that covers $S$.

Happily, the compact subsets of Euclidean spaces can be characterized in a simple way: compact sets are closed and bounded.

I’ve already given you a definition of boundedness for a subset of $\mathbb{R}$. Here’s one for subsets of $\mathbb{R}^n$. 


Definition: For $x \in \mathbb{R}^n$, the norm of $x$, written $||x||$, is the Euclidean distance between $x$ and zero, i.e., $||x|| = \sqrt{\sum_{i=1}^{n} x_i^2}$.

Definition: A set $A \subset \mathbb{R}^n$ is bounded if there exists a real number $b \in \mathbb{R}$ such that $||x|| \leq b$, for all $x \in A$. (Note that this definition of bounded is metric-specific!)

Theorem: (Heine Borel) A set $S \subset \mathbb{R}^n$ is compact with respect to a metric $d$ iff it is closed and bounded with respect to that metric.

It’s pretty straightforward to prove the “if” direction of this theorem, at least for the case of $S \in \mathbb{R}$.

(1) For any $S$ is not closed, one can generalize the construction I used in the above example (for $S = (0,1)$) to construct an open cover for which no finite subcover exists.

(2) Now consider $S$ that is not bounded above. Consider the collection $C = \{(-\infty, n) : n \in \mathbb{N}\}$. Clearly $C$ covers $\mathbb{R}$ so it covers $S$. Now for any finite subset $\mathcal{N}$ of $\mathbb{N}$, consider the subset $\{(-\infty, n) : n \in \mathcal{N}\}$ of $C$. Since $S$ is unbounded, there exists $x \in S$, such that $x > \max\{n \in \mathcal{N}\}$. Therefore, the subset we defined does not cover $S$.

Note that as with everything else in topology, the notion of compactness depends critically on the metric.

Example: Consider the set $[0,1]$. It’s clearly closed and bounded with respect to the Euclidean metric, and so compact in the Euclidean metric. It’s definitely not compact with respect to the discrete metric, however: the collection $C = \{\{x\} : x \in [0,1]\}$ is an open cover of $[0,1]$; it obviously has no finite open subcover.

1.9. The Bolzano-Weierstrass Theorem

We are going to prove one of the most fundamental theorems of analysis:

Theorem: (informally) Every sequence defined on a compact set contains a convergent subsequence.
(1) recall that a sequence in $S$ is called a *convergent sequence* if it converges to an element of the set $S$. E.g., the sequence $(1/n)$ in the set $S = (0, 2) \subset \mathbb{R}$ is *not* a convergent subsequence because the point in $\mathbb{R}$ that is the limit of the sequence does not belong to the set $S$ that contains the sequence.

(2) Recall the definition of a subsequence: a sequence $\{y_1, y_2, \ldots, y_n, \ldots\}$ is a subsequence of $\{x_1, x_2, \ldots, x_n, \ldots\}$ if there exists a *strictly increasing* function $\tau : \mathbb{N} \to \mathbb{N}$ such that for all $n \in \mathbb{N}$, $y_n = x_{\tau(n)}$. That is, you construct a subsequence by throwing out elements of the original sequence, but keeping an infinite number of the original elements *and preserving their order*.

Intuitively, the theorem we’re going to prove is obvious: points in the sequence have to bunch up somewhere. Nowhere for them to escape to. The basic idea of the proof is: because the set is bounded, we can construct a Cauchy subsequence of the original sequence; because the set is closed, the point to which the subsequence is trying to converge actually belongs to the set, and so is indeed a limit of the subsequence.

The proof is going to involve two *inductive constructions*: an inductive construction of a sequence involves:

1. an *initial* step: define the first element of the sequence
2. an *inductive* step: assume that the $n$’th element of the sequence has been defined; now define the $n + 1$’th element in terms of the $n$’th.

For example, a first order difference equation is defined by induction: define $x_0$; then define $x_{t+1} = ax_t + b$.

Back to Bolzano-Weierstrass. Here’s the intuition for the proof, for the case of a sequence in $\mathbb{R}$:

1. the *initial* step: define a set $A_1$. 


(a) Because the set \( S \) is compact, it’s bounded. Pick \( b \in \mathbb{R}_{++} \) such that \( S \) is contained in the interval \([-b/2, b/2]\). (You’ll see why we divide by 2 in a minute.) Now consider any sequence in \( S \).

(b) Divide the interval \([-b/2, b/2]\) into two halves; at least one half contains an infinite number of elements in the sequence (possibly both halves). Pick this half (if there are two such halves, pick either). The set you’ve picked will be called \( A_1 \).

(2) the inductive step: assume that \( A_n \) has been defined, now define \( A_{n+1} \).

(a) The inductive step: take the set you’ve just picked, \( A_n \), split it in half, and pick again, according to the same criterion, to get \( A_{n+1} \). Keep going forever; note that you now have an infinite sequence of nested sets.

(b) Define a subsequence of the original sequence as follows: the first element of the subsequence is the first element of the original sequence that belongs to the first subdivided set.

(c) The second element is the first element in the original sequence that comes after the one you’ve just chosen which lies in the second subdivided set, etc.

**Example:** Take the sequence \([-1, 1, -1, 1, \ldots] \subset [-1, 1]\). Set \( A_1 = [0, 1] \), \( A_2 = [1/2, 1] \), \( A_3 = [3/4, 1] \), etc. To construct a convergent subsequence, set \( \tau(1) = 2 \), \( \tau(2) = 4 \), \( \tau(3) = 6 \), etc., and note that for all \( n \), \( y_n = x_{\tau(n)} \in A_n \)

(d) Now you have constructed an infinite sequence of points with the property that for every \( N \), the entire tail of the sequence (after discarding the first \( N \) points) lies in the \( N \)’th subdivision.

(e) Since the subdivided intervals are getting smaller and smaller, the sequence is a Cauchy sequence.

(f) Every Cauchy sequence defined in \( \mathbb{R} \) converges to a point in \( \mathbb{R} \).

(g) The point in \( \mathbb{R} \) to which the sequence converges is either an element of the sequence or an accumulation point of the original compact set. Since the set is compact, hence closed, it contains its accumulation points, hence the sequence we’ve constructed indeed converges to a point in the set.
(3) It’s important to understand the relationship between our two inductive constructions (there were really three of them, but the second one wasn’t explicit).

(a) first, we constructed a sequence of subsets of the range of the sequence

(b) second, we considered a sequence of tails of the original sequence, i.e., a sequence of subsets of the domain of the sequence. The tricky part was to pick the sequence of tails in a way that accomplished what we wanted to accomplish: specifically, let $T_n$ be a tail such that all points in the tail map into $A_n$; now, toss out the first element of $T_n$ and let $T_{n+1}$ be the subset of ($T_n$ minus its first element) that maps into $A_{n+1}$. Now we consider the sequence of integers, consisting of the first element of each of our tail subsets.

**Theorem:** (Bolzano-Weierstrass) A set $A \subset \mathbb{R}^n$ is compact if and only if every sequence $\{x_n\}_{n=1}^{\infty}$ in $A$ has a convergent subsequence, i.e., there exists a subsequence $\{y_n\}_{n=1}^{\infty}$ of $\{x_n\}_{n=1}^{\infty}$ and a point $y \in A$ such that $\{y_n\}$ converges to $y$.

**Proof:** We’ll do the proof for $A \subset \mathbb{R}$ and only in one direction, i.e., we’ll show that $A$ compact implies each sequence in $A$ has a convergent subsequence.

Suppose that $A$ is bounded above by $b/2$ and below by $-b/2$: that is, the interval $[-b/2, b/2]$ contains the entire (infinite) number of points in the sequence. Split the set $[-b/2, b/2]$ into two closed subsets of equal length, i.e., into $[-b/2, 0]$ and $[0, b/2]$ and observe that at least one of these subsets contains an infinite number of elements of the sequence $\{x_n\}$. Denote by $A_1$ whichever subset this is. If both have this property, then pick either. The length of the subset $A_1$ is equal to $b/2$. Now argue by *induction*. Suppose that we’ve defined a subset $A_n$ of length $b/2^n$ which contains an infinite number of elements of the sequence $\{x_n\}$. (This statement is true for $n=1$ i.e., length of $A_1$ is $b/2 = b/2^1$.) We’ll construct a closed subset $A_{n+1}$ of length $b/2^{n+1}$ which contains an infinite number of elements of $\{x_n\}$: simply divide $A_n$ in half as before: at least one of the subsets contain an infinite number of elements. Now we’ve constructed an infinite sequence of subsets, $\{A_n\}$ with the property that for each $n$, (a) $A_{n+1} \subset A_n$ (i.e., the $A_n$’s are nested) (b) the length of $A_n$ equals $b/2^n$ (c) $A_n$ contains an infinite number of elements of $\{x_n\}$.
Now we’ll define a convergent subsequence \( \{y_n\} \). Define the strictly increasing sequence \( \tau \) as follows: let \( \tau(1) = \min\{k \in \mathbb{N} : x_k \in A_1\} \); now assume that \( \tau(n) \) has been defined and define \( \tau(n + 1) = \min\{k \in \{\tau(n) + 1, \ldots\} : x_k \in A_{n+1}\} \). Note that the sets we’re considering in the previous sentence are subsets of the domain of the sequence, i.e., subsets of \( \mathbb{N} \). Note also that by construction, \( \tau(\cdot) \) is a strictly increasing function of \( n \).

Observe that we now have constructed a subsequence \( \{y_n\} \), i.e., \( y_n = x_{\tau(n)} \), for each \( n \), with the property that for each \( n \), \( y_n \in A_n \). Moreover, since the \( A_n \)'s are nested, it follows that for all \( N \), and \( n, m > N \), \( y_n, y_m \in A_N \). That is, \( \{y_n\} \) is a Cauchy sequence. Now, we know that every Cauchy sequence in \( \mathbb{R} \) converges to a point in \( \mathbb{R} \). i.e., there exists \( y \in \mathbb{R} \) such that for all \( \epsilon > 0 \), \( \exists N \in \mathbb{N} \) such that for all \( n > N \), \( d(y_n, y) < \epsilon \). We need to show that \( y \in A \). If \( y = y_n \), for some \( n \), we’re done, since \( y_n \in A_n \subset A \). If not, then \( y \) is an accumulation point of the set \( A \), since for all \( \epsilon > 0 \), there exists \( n \) such that \( y_n \neq y \) and \( y_n \in B(y, \epsilon) \). But since \( A \) is a closed set, it contains all of its accumulation points. Hence \( y \in A \), and we’ve proved that the subsequence we’ve constructed converges to a point \( y \in A \). □

Perhaps the trickiest part of this proof is the point at which we build the subsequence, i.e., construct the mapping \( \tau \). The method I’ve proposed above is by no means the only way to build a sequence, but it has a distinct merit, which is that it works. It’s important to see what goes wrong with some of the alternative constructions one might think of. For example,

1. Suppose in the inductive step, i.e., once \( \tau(n) \) has been defined, you defined \( \tau(n + 1) = \min\{k \in \mathbb{N} : x_k \in A_{n+1}\} \) rather than, as I did above, \( \min\{k \in \mathbb{N} : k \geq \tau(n) : x_k \in A_{n+1}\} \). Why wouldn’t the former alternative work? Here’s an example: \( x_1 = 0; x_2 = 1; x_n = \frac{3}{8}, \forall n \geq 3 \). This sequence is bounded above by \( b/2 \), for all \( b \geq 2 \). Let \( b = 2 \) and define \( A_1 = [0, 1]; A_2 = [0, \frac{1}{2}]; A_3 = [\frac{4}{7}, \frac{1}{2}] \), etc. Now consider what \( \tau_n = \min\{k \in \mathbb{N} : x_k \in A_{n+1}\} \) delivers: \( \tau(1) = 1; \tau(2) = 1, \tau(3) = 3 \), delivering the sequence \( z = \{0, 0, \frac{3}{8}, \frac{3}{8}, \ldots\} \). This is not, however, a subsequence of \( \{x\} = \{0, 1, \frac{3}{8}, \frac{3}{8}, \frac{3}{8}, \ldots\} \). If, instead, you proceeded as above, and set \( \tau(n + 1) = \min\{k \in \mathbb{N} : k \geq \tau(n) : x_k \in A_{n+1}\} \), the resulting mapping would be \( \tau(1) = 1 \) and for all \( n > 1 \), \( \tau(n) = n + 1 \). This mapping would deliver the sequence \( y = \{0, \frac{3}{8}, \frac{3}{8}, \ldots\} \) which is a subsequence of \( \{x_n\} \).
(2) There was talk in class about picking the sequence element that is the minimum of all sequence elements in \( A_n \), or, more formally, setting \( \tau(n) = \text{argmin}(A_n) \) or \( x_{\tau(n)} = \min \{ x_k : x_k \in A_n \} \). There are a couple of reasons not to do this, but the most basic one is that in general this minimum won’t exist. For example, let \( x_n = 1/n \) and define \( A_n = [0, 1/n] \). Now note that for each \( n \), \( \min \{ x_k : x_k \in A_n \} \) doesn’t exist.