Problem Set #03
Second Calculus Problem Set
Due date: Sep 23

(1) Consider the function $f(x,y,z) = xyz$, with $y = x^2$ and $z = x^{1/3}$.
  (a) Rewrite f as a function $g : \mathbb{R} \to \mathbb{R}$ alone and compute $g'(\cdot)$. Using g', approximate the change in f when x increases by 0.1 units, starting from $(8, 64, 2)$.
  (b) Compute the total derivative of f with respect to x. Using the total derivative, approximate the change in f when x increases by 0.1 units, starting from $(8, 64, 2)$.
  (c) Write down the differential of f at $(8, 64, 2)$. Using the differential, approximate the change in f when x increases by 0.1 units, starting from $(8, 64, 2)$.
  (d) Identify the direction h^* that (x, y, z) moves in, starting from $(8, 64, 2)$, when x increases. Write down the directional derivative of f in the direction $\mathbf{h}^*$, i.e., $f_{\mathbf{h}^*}(\cdot,\cdot,\cdot)$, and evaluate this derivative at $(8, 64, 2)$. Using $f_{\mathbf{h}^*}(8, 64, 2)$, approximate the change in f when x increases by 0.1 units, starting from $(8, 64, 2)$.
  (e) Check to see that all four of these distinct methods give you the same answer!

(2) Recall that a function $f : \mathbb{R}^n \to \mathbb{R}^m$ is nothing more than m functions, $f^1 ... f^m$, each mapping $\mathbb{R}^n \to \mathbb{R}$, and stacked on top of each other.
  (a) Using this fact, write down a formal definition of the directional derivative of f at x_0 in the direction $\mathbf{h} \in \mathbb{R}^n$, for a function $f : \mathbb{R}^n \to \mathbb{R}^m$. Your definition should be of the form

$$\text{blah, blah} \quad = \quad \lim_{k \to \infty} \frac{\text{blah} \qquad\qquad \text{blah}}{\text{blah, blah}}$$

  (b) Consider the function $f : \mathbb{R}^2 \to \mathbb{R}^2$ defined by, for $i = 1, 2$, $f^i(x, y) = x^{i/3} y^{1-i/3}$. Using the formal definition in (a) above, compute $f_{\mathbf{h}^*}(27, 8)$, where $\mathbf{h}^* = (54, 16)$. (Hint: $(27, 8) + (54, 16)/k = (27, 8)(1 + 2/k)$).
  (c) Now compute $f_{\mathbf{h}^*}(27, 8)$ using the differential of f at $(27, 8)$.
  (d) Check to see that all of these three distinct methods give you the same answer!

(3) Consider the function $f(\mathbf{x}) = x_1^\rho + x_2^\rho$, where $\rho \in (-\infty, 1]$. *The whole point here is to use the differential of* $\bigtriangledown f$ *to answer the following questions, ie., to answer all parts of the question, approximate* $\bigtriangledown f(\mathbf{x}+\mathbf{h}) - \bigtriangledown f(\mathbf{x})$ *using the differential of* $\bigtriangledown f$ *at* $\mathbf{x}$*, evaluated at* $\mathbf{h}$*. There are lots of other ways to answer these questions, but the purpose of this question is to give you practice in using the differential of a vector-valued function.*
  (a) Check that, *up to a first order approximation*,[1] f is homothetic (cf the notes for lecture CALCULUS3,[2] specifically the second example in the subsection entitled Four Graphical Examples.[3])
  (b) When $\rho > 0$, does f exhibit increasing, constant or decreasing returns to scale? Is your answer true for all $\rho \in (0, 1]$. (Again, your answer should be in terms of what happens to the *gradient vector* as you move out along a ray.)
  (c) Fix $\mathbf{x} = (\alpha, \alpha)$, and consider $\mathbf{h} = (-0.1, 0.1)$. Approximate $\bigtriangledown f(\mathbf{x}+\mathbf{h})$, for (i) $\rho = 1/2$; (ii) $\rho = -1/2$; (iii) $\rho = -10$.

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[1] The qualifier "up to a first order approximation" means: you should *pretend* that the answer you get using the differential is exactly correct, even though in fact it is only approximately correct, and then only for small $\mathbf{h}$'s, because there are non-zero higher order terms in the Taylor expansion of $\bigtriangledown f$.

[2] In the example in the notes, you don't need the caveat about up to a first order approximation, because the higher order terms in the Taylor approximation are all zero. In this example they are not.

[3] The lecture notes tend to change, and sometimes the problem sets don't keep up. If this reference is no longer current, please notify Leo.

(d) How does the curvature of the level sets of this function change as you move out along a ray through the origin. In particular, discuss the effect of the magnitude of α on the *rate of change* in the direction of $\bigtriangledown f$ as you add $\mathbf{h} = (-\beta, \beta)$ to $\mathbf{x} = (\alpha, \alpha)$. (Hint: Think of the angle θ between $\bigtriangledown f(\mathbf{x})$ and $\bigtriangledown f(\mathbf{x}+\mathbf{h})$, how would you measure that? You might try using the fact that $\bigtriangledown f(\mathbf{x}) \cdot \bigtriangledown f(\mathbf{x}+\mathbf{h}) = || \bigtriangledown f(\mathbf{x}) || \times || \bigtriangledown f(\mathbf{x}+\mathbf{h}) || \times \cos(\theta)$).

(4) Suppose that $f : \mathbb{R} \to \mathbb{R}$ is (n+1) times continously differentiable.

(a) Show that a sufficient condition for f to a attain a strict (local) maximum at x_0 is that for some even number n. the derivatives $f^{(k)}(x_0)$ are zero for $k = 1...n-1$, and $f^{(n)}(x_0)$ is negative

(b) If $f^{(k)}(x_0)$ is zero for $k = 1...n-1$ and $f^{(n)}(x_0)$ is non-zero, show that there exists an ϵ-neighborhood around x_0 where the absolute value of the n^{th}-order Taylor expansion is bigger than the absolute value of the remainder term $R_n(x)$.

(c) Give a counter example to show that the result in part (a) would be false if the words "for some even n" were replaced with "for some $n > 0$".

(d) Explain carefully, but in as few a words as possible, why the argument in (a) works for even n but not for odd n.

(e) Show that the n^{th}-order Taylor expansion around any point x_0 of a polynomial of degree n (i.e. a function of the form $f(x) = \sum_{k=0}^{n} \alpha_k x^k$) is perfectly accurate, regardless of the magnitude of dx.

(f) Show that if f is an arbitrary polynomial of degree 2, i.e., $f(x) = ax^2 + bx + c$, then for any point x_0, if you add to $f(x_0)$ the 2^{nd}-order Taylor expansion around f, the expression you get is precisely the original function f. More precisely, show by writing out the Taylor expansion explicitly, that for arbitrary dx, $f(x_0 + dx) = f(x_0) + f'(x_0)dx + 0.5f'(x_0)dx^2$.