B) (a) The condition is $\nabla f(x) \neq 0 \quad \forall x \in \mathbb{R}^2$. (C3).

ie the gradient of $f$ never vanishes.

(b)
(3) \( \forall x, x' \in \mathbb{R}^2 \quad f(x') > f(x) \implies \nabla f(x) \cdot (x' - x) > 0 \)

(b)

(4) \((C2)\) and \((C3) = \text{M.K.G. which is a sufficient condition for a global constrained max under } g \text{ quasiconvex}\)

\((C1)\)

So it is better to have the weakest possible sufficient condition

ie M.K.G better than \((C2) + (C3) \) because it requires fewer assumptions and still guarantees a global constrained max.

D) If \( f \) is concave and differentiable on \( \mathbb{R}^2 \)

\( \forall x, x' \in \mathbb{R}^2 \quad f(x') \leq f(x) + \nabla f(x) \cdot (x' - x) \)

MKG: \( \forall x, x' \in \mathbb{R}^2 \quad f(x') > f(x) \implies \nabla f(x) \cdot (x' - x) > 0 \)

Suppose \( x, x' \in \mathbb{R}^2 \) such that \( f(x') > f(x) \)

Want to prove: \( \nabla f(x) \cdot (x' - x) > 0 \).

If \( f \) concave and differentiable, hence \( f(x') \leq f(x) + \nabla f(x) \cdot (x' - x) \)

but \( f(x') > f(x) \)

hence \( f(x) < f(x) + \nabla f(x) \cdot (x' - x) \)

ie \( \nabla f(x) \cdot (x' - x) > 0 \).
A) \( h(0,0; x) = e^{\alpha(0+0)} + 0 + 0 = 1 \)

B) if \((x,y) \notin [-1,1]^2\)
\[ \text{then } h(x,y; x) > 1 \text{ since } x^2 > 1 \]
\[ y^4 > 1 \]
\[ e^{x(x+y)} > 0 \]

So you can restrict the search to the compact set \([-1,1]^2\).

C) \( h( , ; x) \) is continuous.

Since \([-1,1]^2\) is a compact set \( h \) attains a max and a min on \([-1,1]^2\).

So (2*) has at least one solution.

D) \[ \frac{\partial h}{\partial x}(x,y; x) = \alpha e^{\alpha(x+y)} + 2x \]
\[ \frac{\partial h}{\partial y}(x,y; x) = \alpha e^{\alpha(x+y)} + 4y^3 \]
\[ \frac{\partial^2 h}{\partial x^2}(x,y; x) = \alpha e^{\alpha(x+y)} + 2 \]
\[ \frac{\partial^2 h}{\partial x \partial y}(x,y; x) = 2 \alpha e^{\alpha(x+y)} \]

\[ D^2 h_{x,y}(x,y; x) = \begin{bmatrix} \alpha e^{\alpha(x+y)} + 2 & \alpha^2 e^{\alpha(x+y)} \\ \alpha^2 e^{\alpha(x+y)} & \alpha^2 e^{\alpha(x+y)} + 12y^2 \end{bmatrix} \]

E) \[ |D^2 h_{x,y}(x,y; x)| = (\alpha^2 e^{\alpha(x+y)}) + (\alpha^2 e^{\alpha(x+y)} + 12y^2) - \alpha^4 e^{2\alpha(x+y)} \]
\[ D^2 h_{x,y} \text{ is positive definite} \]

So \( h \) is strictly convex \( \rightarrow \) it has at most one minimizer.

So we have proved unicity and in c) we proved existence.

\( \Rightarrow (2*) \) has a unique solution.
a) \[ \min h(x, y; \alpha) \text{ on } \mathbb{R}^2 \]

\[
\begin{align*}
\frac{\partial h}{\partial x} &= 0 \\
\frac{\partial h}{\partial y} &= 0 \\
\alpha e^{x(x+y)} + 2x &= 0 \\
\alpha e^{x(x+y)} + 4y^3 &= 0
\end{align*}
\]

Existence

For an unconstrained minimization problem, FOC are necessary (there is no C.Q. condition). So since in E we proved existence of a unique solution, it means that at that solution (call it \( (x^*, y^*) \)) \( \nabla h(x^*, y^*) = 0 \).

So \( (x^*, y^*) \) satisfies FOC. i.e., these FOC have a solution.

Unicity

Since \( h \) is strictly convex, FOC are also sufficient. So if FOC had another solution, it would also be a solution to \((2x)\). But in E) we showed that \((2x)\) had a unique solution.

So FOC can only have one solution.

Conclusion

FOC have a unique solution identical to the solution of \((2x)\).

c) Consider \( f: \mathbb{R}^3 \rightarrow \mathbb{R}^2 \)

\[
(x, y, \alpha) \rightarrow \begin{pmatrix} f_1(x, y; \alpha) \\ f_2(x, y; \alpha) \end{pmatrix} = \begin{pmatrix} f^1(x, y; \alpha) \\ f^2(x, y; \alpha) \end{pmatrix} = \begin{pmatrix} \alpha e^{x(x+y)} + 2x \\ \alpha e^{x(x+y)} + 4y^3 \end{pmatrix}
\]

The unique solution to FOC is the level set of \( f \) at \( \alpha \).

g) We have \( f(x^*, y^*; \alpha) = 0 \).

And \( D_{x,y} f(x^*, y^*) = \begin{pmatrix} \frac{\partial f^1}{\partial x} (x^*, y^*) \\ \frac{\partial f^1}{\partial y} (x^*, y^*) \\ \frac{\partial f^2}{\partial x} (x^*, y^*) \\ \frac{\partial f^2}{\partial y} (x^*, y^*) \end{pmatrix} = \begin{pmatrix} \alpha^2 e^{x^*(x^*+y^*)} + 2 \alpha e^{x^*(x^*+y^*)} \\ \alpha^2 e^{x^*(x^*+y^*)} + 4 \alpha e^{x^*(x^*+y^*)} \end{pmatrix} \)
Hence \[ D_{x,y} f(x^*,y^*) = \left( \alpha^2 e^{\alpha(x^*+y^*)} + 2 \right) \left( \alpha^2 e^{\alpha(x^*+y^*)} + 12y^* + 2 \right) - \alpha^4 e^{2\alpha(x^*+y^*)} \]

\[ = 2\alpha(y^*)^2 + \alpha^2 e^{\alpha(x^*+y^*)} \left( \alpha + 12y^* + 2 \right) \]

So we can apply the implicit function theorem.

H) From 6) there exists a neighborhood around \( \bar{x} \), \( N(\bar{x}) \)

and a continuously differentiable function \( \phi : N(x) \to \mathbb{R}^2 \)

such that:

(i) \( \phi(\bar{x}) = (x^*, y^*) \)

(ii) \( \forall \alpha \in N(x) : \phi(\phi(\alpha); \alpha) = 0 \)

(iii) \( \phi'(\bar{\alpha}) = -\left[ D_{x,y} f(x^*,y^*) \right]^{-1} D_{x,y} f(x^*,y^*,\bar{\alpha}) \)

\[ \phi'(\bar{\alpha}) = \begin{bmatrix}
\frac{d x^*}{d \alpha} \\
\frac{d y^*}{d \alpha}
\end{bmatrix} \]

\[ \frac{d x^*}{d \alpha} = -\frac{1}{\left| D_{x,y} f(x^*,y^*,\bar{\alpha}) \right|} \left[ e^{\alpha(x^*+y^*)} + (x^*+y^*) \alpha e^{\alpha(x^*+y^*)} \right] \]

\[ = \frac{-1}{2\alpha(y)^2 + \alpha^2 e^{\alpha(x^*+y^*)} \left( 2 + 12y^* + 2 \right)} \left[ e^{\alpha(x^*+y^*)} \left[ 1 + \alpha(x^*+y^*) \right] \left[ \alpha^2 e^{\alpha(x^*+y^*)} + 12y^* \right] \right] \]

\[ = -\frac{e^{\alpha(x^*+y^*)} \times 12y^*}{2\alpha(y)^2 + \alpha^2 e^{\alpha(x^*+y^*)} \left( 2 + 12y^* + 2 \right)} \left[ 1 + \alpha(x^*+y^*) \right] \]

\[ \frac{d y^*}{d \alpha} = \frac{-\alpha^2 e^{\alpha(x^*+y^*)} \left[ 1 + \alpha(x^*+y^*) \right] \left[ \alpha^2 e^{\alpha(x^*+y^*)} + 12y^* \right]}{2\alpha(y)^2 + \alpha^2 e^{\alpha(x^*+y^*)} \left( 2 + 12y^* + 2 \right)} \left[ -\alpha^2 e^{\alpha(x^*+y^*)} \left[ 1 + \alpha(x^*+y^*) \right] \left[ \alpha^2 e^{\alpha(x^*+y^*)} + 12y^* \right] \right] \]

\[ = -\frac{e^{\alpha(x^*+y^*)} \times 12y^*}{2\alpha(y)^2 + \alpha^2 e^{\alpha(x^*+y^*)} \left( 2 + 12y^* + 2 \right)} \left[ 1 + \alpha(x^*+y^*) \right] \]