(1) Consider the function $f(x) = x_1^2 + x_2^2$.

(a) Evaluate the gradient of $f$ at the point $(1, 1)$

$$\nabla f(x_1, x_2) = (2x_1, 2x_2) \text{ so that } \nabla f(1, 1) = (2, 2).$$

(b) Compute the directional derivative of $f$ at $(1, 1)$ in the direction $h = (1, 3/4)$.

The unit length vector pointing the direction $h$ is $(4/5, 3/5)$. So the directional derivative in this direction is $\nabla f(1, 1)(4/5, 3/5)' = 14/5$.

(c) Using a first order Taylor expansion, approximate the gradient of $f$ at the point $(1 + \epsilon, 1 - \epsilon)$.

The Hessian of $f$ at $(1, 1)$ is $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$, so the first order Taylor approximation to $d\nabla f = \nabla f(1 + \epsilon, 1 - \epsilon) - \nabla f(1, 1) = \begin{bmatrix} 2 \\ 0 \end{bmatrix} [\epsilon, -\epsilon] = \begin{bmatrix} 2\epsilon \\ -2\epsilon \end{bmatrix}$

(d) How does the quality of your approximation depend on $\epsilon$? Explain your answer.

This approximation is exact: from above, $\nabla f(1 + \epsilon, 1 - \epsilon) = (2(1 + \epsilon), 2(1 - \epsilon))$ so that $\nabla f(1 + \epsilon, 1 - \epsilon) - \nabla f(1, 1) = 2(\epsilon, -\epsilon)$. The reason why the approximation is exact is that $f$ is a quadratic, so that all terms in the Taylor expansion of $\nabla f$ after the first are zero.
(2) Cecile hates paying too much for things. Not just because the more she pays, the less she can buy. But because she gets a bigger knot in her stomach the higher are the prices she has to pay. Her utility function is 
\[ u(x, p) = (x_1 - 2)(2 + x_2) - p_1 p_2. \] Suppose that \( p_1 = p_2 = 1 \), and that she has one unit of income. She cannot consume negative quantities of either good.

(a) Set up the Lagrangian to solve her utility maximization problem

\[ L(x, p, \lambda) = (x_1 - 2)(2 + x_2) - p_1 p_2 + \lambda_1 (0 - (-x_1)) + \lambda_2 (0 - (-x_2)) + \lambda_3 (1 - x_1 - x_2) \]

(b) Solve her problem graphically.

The objective function in a neighborhood of \((1, 0)\) is locally quasi-convex!

(c) Verify graphically that the mantra is satisfied.

Clearly \( \nabla f \) belongs to the positive cone defined by the two constraints \( \nabla g_2 \) and \( \nabla g_3 \) that are satisfied with equality at \((1, 0)\).

(d) Verify mathematically that the KKT necessary conditions are satisfied.

\[ \nabla f(1, 0) = (2, -1) = \lambda_2 (0, -1) + \lambda_3 (1, 1), \] where \( \lambda_2 = 3, \lambda_3 = 2 \).

(e) Now for \( i = 1, 2 \), suppose that the price of good \( i \) increases. Compare the effect of these increases on Cecile’s utility

(i) with the aid of graphs.
As the graph illustrates an increase in $p_1$ has an effect on the solution through the budget constraint $g_3$, while an increase in $p_2$ has no effect through this constraint. However, both prices have an identical effect through the objective function, though this cannot be seen in the figure.

(ii) using the envelope theorem.

\[ \frac{du(x^* p^*)}{dp_1} = \frac{\partial u(x^* p^*)}{\partial p_1} - \lambda_3 x_i \]

\[ = -(p_j + \lambda_3 x_i) \]

Thus \( \frac{du(x^* p^*)}{dp_1} = -(1 + 2 \cdot 1) = -3 \); while \( \frac{du(x^* p^*)}{dp_2} = -(1 + 0) = -1 \).

Explain how the mathematics in your answer to 2(e)ii relates to the graphs in your answer to 2(e)i

The difference between the left and right panels matches the difference between -3 and -1. The budget set shrinkage affects Cecile's utility only for $p_1$ and at a rate $-2$. The remaining -1 in each case affects the vertical height above the solution level set, but doesn't show up in the figure, of course.
(3) Consider the function \( f(x) = x_1 x_2 \).

(a) Using the variables \( x_1, x_2 \) and \( \lambda \), write down a vector that is orthogonal to \( \nabla f(x) \). Verify that it is indeed orthogonal.

For \( \lambda \in \mathbb{R}_+ \), let \( dx = \lambda(x_1, -x_2) \). Since \( \nabla f(x) = (x_2, x_1) \), we have \( \nabla f(x) dx = \lambda(x_1 x_2 - x_1 x_2) = 0 \), so that \( dx \) is indeed orthogonal to \( \nabla f(x) \).

(b) Show that any vector that is orthogonal to \( x \) can be written in the form you identified in (3a).

That is, show that if \( y \) is orthogonal to \( x \), then there exists \( \lambda \in \mathbb{R} \) such that \( y \) can be written in terms only of \( x_1, x_2 \) and \( \lambda \).

Suppose that \( \nabla f(x) y = 0 \). That is, suppose that \( y_1 x_2 + y_2 x_1 = 0 \), so that \( y_2 = -\frac{y_1 x_2}{x_1} \). Let \( \lambda = \frac{y_1}{x_1} \) and observe that \( y = (\lambda x_1, -\lambda x_2) \).

(c) Using your answer to the preceding part, and without using a bordered Hessian argument, verify that \( f(\cdot) \) is strictly quasi-concave on the strictly positive orthant, i.e., \( \mathbb{R}^{++} \).

\( f \) is strictly quasi-concave if for all \( x \) and all \( dx \) such that \( \nabla f(x) dx = 0, \) \( dx' H f(x) dx < 0. \) Now fix \( x \) arbitrarily and pick \( dx = \lambda(x_1, -x_2) \) such that \( \nabla f(x) dx = 0. \) \( H f(x) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \) so that, since \( x \) is strictly positive,

\[
dx' H f(x) dx = \lambda^2 \begin{bmatrix} x_1 & -x_2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ -x_2 \end{bmatrix} = \lambda^2 \begin{bmatrix} x_1 & -x_2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -x_2 \\ x_1 \end{bmatrix} = -2\lambda^2 x_1 x_2 < 0
\]

(d) What can you say about the extension of \( f \) to the nonnegative orthant, i.e., \( \mathbb{R}_+ \). Again, your answer shouldn’t invoke Bordered Hessians.

Once you extend to the nonnegative orthant, \( dx' H f(x) dx \) will be positive whenever \( x \) is strictly positive and zero if either component of \( x \) is positive. Hence the extension of \( f \) to the nonnegative orthant is negative semi-definite.
(4) Consider the following system of equations,

\[
\begin{align*}
3y + 2w &= z + 1 \\
3w + 2z &= 8 - 5y
\end{align*}
\]

We are going to use the implicit function theorem to find \( \frac{dy}{dz} \).

(a) Set this problem up in the format of the implicit function theorem, i.e., identify \( f, x \) and \( \alpha \).

\[
\begin{align*}
x &= (w, y), \quad \alpha = z, \quad f(x, \alpha) = \begin{bmatrix} 2 & 3 & -1 \\ 3 & 5 & 2 \\ y & z \end{bmatrix} \begin{bmatrix} w \\ y \\ z \end{bmatrix}
\end{align*}
\]

(b) Identify the level set of \( f \) that we will stay on as we vary \( z \).

We will stay on the level set \( f(x, \alpha) = \begin{bmatrix} 1 \\ 8 \end{bmatrix} \).

(c) Verify that the condition required to apply the implicit function theorem is satisfied.

The required condition is that the matrix \( \begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix} \) is nonsingular. Its determinant is 1, so indeed it has full rank.

(d) Apply the implicit function theorem (and Cramer’s rule) to obtain \( \frac{dy}{dz} \).

\[
\begin{bmatrix}
\frac{\partial w}{\partial z} \\
\frac{\partial y}{\partial z}
\end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix}^{-1} \begin{bmatrix} -1 \\ 2 \end{bmatrix} \quad \text{so that} \quad \frac{dy}{dz} = \det \left( \begin{bmatrix} 2 & -1 \\ 3 & 2 \end{bmatrix} \right) / \det \left( \begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix} \right) = -7
\]

(5) Consider the function \( f(x) = 2x - 5x^2 + \frac{2}{3}x^3 \). This function has one strict local maximum \( \bar{x} \) and a one strict local minimum \( \bar{x} \).

(a) Find \( \bar{x} \) and \( \underline{x} \) and distinguish between them using second order conditions

\[
f'(x) = 2 - 10x + 8x^2. \quad f''(x) = 16x - 10. \quad f'''(x) = 16 \quad f'(x) = 0 \implies (1 - 5x + 4x^2) = 0, \quad \text{i.e.,} \quad x = 1/4 \quad \text{or} \quad x = 1. \quad f'''(1/4) = 4 - 10 = -6 < 0 \quad \text{so} \quad \underline{x} = 1/4 \quad \text{is a local max.} \quad f'''(1) = 16 - 10 = 4 > 0 \quad \text{so} \quad \bar{x} = 1 \quad \text{is a local min.}
\]

(b) Does the function have a global max and/or a global min. If not, why not?

As \( x \) becomes large and positive, the cubed term dominates and the function increases without bound. Similarly As \( x \) becomes large in absolute value and negative, the cubed term again dominates and the function decreases without bound. Conclude that the function has no global max or min.

(c) Using a Taylor expansion, find the largest \( \epsilon \) such that \( f \) has a strict global maximum at \( \bar{x} \) when \( f \) is restricted to the interval \( (\bar{x} - \epsilon, \bar{x} + \epsilon) \).

The third order Taylor expansion of \( f \) about \( \bar{x} \) is exact since there are no higher order terms. Hence we have

\[
f(x) - f(1/4) = f'(1/4)dx + f''(1/4)dx^2/2 + f'''(1/4)dx^3/6 = 0 - 3dx^2 + 8/3dx^3
\]

Thus \( f(x) - f(1/4) \geq 0 \) if \( 8dx/3 > 3 \) or \( dx > 9/8 \). Therefore, \( f \) has a strict global maximum at \( \bar{x} = 0.25 \) when \( f \) is restricted to the interval \( (\bar{x} - 9/8, \bar{x} + 9/8) \).