This is the final exam for ARE202. As announced earlier, this is an open-book exam. However, use of computers, calculators, Palm Pilots, cell phones, Blackberries and other comparable objects is forbidden.

If a question says “prove formally” then we mean it: a purely verbal answer is unlikely to be given full marks. However, if there’s a step in your answer that involves a theorem that’s given in the lecture notes, then you may state the theorem and reference the notes.

Allocate your 120 minutes in this exam wisely. Make sure that you first do all the easy parts, before you move onto the hard parts. The questions are designed so that, to some extent, even if you cannot answer some parts, you can still be able to answer later parts. So don’t hesitate to leave a part out. You don’t have to answer questions and parts of questions in the order that they appear on the exam, provided that you clearly indicate the question/part-question you are answering.
Problem 1 (50 points).
Let \( f : \mathbb{R}^2 \to \mathbb{R} \) and \( g : \mathbb{R}^2 \to \mathbb{R} \) be two differentiable functions. Consider the following NPP problem, which we shall refer to as \((1*)\).

\[
\max_{x \geq 0} f(x) \text{ s.t. } g(x) \leq 0 \quad \text{(1*)}
\]

where

(C0) \( g \) is a quasi-convex function.

A) (4 points). Draw a constraint set that is consistent with assumption (C0).

B) (12 points). Assume in this part that \( f \) is not quasi-concave.

(a) A local solution to \((1*)\) is a point \( x \in \mathbb{R}^2 \) such that for some neighborhood \( U \) of \( x \), \( f(x) \geq f(x') \), for all \( x \neq x' \in U \) such that \( g(x') \leq 0 \). Say that a local solution is a strict local solution if \( f(x) > f(x') \), for all \( x \neq x' \in U \) such that \( g(x') \leq 0 \).

Show graphically that a point \( x \) may exist that is a strict local solution to \((1*)\) but not a solution.

(b) Show graphically that a point \( x \) may exist that satisfies the KT conditions but is not a local solution to \((1*)\).

C) (12 points). Fix \( x \in \mathbb{R}^2 \) and suppose that

(C1) at \( x \) the KT conditions are satisfied.

(C2) \( f \) is quasi-concave

(a) Give a condition of \( \nabla f(\cdot) \) which,

- differs from M.K.9
- together with (C1) and (C2), guarantees that \( x \) is a solution to \((1*)\).

Denote this condition by (C3).

(b) Provide a graphical example demonstrating that if (C1) and (C2) are satisfied but (C3) is not, then \( x \) may fail to be a solution to \((1*)\).

D) (12 points). An alternative to conditions (C2) and (C3) is the one referred to in the lecture notes as condition M.K.9. If \( f \) satisfies M.K.9 and \( x \) satisfies (C1), then \( f \) is a solution to \((1*)\).

(a) write down the assumption M.K.9

(b) explain why it is preferable to assume M.K.9 than to assume both (C2) and (C3). Your explanation should include an example. If you prefer, your example can be constructed using functions \( f \) and \( g \) that map \( \mathbb{R} \) to \( \mathbb{R} \).

E) (10 points). Prove formally that if \( f \) is a concave function, then \( f \) satisfies M.K.9.

**Hint:** Use the fact that \( f \) is differentiable.
Problem 2 (50 points).

Fix $\alpha > 0$ and define $h(\alpha; x, y) = e^{\alpha(x+y)} + x^2 + y^4$. Now consider the unconstrained maximization problem

$$\minimize h(\alpha; \cdot, \cdot) \text{ on } \mathbb{R}^2 \quad (2*)$$

For your convenience, Fig. 1 on the last page of the exam plots $z = e^\theta$. Also recall that $\frac{de^\theta}{d\theta} = e^\theta$.

A) (6 points). Compute $h(\alpha; 0, 0)$.

B) (6 points). Use your answer to part A) to prove that the solution to $2*$ is the same as the following constrained minimization problem: minimize $h(\alpha; \cdot, \cdot)$ such that $x \in [-1, 1]$ and $y \in [-1, 1]$.

C) (6 points). Use your answer to part B) to prove that $2*$ has a solution. Here and later in the question, you may take it as given that $h$ is a continuously differentiable function on $\mathbb{R} \times \mathbb{R} \times \mathbb{R}_+^+$.

D) (6 points). For arbitrary $x, y \in \mathbb{R}^2$, compute the Hessian of $h(\alpha; \cdot, \cdot)$ at $(x, y)$.

E) (6 points). Use your answers to C) and D) to prove that $2*$ has a unique solution.

F) (6 points).

a) Write down the first order conditions for $2*$.

b) Prove that a solution to these first order conditions exist and is unique.

c) Show that the unique solution to the FOC can be written as the level set corresponding to 0 of a function from $\mathbb{R}^3$ to $\mathbb{R}^2$ (to be denoted by $f$).

G) (6 points). Show that for all $\alpha > 0$, you can apply the implicit function theorem to the function $f$ given in Fc).

H) (8 points). Given $\bar{\alpha} > 0$, let $(x^*(\bar{\alpha}), y^*(\bar{\alpha}))$ denote the solution to $2*$. Use Cramer’s Rule to find an expression for $\frac{dx^*(\bar{\alpha})}{d\alpha}$.
Figure 1. The graph of $z = e^\theta$