PROBLEM SET #06- ANSWER KEY
FIRST COMPSTAT PROBLEM SET

(1) In problem 3 on your last problem set, you found the maximum and minimum distance from the origin to the ellipse $x_1^2+x_1x_2+x_2^2=3$. Generalize this problem to "minimize/maximize the distance from the origin to the ellipse $x_1^2+x_1x_2+\alpha x_2^2=3$" and use the Envelope theorem (starting from $\alpha=1$) to estimate the maximum and minimum distance from the origin to the following
ellipse,

$$x_1^2+x_1x_2+0.9x_2^2=3$$

.

Ans: From problem 3 on the preceding problem set we know that the Lagrangian was:

$$L=x_1^2+x_2^2+\lambda(3-x_1^2-x_1x_2-x_2^2)$$

Let's parameterize the Lagrangian accordingly to estimate the change.

$$L=x_1^2+x_2^2+\lambda(3-x_1^2-x_1x_2-\alpha x_2^2)$$

Initially, $\alpha=1$. From the envelope theorem we know that the first derivative of the squared distance $M_2(\alpha)$ w.r.t the parameter α is $\frac{dM_2(\alpha)}{d\alpha}=\frac{\delta L}{\delta\alpha}=-\lambda x_2^2$
Using a first order Taylor expansion we know that:

$$M_2(\alpha+d\alpha)\approx M_2(\alpha)+\frac{\delta M_2}{\delta\alpha}d\alpha=M_2(\alpha)-\lambda x_2^2 d\alpha$$

In the following $\alpha=1, d\alpha=-0.1$

(1) $x_1^{(1)}=1,\ x_2^{(1)}=1,\ \lambda^{(1)}=\frac{2}{3},\ M(\alpha=1)=\sqrt{2}\approx 1.4142$
$M_2(0.9)\approx 2-\frac{2}{3}*1^2*(-0.1)=\frac{31}{15}$
$\Rightarrow M(0.9)=\sqrt{M_2(0.9)}=\sqrt{\frac{31}{15}}\approx 1.4376$

(2) $x_1^{(2)}=-1,\ x_2^{(2)}=-1,\ \lambda^{(2)}=\frac{2}{3},\ M(\alpha=1)=\sqrt{2}\approx 1.4142$
$M_2(0.9)\approx 2-\frac{2}{3}*(-1)^2*(-0.1)=\frac{31}{15}$
$\Rightarrow M(0.9)=\sqrt{M_2(0.9)}=\sqrt{\frac{31}{15}}\approx 1.4376$

(3) $x_1^{(3)}=\sqrt{3},\ x_2^{(3)}=-\sqrt{3},\ \lambda^{(3)}=2,\ M(\alpha=1)=\sqrt{6}\approx 2.4495$
$M_2(0.9)\approx 6-2*(-\sqrt{3})^2(-0.1)=6.6$
$\Rightarrow M(0.9)=\sqrt{M_2(0.9)}=\sqrt{6.6}\approx 2.5690$

(4) $x_1^{(4)}=-\sqrt{3},\ x_2^{(4)}=\sqrt{3},\ \lambda^{(4)}=2,\ M(\alpha=1)=\sqrt{6}\approx 2.4495$
$M_2(0.9)\approx 6-2*\sqrt{3}^2(-0.1)=6.6$
$\Rightarrow M(0.9)=\sqrt{M_2(0.9)}=\sqrt{6.6}\approx 2.5690$

(2) a) Prove that the expression $\alpha^2 - \alpha x^3 + x^5 = 17$ defines x implicitly as a function of α.

Ans: The partial derivative of the expression w.r.t. x is $\frac{\delta f}{\delta x} = -3\alpha x^2 + 5x^4$
Evaluated at the point $(\bar{\alpha}, \bar{x}) = (5, 2)$: $\frac{\delta f}{\delta x} = -3*5*4+5*16 = 20 \neq 0$ Since $\frac{\delta f(\bar{\alpha},\bar{x})}{\delta x} \neq 0$ there exists a neighborhood around $(\bar{\alpha}, \bar{x})$ where x can be written as an implicit function of α, i.e., $x = g(\alpha)$. in a neighborhood of $(\bar{\alpha}, \bar{x}) = (5, 2)$

b) Estimate the x-value which corresponds to $\alpha = 4.8$ using a first order approximation.

Ans: The partial derivative of f w.r.t. α is $\frac{\delta f}{\delta \alpha} = 2\alpha - x^3$
Evaluated at the point $(\bar{\alpha}, \bar{x}) = (5, 2)$: $\frac{\delta f}{\delta \alpha} = 2 * 5 - 8 = 2$
From the implicit function theorem we know $\frac{\delta g}{\delta \alpha} = -\frac{\frac{\delta f}{\delta \alpha}}{\frac{\delta f}{\delta x}} = -\frac{2}{20} = -0.1$
Using a first order Taylor expansion: $g(\alpha + d\alpha) \approx g(\alpha) + \frac{\delta g}{\delta \alpha} d\alpha$
Hence for $\alpha = 5, d\alpha = -0.2$, $\underline{\underline{g(4.8) \approx 2 - 0.1 * (-0.2) = 2.02}}$

(3) (a) Consider the function $f(x, y, \gamma) = xy + \gamma y$ subject to the following constraints: $g(x, y, \gamma) \leq 1$, $x \geq 0$, $y \geq 0$, where $g(x, y) = x^2 + \gamma y$.

(i) For $\gamma = 1$, solve this maxmization problem using either the Lagrangian or KKT method.

Ans: The KKT conditions are

$$\begin{bmatrix} y & x + \gamma \end{bmatrix} = \lambda \begin{bmatrix} 2x & \gamma \end{bmatrix} \tag{1}$$

We'll try to solve this assuming that the first constraint is binding and the nonnegativity constraints are slack. In this case, $y = 1 - x^2$, so that when $\gamma = 1$, the KKT conditions become

$$\begin{bmatrix} 1 - x^2 & x + 1 \end{bmatrix} = \lambda \begin{bmatrix} 2x & 1 \end{bmatrix}$$

Solving the second equation, we obtain $\lambda = x + 1$. Substituting into the first, we get $1 - x^2 = (x + 1)2x$ or $3x^2 + 2x - 1 = 0$ or $(3x - 1)(x + 1) = 0$. The unique positive solution to this equation is $x = 1/3$, hence $y = 8/9$, $\lambda = 4/3$. Double-checking the KKT conditions for arithmetic errors, the l.h.s. of (1) is $\begin{bmatrix} 8/9 & 4/3 \end{bmatrix}$ while the r.h.s. is $4/3 \begin{bmatrix} 2/3 & 1 \end{bmatrix} = \begin{bmatrix} 8/9 & 4/3 \end{bmatrix}$. We've established, then, that the KKT conditions are indeed satisfied at $(x^*, y^*, \lambda^*) = (1/3, 8/9, 4/3)$.

(ii) Now, use the envelope theorem to estimate the maximized value of f when $\gamma = 1.2$

Ans: To estimate the required value, we will use a first order Taylor expansion, i..e, $f(x^*(1), y^*(1), 1) + \frac{df(x^*(1),y^*(1),1)}{d\gamma} d\gamma$, where $d\gamma = 0.2$. By the envelope theorem, $\frac{df(x^*(\gamma),y^*(\gamma),\gamma)}{d\gamma} = \frac{\partial f(x^*(\gamma),y^*(\gamma)),\gamma)}{\partial \gamma} + \lambda^*(\gamma) \frac{\partial g(x^*(\gamma),y^*(\gamma)),\gamma)}{\partial \gamma}$. Now $\frac{\partial f(\cdot,\cdot,\cdot)}{\partial \gamma} = \frac{\partial g(\cdot,\cdot,\cdot)}{\partial \gamma} = y$, so that $\frac{df(x^*(\gamma),y^*(\gamma),\gamma)}{d\gamma} = y^*(\gamma)(1 - \lambda^*(\gamma))$. Plugging in the solution values we have just obtained, we have $f(x^*(1), y^*(1), 1) = 8/9 \times (1 + 1/3) = 32/27$ while $\frac{df(x^*(\gamma),y^*(\gamma),\gamma)}{d\gamma} = y^*(\gamma)(1 - \lambda^*(\gamma)) = 8/9 \times -1/3 = -8/27$. Hence, our first order approximation to $f(x^*(1.2), y^*(1.2), 1.2)$ is $32/27 - 0.2 \times 8/27 = 152/135$.

(b) Consider the problem $\max_x \; f(x;\alpha) \quad \text{s.t.} \quad g(x;\alpha) \leq b$, where $x, \alpha, b \in \mathbb{R}$, f and g are twice continuously differentiable, $g_x(\cdot,\alpha) > 0$. Let $x^*(\alpha)$ denote the solution to this problem, given α. Use the implicit function theorem to identify sufficient conditions for $x^*(\cdot)$ to be everywhere strictly increasing in α. Are the conditions you identified necessary as well? If so prove it. If not, provide a counter-example.

Ans: The Lagrangian for this problem is $L(x,\lambda;\alpha) = f(x;\alpha) + \lambda(b - g(x,\alpha))$. To determine the relationship between x and α we apply the implicit function theorem to the zero level set of the first order conditions of the Lagrangian. We have

$$\begin{aligned} L_x &= 0 = f_x(x,\alpha) - \lambda g_x(x,\alpha) \\ L_\lambda &= 0 = b - g(x,\alpha) \end{aligned}$$

Applying the implicit function theorem to these conditions, we have

$$\begin{bmatrix} L_{x,x} & L_{x,\lambda} \\ L_{\lambda,x} & L_{\lambda,\lambda} \end{bmatrix} = \begin{bmatrix} (f_{xx} - \lambda g_{xx}) & -g_x \\ -g_x & 0 \end{bmatrix}$$

while

$$\begin{bmatrix} L_{x,\alpha} \\ L_{\lambda,\alpha} \end{bmatrix} = \begin{bmatrix} (f_{x,\alpha} - \lambda g_{x,\alpha}) \\ -g_\alpha \end{bmatrix}$$

and

$$\begin{bmatrix} dx/\,d\alpha \\ d\lambda/\,d\alpha \end{bmatrix} = -\begin{bmatrix} L_{x,x} & L_{x,\lambda} \\ L_{\lambda,x} & L_{\lambda,\lambda} \end{bmatrix}^{-1} \begin{bmatrix} L_{x,\alpha} \\ L_{\lambda,\alpha} \end{bmatrix}$$

We can now apply Cramer's rule to obtain

$$\begin{aligned} dx/\,d\alpha &= -\det\left(\begin{bmatrix} L_{x,\alpha} & L_{x,\lambda} \\ L_{\lambda,\alpha} & L_{\lambda,\lambda} \end{bmatrix}\right) \Big/ \det\left(\begin{bmatrix} L_{x,x} & L_{x,\lambda} \\ L_{\lambda,x} & L_{\lambda,\lambda} \end{bmatrix}\right) \\ &= -\det\left(\begin{bmatrix} (f_{x,\alpha} - \lambda g_{x,\alpha}) & -g_x \\ -g_\alpha & 0 \end{bmatrix}\right) \Big/ -g_x^2 \\ &= -\left(g_x g_\alpha/\,g_x^2\right) = -\left(g_\alpha/\,g_x\right) \end{aligned}$$

Since g_x is positive by assumption, it follows that $dx/\,d\alpha$ will be positive iff $g_\alpha < 0$.

This condition is not necessary however for $x^*(\cdot)$ to be strictly increasing in α. For example, let $f(x;\alpha) = x$, and let $g(x;\alpha) = x - \alpha^3$. Our maximization problem is now $\max_x \; x$ s.t. $x \leq b + \alpha^3$. Clearly the solution to this problem is globally strictly increasing in α. However, when $\alpha = 0$, then $g_\alpha = 0$. Hence the condition $g_\alpha < 0$ is not necessary for $x^*(\cdot)$ to be strictly increasing in α.

(4) Consider the equation $\alpha_1^3 + 3\alpha_2^2 + 4\alpha_1 x^2 - 3x^2\alpha_2 = 1$. Does this equation define x as an implicit function of α_1, α_2

Ans: The partial derivative of f w.r.t. x is $\frac{\delta f}{\delta x} = 8\alpha_1 x - 6\alpha_2 x$
The partial derivative of f w.r.t. α_1 is $\frac{\delta f}{\delta \alpha_1} = 3\alpha_1^2 + 4x^2$
The partial derivative of f w.r.t. α_2 is $\frac{\delta f}{\delta \alpha_2} = 6\alpha_2 - 3x^2$

a) in a neighborhood of $(\bar{\alpha_1}, \bar{\alpha_2}) = (1, 1)$

Ans: $f(x,1,1) = 1 + 3 + 4x^2 - 3x^2 - 1 = 0 \iff x^2 = -3$
which will not be satisfied for any real x. Hence, $f(x, \alpha_1, \alpha_2)$ does not define x as an implicit function of α_1, α_2.

b) in a neighborhood of $(\bar{\alpha_1}, \bar{\alpha_2}) = (1, 0)$

Ans: $f(x,1,0) = 1 + 0 + 4x^2 + 0 - 1 = 0 \iff x^2 = 0 \iff x = 0$
$\frac{\delta f(0,1,0)}{\delta x} = 0 + 0 = 0$ and we cannot use the implicit function theorem to express x as a function of α_1, α_2. However, the implicit function theorem is only a sufficient but not a necessary condition that there exists an implicit function.
So does there exists a neighborhood around (1,0) where x can be expressed as an implicit function of α_1, α_2? Well consider $\alpha_1 = 1 + \delta$ where $\delta > 0$. Then
$f(x, 1+\delta, 0) = (1+\delta)^3 + 0 + 4(1+\delta)x^2 + 0 - 1 = 0$
$\iff x^2 = \frac{1-(1+\delta)^3}{4(1+\delta)} < 0.$
Which does not have a real solution for x. Hence for any $\alpha_1 > 1$ there does not exist a x such that $f(x(\alpha_1, \alpha_2), \alpha_1, \alpha_2) = 0$. Consequently, there does not exist a neighborhood where x can be expressed as an implicit function of α_1, α_2.

c) in a neighborhood of $(\bar{\alpha_1}, \bar{\alpha_2}) = (0.5, 0)$

Ans: $f(x, \frac{1}{2}, 0) = \frac{1}{8} + 0 + 2x^2 + 0 - 1 = 0 \iff x^2 = \frac{7}{16} \iff x = \pm\frac{\sqrt{7}}{4}$
As $\frac{\delta f(\frac{\sqrt{7}}{4}, \frac{1}{2}, 0)}{\delta x} = 8 * \frac{1}{2} * \frac{\sqrt{7}}{4} = \sqrt{7} \neq 0$ and $\frac{\delta f(\frac{-\sqrt{7}}{4}, \frac{1}{2}, 0)}{\delta x} = -\sqrt{7} \neq 0$ there exist a neighborhood around $(\frac{\sqrt{7}}{4}, \frac{1}{2}, 0)$ and $(\frac{-\sqrt{7}}{4}, \frac{1}{2}, 0)$ where x can be expressed as an implicit function of α_1, α_2, i.e., $x = g(\alpha_1, \alpha_2)$. You can pick either one.
Let's pick $(\frac{\sqrt{7}}{4}, \frac{1}{2}, 0)$. We know that:
$\frac{\delta f(\frac{\sqrt{7}}{4}, \frac{1}{2}, 0)}{\delta \alpha_1} = 3 * \frac{1}{4} + 4 * \frac{7}{16} = \frac{5}{2}$
$\frac{\delta f(\frac{\sqrt{7}}{4}, \frac{1}{2}, 0)}{\delta \alpha_2} = 0 - 3 * \frac{7}{16} = -\frac{21}{16}$
Hence, using the implicit function theorem:

$$\frac{\delta g(\frac{\sqrt{7}}{4}, \frac{1}{2}, 0)}{\delta \alpha_1} = -\frac{\frac{\delta f(\frac{\sqrt{7}}{4}, \frac{1}{2}, 0)}{\delta \alpha_1}}{\frac{\delta f(\frac{\sqrt{7}}{4}, \frac{1}{2}, 0)}{\delta x}} = -\frac{\frac{5}{2}}{\sqrt{7}} = \underline{\underline{-\frac{5}{2\sqrt{7}}}}$$

$$\frac{\delta g(\frac{\sqrt{7}}{4}, \frac{1}{2}, 0)}{\delta \alpha_2} = -\frac{\frac{\delta f(\frac{\sqrt{7}}{4}, \frac{1}{2}, 0)}{\delta \alpha_2}}{\frac{\delta f(\frac{\sqrt{7}}{4}, \frac{1}{2}, 0)}{\delta x}} = -\frac{-\frac{21}{16}}{\sqrt{7}} = \underline{\underline{\frac{3\sqrt{7}}{16}}}$$

If so, compute $\frac{\partial x}{\partial \alpha_1}$ and $\frac{\partial \delta x}{\partial \alpha_2}$ at this point.

(5) The economy of Iceland can be expressed by the following three variables:
x : hard-core liquor to survive dark cold winters

y : imported oil for heating in the cold winter
z : beaver pelts for people crazy enough to leave the house
The equilibrium is given by the two equations:

$$2xz + xy + z - 2\sqrt{z} = 11 \tag{2}$$
$$xyz = 6 \tag{3}$$

Assume the economy finds itself at the initial equlibrium where $x = 3, y = 2, z = 1$. The government fixes the number of allowances to hunt beaver exogenously.

Ans: There are several ways to solve a system of several implicit functions. You can either just use the formula of the Jacobians or totally differentiate each equation and then use Cramer's rule to come up with the same conclusion. I will follow the second approach. Totally differentiate the two equilibrium identities:

$$\begin{array}{cccccccc} \{2z+y\} & dx+ & \{x\} & dy+ & \{2x+1-\frac{1}{\sqrt{z}}\} & dz = & 0 \\ \{yz\} & dx+ & \{xz\} & dy+ & \{xy\} & dz = & 0 \end{array}$$

a) If the prime minister raises z to 1.1, calculate the change in x and y.

Ans: Rewrite the totally differentiated equations from above such that the exogenous variables appear on the right hand side:

$$\begin{bmatrix} 2z+y & x \\ yz & xz \end{bmatrix} \begin{bmatrix} dx \\ dy \end{bmatrix} = \begin{bmatrix} -2x-1+\frac{1}{\sqrt{z}} \\ -xy \end{bmatrix} dz$$

Plug in the current equilibrium $(x, y, z) = (3, 2, 1)$

$$\begin{bmatrix} 4 & 3 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} dx \\ dy \end{bmatrix} = \begin{bmatrix} -6 \\ -6 \end{bmatrix} dz$$

You can use the implicit function theorem if the determinant of the Jacobian w.r.t the endogenous variables (the matrix on the left hand side above) is different from zero. In our example the determinant is $12 - 6 = 6 \neq 0$. Hence we can use Cramer's rule to derive the partial derivatives:

$$(\frac{\delta x}{\delta z} = \frac{\begin{vmatrix} -6 & 3 \\ -6 & 3 \end{vmatrix}}{\begin{vmatrix} 4 & 3 \\ 2 & 3 \end{vmatrix}} = \frac{-18+18}{12-6} = \frac{0}{6} = 0$$

$$\frac{\delta y}{\delta z} = \frac{\begin{vmatrix} 4 & -6 \\ 2 & -6 \end{vmatrix}}{\begin{vmatrix} 4 & 3 \\ 2 & 3 \end{vmatrix}} = \frac{-24+12}{12-6} = \frac{-12}{6} = -2$$

Using a first order Taylor approximation we therefore know that
$dx = \frac{\delta x}{\delta z} * dz = 0 * 0.1 = \underline{\underline{0}}$ and
$dy = \frac{\delta y}{\delta z} * dz = -2 * 0.1 = \underline{\underline{-0.2}}$

b) The anti-drug alliance argues that too much alcohol is consumed in the country. They therefore argue to fix the amount of alcohol consumed exogenously by law at 2.95 and instead abolish the beaver hunting constraint. Can they use the implicit function theorem to estimate the changes in y and z?

Ans: Rewrite the totally differentiated equations from above such that the exogenous variables appear on the right hand side:

$$\begin{bmatrix} x & 2x+1-\frac{1}{\sqrt{z}} \\ xz & xy \end{bmatrix} \begin{bmatrix} dy \\ dz \end{bmatrix} = \begin{bmatrix} -2z-y \\ -yz \end{bmatrix} dx$$

Plug in the current equilibrium $(x, y, z) = (3, 2, 1)$

$$\begin{bmatrix} 3 & 6 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} dx \\ dy \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix} dz$$

Now the determinant of the Jacobian w.r.t the endogenous variables (the matrix on the left hand side above) is $-18+18=0$. Hence we can't use the implicit function theorem.

(6) Assume you are given a competitive industry of identical firms with open entry. (i.e., two equations characterize the system: (1) producers are profit maximizers and the derivative of the price w.r.t a firm's own quantity is zero $\frac{\delta p(Q)}{\delta q_i} = 0$ and (2) new entry into the market occurs until profits are zero).

The government is thinking about imposing a lump-sum tax on firms. What would be the effects on a firm's individual output q, it's profit π, the total market output Q, the price $p(Q)$ and the number of firms in the industry n?

This problem involves one of the many internal contradictions that economists make all the time. First you assume that $\frac{\delta p(Q)}{\delta q_i} = 0$, i.e., that individual firms face perfectly elastic demand curves. Then you asume that the industry *faces a downward sloping demand curve, i.e., that $\frac{dp(Q)}{dQ} < 0$. Mathematically this is of course absurd, since $Q = nq_i$, but economists have been doing this shamelessly for centuries. There's a way of "fixing" this contradiction, but we're not going to get into it. So just do it and shudder quietly to yourselves.*

Ans: We will follow the standard approach to solving comparative statitics problems:

a) *Define your variables*:

n	number of firms in the industry
q_i	output of each individual firm i=1...n
$c_i(q_i)$	cost function of firm i as a function of its output
Q	market output $Q = \sum_{i=1}^{n} q_i$
$p(Q)$	price as a function of market output (inverse demand curve)
π_i	profit of each individual firm i=1...n
T	lump-sum tax imposed by the government

b) *Write down the identities that characterize your system*:

First, profit maximization on behalf of firms gives us the following equalities. The profit of an individual firm i is:

$$\pi_i = p(Q)q_i - c_i(q_i) - T \tag{4}$$

The FOC for profit maximization is (recall that in a competive industry firms are price takers and thus the derivative of the price with respect to its own quantity is zero).

$$\frac{\delta \pi_i}{\delta q_i} = p(Q) - c_i'(q_i) = 0 \tag{5}$$

Always write down the second order conditions as it later on helps you to sign things. The SOC for profit maximization becomes

$$\frac{\delta^2 \pi_i}{\delta q_i^2} = -c_i''(q_i) < 0 \tag{6}$$

Which simply implies that the marginal cost curve is rising at the equilibrium level.

Second, the free entry condition implies that profits are driven down to zero. Using equation (4) we know

$$\pi_i = p(Q)q_i - c_i(q_i) - T = 0 \tag{7}$$

Our system is characterized by the two identies given in equation (5) and (7).

Ans:

c) *Write down what other assumptions / information you know about the economic system.* First we are given that all firms are identical and we hence know that the output, cost-function and profit functions are identical

$$q_i = q \quad \forall i = 1...n \tag{8}$$
$$c_i(q_i) = c(q) \quad \forall i = 1...n \tag{9}$$
$$\pi_i = \pi \quad \forall i = 1...n \tag{10}$$

Second, we usually assume that the demand function is downward sloping

$$p'(Q) < 0 \tag{11}$$

d) This step should help you to organize things. It is the most difficult step for most people. Once you have finished it, the rest is pure mechanics.
Separate your endogenous variables in primary and secondary endogenous variables. You should have as many primary endogenous variables as equations in part (b) and the equations in part (b) can only contain primary endogenous variables as well as the exogenous variables: Exogenous variable(s): T
Primary endogenous variable(s): n, q
Secondary endogenous variable(s): Q, π Hint: try to reduce your system to as few equations and primary endogenous variables as possible because it makes things much easier later on. (I.e., instead of including a third equation $Q = nq$ and a third endogenous variable Q, substitute in the relationships $Q = nq$). So our two identities from part (b) that are given in equation (5) and (7) can be rewritten using equation (8) to (11) and only using the primary endogenous variables:

$$p(nq) - c'(q) = 0 \tag{12}$$
$$p(nq)q - c(q) - T = 0 \tag{13}$$

The relating equations for the secondary endogenous variables are

$$Q = nq \tag{14}$$
$$\pi = p(Q)q - c(q) - T \tag{15}$$

Ans:

e) *Totally differentiate your identities containing the primary endogenous variables in part (d) w.r.t the exogenous variable(s) and your primary endogenous variable(s)*
Note: up until know we always included the arguments of functions (e.g., p(nq)) because when we differentiate we need to know what the arguments of a functions are. After this step you won't have to differentiate again and it makes things much easier if you **drop all arguments**

$$\begin{array}{cccccccc} \{qp'\} & dn+ & \{np'-c''\} & dq+ & \{0\} & dz= & 0 \\ \{q^2p'\} & dn+ & \{nqp'+p-c'\} & dq+ & \{-1\} & dT= & 0 \end{array}$$

Rewrite the totally differentiated equations such that the exogenous variable appear on the right hand side: (recall from equation (5) that $p-c'=0$).

$$\begin{bmatrix} qp' & np'-c'' \\ q^2p' & nqp' \end{bmatrix} \begin{bmatrix} dn \\ dq \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} dT$$

Note that the determinat of the Jacobian w.r.t the primary endogenous variables (the matrix on the left hand side) will appear several times. Therefore, let's denote it by D and look at it first:

$$\begin{aligned} D &= \begin{vmatrix} qp' & np'-c'' \\ q^2p' & nqp' \end{vmatrix} \\ &= qp' * nqp' - q^2p' * (np'-c'') \\ &= nq^2(p')^2 - nq^2(p')^2 + q^2p'c'' \\ &= \underbrace{q^2}_{>0} * \underbrace{p'}_{<0} * \underbrace{c''}_{>0} < 0 \end{aligned} \tag{16}$$

Use Cramer's rule and the implicit function theorem to sign the partial derivatives of the primary endogenous variable(s) with respect to the exogenous variable.

$$\frac{\delta n}{\delta T} = \frac{\begin{vmatrix} 0 & np'-c'' \\ 1 & nqp' \end{vmatrix}}{D} = \frac{-n * \overbrace{p'}^{<0} + \overbrace{c''}^{>0}}{\underbrace{D}_{<0}} < 0 \tag{17}$$

$$\frac{\delta q}{\delta T} = \frac{\begin{vmatrix} qp' & 0 \\ q^2p' & 1 \end{vmatrix}}{D} = \frac{\overbrace{q}^{>0} * \overbrace{p'}^{<0}}{\underbrace{D}_{<0}} > 0 \tag{18}$$

f) *Totally differentiate your secondary endogenous equations in part (d) w.r.t the exogenous variable*
We know that $Q = nq$ and hence $\frac{\delta Q}{\delta T} = q\frac{\delta n}{\delta T} + n\frac{\delta q}{\delta T}$.
Using equations (17) and (18):

$$\begin{aligned} \frac{\delta Q}{\delta T} &= q\frac{\delta n}{\delta T} + n\frac{\delta q}{\delta T} \\ &= \frac{-qnp' + qc''}{D} + \frac{qnp'}{D} = \frac{\overbrace{q}^{>0} * \overbrace{c''}^{>0}}{\underbrace{D}_{<0}} < 0 \end{aligned} \tag{19}$$

Furthermore as $p = p(Q)$ we know by the chain rule that $\frac{\delta p}{\delta T} = p' \frac{\delta Q}{\delta T}$.
Using equations (11) and (19) we therefore know that

$$\frac{\delta p}{\delta T} = \underbrace{p'}_{<0} * \underbrace{\frac{\delta Q}{\delta T}}_{<0} > 0 \tag{20}$$

Finally, from equation (7) we know that the free entry condition always drives profit down to zero and consequently

$$\frac{\delta \pi}{\delta T} = 0 \tag{21}$$

In summary, a lump-sum tax will increase the output of each individual firm but decrease the number of firms in the industry. The overall market output will also decrease as the decrease in the number of firms outweights the increase of each company's output. The reduced market output will lead to a rise in the price.