1. Analysis (cont)

Before proceeding to the next topic, we’ll prove a completely obvious Fact about sequences and subsequences. (We could call it a Lemma, but that’s glorifying it.) Recall that to show that \((y_n)\) is a subsequence of \((x_n)\), you always have to show the existence of a strictly increasing function \(\tau : \mathbb{N} \to \mathbb{N}\) such that for all \(n \in \mathbb{N}\), \(y_n = x_{\tau(n)}\). Now note:

**Fact:** If \(\tau : \mathbb{N} \to \mathbb{N}\) is a strictly increasing mapping, then for all \(n\), \(\tau(n) \geq n\).

To prove this, we’ll argue by induction.

- **Initial step:** \(\tau(1) \geq 1\) (duh).
- **Inductive step:** suppose that \(\tau(n) \geq n\); then \(\tau(n + 1) \geq n + 1\).

  **Proof of the Inductive step:** Let \(\tau(n) = k \in \mathbb{N}\); since \(\tau(n + 1) > \tau(n)\) and \(\tau(n + 1) \in \mathbb{N}\), \(\tau(n + 1) \geq k + 1\). Since by assumption \(k \geq n\), \(\tau(n + 1) \geq k + 1 \geq n + 1\).

1.9. Continuous Functions

A function is continuous if it maps nearby points to nearby points. Draw the graph without taking pen off paper. Graph is connected. Formally:
Lemma: Consider \( f : X \to \mathbb{R}^k \). Fix \( x_0 \in X \). The function \( f \) is continuous at \( x_0 \) if whenever \( \{x_m\}_{m=1}^{\infty} \) is a sequence in \( X \) which converges to \( x_0 \), then \( \{f(x_m)\}_{m=1}^{\infty} \) converges to \( f(x_0) \). The function \( f \) is continuous if it is continuous at \( x \), for every \( x \in X \).

We’ll first prove a lemma that we need to prove today’s main result.

Proof of the Lemma: We’ll just show that the function is bounded above, by proving that if \( X \) is compact and \( f \) isn’t bounded, then \( f \) cannot be continuous. Assume that \( f \) isn’t bounded, i.e., for all \( m \in \mathbb{N} \), \( \exists x_m \) such that \( f(x_m) > m \). Since \( X \) is compact, the sequence \( \{x_n\} \) contains a convergent subsequence. Call this subsequence \( \{y_n\} \) and let \( y \in X \) denote its limit. Define \( \tau : \mathbb{N} \to \mathbb{N} \) by \( y_n = x_{\tau(n)} \) for all \( n \). Since \( f \) is defined on \( X \), \( f(y) \in \mathbb{R} \), that is, \( f(y) < N \), for some \( N \in \mathbb{N} \). Now pick \( n \geq N + 1 \) and note that by the Fact above, \( \tau(n) \geq n \geq N + 1 \). Moreover, by assumption, \( f(\tau(n)) = f(x_{\tau(n)}) \geq \tau(n) \geq N + 1 \). Hence for all \( n > N + 1 \), \( f(y_n) - f(y) > 1 \), so that \( f \) is not continuous at \( y \). Similarly, \( f \) is bounded below.

Now for the main result.

Theorem: (Weierstrass) Consider a function \( f : X \to \mathbb{R}^1 \), where both \( X \) and \( \mathbb{R} \) are endowed with the Pythagorean metric. If \( X \) is a compact set and \( f \) is continuous on \( X \), then \( f \) attains a global maximum and a global minimum on \( X \).

Sketch of the Proof:

- show that the image of the function must be bounded.
- let \( \bar{f} \) denote the supremum of the image of the function.
- pick a sequence \( \{x_n\} \) such that the sequence \( \{f(x_n)\} \) gets closer to the supremum.
- while the sequence \( \{x_n\} \) needn’t converge, it follows from the compactness of \( X \) that there must exist a subsequence \( \{y_n\} \) of \( \{x_n\} \) such that \( \{y_n\} \) converges to \( y \in X \).
- since \( f \) is continuous, the sequence \( \{f(y_n)\} \) must converge to \( f(y) \). But by defn of the supremum, \( \{f(y_n)\} \) converges also to \( \bar{f} \). Hence \( f(y) = \bar{f} \).
- Since \( \bar{f} \) is the supremum of the image of \( X \) under \( f \), then \( f(y) = \bar{f} \geq f(x) \), for all \( x \in X \).

Proof: Let \( \bar{f} \) denote the supremum of the image of \( X \) under \( f \), i.e., the set \( \{f(x) : x \in X\} \). By the Lemma above, \( \bar{f} \in \mathbb{R} \). By definition of the supremum, for all \( n \), there exists \( x_n \) such that \( f(x_n) > \bar{f} - 1/n \). Since \( X \)
is compact, the sequence \( \{x_n\} \) contains a convergent subsequence. Call this subsequence \( \{y_n\} \) and let \( y \in X \) denote its limit. Since \( f \) is continuous, the sequence \( \{f(y_n)\} \) converges to \( f(y) \in \mathbb{R} \). To complete the proof, we’ll show that \( \{f(y_n)\} \) also converges to \( \bar{f} \). Since a sequence has at most one limit\(^1\), this will imply that \( f(y) = \bar{f} \geq f(x) \), for all \( x \in X \), and hence imply that \( f \) attains a global maximum at \( y \).

Since \( \{y_n\} \) is a subsequence of \( \{x_n\} \), there exists a strictly increasing mapping \( \tau : \mathbb{N} \to \mathbb{N} \) such that for all \( n \), \( y_n = x_{\tau(n)} \). To prove that \( \{f(y_n)\} \) converges to \( \bar{f} \), we need to show that for all \( \varepsilon > 0 \), there exists \( N \in \mathbb{N} \) such that for all \( n > N \), \( f(y_n) \in B(\bar{f}, \varepsilon) \). Note first that for all \( n \), \( f(y_n) \leq \bar{f} \), since \( \bar{f} \) is an upper bound for the set \( \{f(x) : x \in X\} \). Moreover, it follows from the construction of \( \{x_n\} \) that for all \( n \), \( f(y_n) = f(x_{\tau(n)}) > \bar{f} - 1/\tau(n) \geq \bar{f} - 1/n \) (by the above Fact). It follows therefore that for an arbitrarily chosen \( \varepsilon > 0 \), we can pick \( N_{\varepsilon} \in \mathbb{N}, N_{\varepsilon} > 1/\varepsilon \), so that \( 1/\tau(N_{\varepsilon}) \leq 1/N_{\varepsilon} < \varepsilon \). To complete the proof, observe that for all \( n > N_{\varepsilon} \),

\[
 f(y_n) \in (\bar{f} - 1/\tau(n), \bar{f}] \subset (\bar{f} - 1/\tau(N_{\varepsilon}), \bar{f}] \subset (\bar{f} - \varepsilon, \bar{f}] \subset (\bar{f} - \varepsilon, \bar{f} + \varepsilon) = B(\bar{f}, \varepsilon)
\]

So we have, \( (f(y_n)) \to f(y) \in \mathbb{R} \) and \( (f(y_n)) \to \bar{f} \in \mathbb{R} \). Since a function can have only one limit, \( f(y) = \bar{f} \).

Since \( \bar{f} \) is an upper bound for \( \{f(x) : x \in X\} \), we now have \( f(y) \geq f(x) \), for all \( x \in X \). \( \square \)

A common source of puzzlement is: since \( f(x_n) \) already converges to \( \bar{f} \), why do I need to pick a subsequence \( \{y_n\} \) and show that \( f(y_n) \) also converges to \( \bar{f} \)? The reason is that \( (x_n) \) doesn’t necessarily converge to some \( x \in X \), so \( y \) can’t invoke continuity to establish that \( f(x_n) \) converges to \( f(x) \), for some \( x \in X \). Here are two examples that illustrate conclusively (I hope) why you absolutely have to pick the subsequence.

1. Let \( X = (0, 1) \) and consider \( f : X \to \mathbb{R} \), defined by \( f(x) = x \). Clearly \( f \) doesn’t attain a maximum on \( X \). The theorem doesn’t apply because \( X = (0, 1) \) isn’t compact. Here’s where the proof would break down if we tried to apply it. Using the notation of the proof, \( \bar{f} = 1 \). Pick \( x_n = 1 - 1/(n + 1) \) and note that for all \( n \), \( f(x_n) > \bar{f} - 1/n \). But we can’t go past this point, because without compactness, we can’t pick a subsequence \( \{y_n\} \) and \( y \in X \) such that \( y_n \to y \). The point of the example is that having the sequence \( x_n \) such that the \( f(x_n) \)’s approach \( \bar{f} \) doesn’t do us much good, without further help.

2. Consider the function \( f \) and sequence \( \{x_n\} \) graphed in Fig. 1 below. The example illustrates the point that even though the sequence \( \{f(x_n)\} \) converges to \( \bar{f} \), the sequence \( \{x_n\} \) doesn’t converge. However, \( \{x_n\} \) has two convergent subsequences, each of which work fine.

Our last result in the analysis section establishes a useful alternative definition of continuity.

\(^1\) We noted this in Lecture 6 but didn’t prove it. Prove it as an exercise.
Definition: Given a mapping $f : X \to Y$, and $O \subset Y$, $f^{-1}(O)$ is the subset of $X$ that $f$ maps into $Y$, i.e., $f^{-1}(O) = \{ x \in X : f(x) \in O \}$. $f^{-1}(O)$ is called the inverse image of $O$ under $f$.

The result is that a function is continuous iff the inverse image of every open subset of the range of the function is an open set in the domain.

Theorem: A function $f : X \to Y$ is continuous iff for every open set $O \subset Y$, $f^{-1}(O)$ is an open subset of $X$.

Proof:

(1) first prove that continuity implies inverse images of open sets are open. Fix an arbitrary set $S \subset Y$ such that $f^{-1}(S)$ isn’t open in $X$. We’ll argue that $S$ isn’t open in $Y$. This will prove that when $f$ is continuous, $S$ open in $Y$ implies $f^{-1}(S)$ is open in $X$.

- if $f^{-1}(S)$ isn’t open there must exist a point $x$ in $f^{-1}(S)$ (i.e. such that $f(x) \in S$) which is a boundary point of $f^{-1}(S)$. 

Figure 1. Why $(x_n)$ isn’t enough: you need to pick a subsequence $(y_n)$
• i.e., there’s a sequence of points $x^n$ converging to $x$ all of which get mapped to points outside of $S$, that is, for all $n$, $f(x^n) \notin S$.
• since $f$ is continuous, $f(x^n)$ must converge to $f(x)$.
• but this means that $f(x)$ is a boundary point of $S$.
• conclude that $S$ isn’t open

(2) now prove that inverse images of open sets are open implies continuity, for the case $Y = \mathbb{R}$. We’ll show that if $f$ is not continuous, then there exists an open set $O \subset Y$ such that $f^{-1}(O)$ isn’t open in $X$.

• if $f$ isn’t continuous, there exists $x \in X$ and a sequence $\{x_n\}$ which converges to $x$ such that $f(x^n)$ doesn’t converge to $f(x)$, i.e., there exists $\epsilon > 0$ and a subsequence $\{y^n\}$ of $\{x^n\}$ such that for each $n$, $|f(y^n) - f(x)| > \epsilon$. Let $O = (f(x) - \epsilon, f(x) + \epsilon)$. Clearly $f(x) \in O$ so that $x \in f^{-1}(O)$. We’ll show that $x$ is not an interior point of $f^{-1}(O)$ and conclude that $f^{-1}(O)$ is not open.

• pick an arbitrary open set $W$ containing $x$. Since $\{y_n\}$ converges to $x$, there exist $n$ sufficiently large that $y^n \subset W$. But since by assumption, $f(y^n) \notin O$, it follows that $y^n \notin f^{-1}(O)$. Since $W$ was chosen arbitrarily, we have established that there does not exist an open set which contains $x$ and is itself contained in $f^{-1}(O)$. Conclude that $f^{-1}(O)$ is not open in $X$. 