6. Foundations of Comparative Statics (cont)

6.3. Implicit function Theorem.

Hardest concept in the course. It is also the basis for all of comparative statics. If you know how to do KTC and use the implicit function theorem you can read 90% of the journals.

A notation problem: Dealing with the implicit function theorem creates an inescapable notation problem. It is all about endogenous variables and exogenous variables.

- Economists always, always, always write the endogenous variables first, i.e., \( f(x; \alpha) \) where \( \alpha \) is the vector of exogenous variables.
- The implicit function theorem is all about writing \( x \) as a function of \( \alpha \).
- Mathematicians always, always, always write \( x_{n+1} \) as a function of the first \( n \) variables. Translating to economic language, they always, always, always put the exogenous variables first. That is, if they used the \( \alpha \) and \( x \) notation, they would write \( f(\alpha; x) \)
- So we have a problem either way.
- I’ll go with the mathematicians, since you will be reading math textbooks to understand the implicit function theorem, i.e., I’ll write \( f(\alpha; x) \).
The implicit function theorem and comparative statics: The solution to any economic system can be represented as the level set of some function. Here’s a simple economic model: $S = S(p, t), D = D(p, y), S = D$, where $p$ denotes market price, $t$ denotes a tax rate paid by the producer and $y$ denotes consumer income level. The solution to this model can be represented as the level set $f(\alpha, x) = 0$, where $f = S - D, \alpha = t, y$ and $x = p$. When we do comparative statics, we change $\alpha$ and ask how $x$ has to change in order to stay on the same level set.

Implicit function theorem: motivation. Given a level set of a function $f : \mathbb{R}^2 \to \mathbb{R}$ and a starting point in $\mathbb{R}^2$, under what conditions:

1. can I characterize the level set locally (i.e., in a nbd of the starting point) as the graph of a function from $\alpha$ to $x$? That is, is there an alternative way to represent $\{(\alpha, x) : f(\alpha, x) = c\}$ as the graph of some function $x = g(\alpha)$?

2. can I characterize the level set as the graph of a differentiable function from $\alpha$ to $x$?

3. If I can write $x$ as a differentiable function of $\alpha$, what’s an easy way to compute the slope of this function?

Typically, in economics, we are interested not so much in the explicit functional relationship between $x$ and $\alpha$ as in the slope of this function; e.g., you often want to know the marginal rate of substitution, or marginal rate of transformation, but you rarely write capital explicitly as a function of labor, etc.

Look at the next diagram:

1. In Case 1, it’s transparent that $x$ is a perfectly nice, well behaved function of $\alpha$.

2. In Case 2, $x$ is certainly a function of $\alpha$, but it is not so well-behaved. Specifically, it isn’t differentiable. Look at the picture of the level set and say what’s wrong with it. Answer: $f_2(\tilde{\alpha}, \tilde{x}) = 0$.

3. Case 3 is a little more subtle: we can’t write $x$ globally as a function of $\alpha$. Most of the time, however, what we can do is to restrict attention to a small neighborhood of $(\alpha, x)$, look at the level set restricted to that neighborhood, and characterize this level set as the graph of a function mapping $\alpha$ to $x$. In fact, when we apply the implicit function theorem, that’s all we need to do. we only care about local details of the level set, say in a neighborhood of the tangency. Note also that I can’t always do this: at the point $(\tilde{\alpha}, \tilde{x})$ it doesn’t work.
Case 1: $x$ is a well-defined, differentiable function of $\alpha$

Case 2: $x$ is a well-defined, but not differentiable function of $\alpha$

Case 3: Locally, $x$ is a well-defined, differentiable function of $\alpha$, except at $(\bar{\alpha}, \bar{x})$

FIGURE 1. Deriving $x$ as a function of $\alpha$ from level sets

(4) Next point is about computation. In principle, could solve for $x$ explicitly, then take derivative of this function that I’ve computed, but this could be a mess. Example: $u(\alpha, x) = \sqrt{\alpha x}$; take the level set $\sqrt{\alpha x} = 4.5$ solve to get $x$ as a function of $\alpha$, then take derivative. Implicit function theorem says you don’t have to go to all this trouble; can calculate $g'(\cdot)$ directly from the derivatives of $f$, but only under certain conditions.
Back to my problem: implicit function theorem tells me that I can almost always get my graph if I’m willing to take a small enough nbd and give the computer a good enough starting point. But sometimes, when the level set I’m interested in happens to be vertical, there’s nothing I can do except switch the axes of my graph.

**Implicit function theorem (single variable version):** Given $f : \mathbb{R}^2 \to \mathbb{R}^1$ continuously differentiable and $(\bar{\alpha}, \bar{x}) \in \mathbb{R}^2$, if $\frac{\partial f(\alpha, x)}{\partial x} \neq 0$, then there exist neighborhoods $U^\alpha$ of $\bar{\alpha}$ and $U^x$ of $\bar{x}$ and a continuously differentiable function $g : U^\alpha \to U^x$ such that for all $\alpha \in U^\alpha$,

$$f(\alpha, g(\alpha)) = f(\bar{\alpha}, \bar{x}) \text{ i.e., } (\alpha, g(\alpha)) \text{ is on the level set of } f \text{ through } (\bar{\alpha}, \bar{x})$$

$$g'(\alpha) = -\frac{\frac{\partial f(\alpha, g(\alpha))}{\partial \alpha}}{\frac{\partial f(\alpha, g(\alpha))}{\partial x}}$$

Note that I don’t put bar’s on the $\alpha$’s. Why not? Because $g$ is a function, and its derivative everywhere on the nbd is defined by the above condition. Once again, $g'(\alpha)$ is the slope of the function that *locally* represents the level set through $(\bar{\alpha}, \bar{x})$.

If you just mindlessly do the math, then the whole thing is trivial, a complete nobrainer.

$$\frac{df(\alpha, g(\alpha))}{d\alpha} \equiv 0$$

$$= \frac{\partial f(\alpha, g(\alpha))}{\partial \alpha} + \frac{\partial f(\alpha, g(\alpha))}{\partial x}g'(\alpha)$$

Hence

$$g'(\alpha) = -\frac{\frac{\partial f(\alpha, g(\alpha))}{\partial \alpha}}{\frac{\partial f(\alpha, g(\alpha))}{\partial x}}$$

But in fact there are lots of subtleties that become apparent from the picture.

- The neighborhood condition: without further qualification, the condition that $(\alpha, x)$ lies on a given level set doesn’t *globally* associate a unique $x$ to $\alpha$.
  - Look at the picture with two $x$’s to each $\alpha$: the implicit function theorem identifies *two* functions, quite different, each of which represents a different piece of the same level set.
  - ask the usual Econ 1 question: by how does $x$ need to change in order to stay on the same indifference curve when $\alpha$ changes.
If you feed the computer just your starting value of $\alpha$ but don’t tell it your starting value of $x$, then the computer can’t tell which function to give you. Can’t tell whether the slope you are interested in is positive or negative: it depends on $x$.

Once you know both $x$ and $\alpha$, however, then you can almost always identify a little neighborhood on which there is a unique relationship between $x$ and $\alpha$.

Fortunately, this isn’t usually a problem for practical purposes, because we always do know where the starting point is; doing comparative statics which is a local business.

The condition on $\frac{\partial f(\bar{\alpha}, \bar{x})}{\partial x}$:
- Given the formula, it’s obvious that you can’t have $\frac{\partial f(\bar{\alpha}, \bar{x})}{\partial x} = 0$, else you wouldn’t be able to define the ratio.
- What does the derivative condition mean? The level set could be vertical at $(\bar{\alpha}, \bar{x})$, in which case there isn’t even a neighborhood on which the condition that $(\bar{\alpha}, \bar{x})$ lies on a given level set associates a unique $x$ to $\alpha$. If level set is vertical at $\bar{x}$, then $f_2(\bar{\alpha}, \bar{x}) = 0$; e.g., $f(\alpha, x) = \alpha^2 + x^2$; level sets are circles; level set for $f = 1$ is the unit circle; observe that $f_2(\alpha, x) = 2x = 0$, when $x = 0$.
- Hence the appropriate condition to avoid verticalness is $f_2(\bar{\alpha}, \bar{x}) \neq 0$.

Here’s an example using more familiar notation: the computation of the MRS. We have a utility function $u : \mathbb{R}^2 \to \mathbb{R}^1$, and want to know the slope of an indifference curve through $(\bar{z}_1, \bar{z}_2)$. In this case, fitting our specific example into the general notation of the implicit function theorem,

- $\alpha$ is $z_1$,
- $x$ is $z_2$.
- $f(\alpha, x)$ is $u(z_1, z_2) - u(\bar{z}_1, \bar{z}_2)$.

hence we have $f(\bar{z}_1, \bar{z}_2) = 0$, and we want to vary $\alpha$ (in our example, $z_1$) and see how $x$ (in our example, $z_2$) has to change in order to keep us on the level set of $f$ corresponding to $0$.

We’ll write $z_2$ as a function of $z_1$, i.e., at this point,

$$\left. \frac{dz_2}{dz_1} \right|_{u \text{ constant}} = -\frac{\frac{\partial u(z_1, z_2)}{\partial z_1}}{\frac{\partial u(z_1, z_2)}{\partial z_2}}$$

Another example that illustrates the computational value of the theorem. $f(\alpha, x) = \alpha x^{15} + \alpha^{13} + x^{95}$.
• clearly difficult to write $g$ explicitly.
• $f_1(\alpha, x) = x^{15} + 13\alpha^{12}$
  $f_2(\alpha, x) = 15\alpha x^{14} + 95x^{94}$;
• Implicit function theorem says that $g'(\alpha) = -f_1(\alpha, g(.))/f_2(\alpha, g(.)) = -\frac{(x^{15} + 13\alpha^{12})}{(\alpha x^{14} + 95x^{94})}$