ARE201-Simon, Fall2015

LINALGEBRA2: TUE, SEP 1, 2015 PRINTED: SEPTEMBER 1, 2015 (LEC# 2)

Contents

1. Linear Algebra (cont)

1.6. **Vector Spaces**

A key concept in linear algebra is called a *vector space.*

Verbal Defn: a nonempty set of vectors V is called a *vector space* if it satisfies the following property: given any two vectors that belong to the set V, *every* linear combination of these vectors is also in the set.

Math Defn: a nonempty set of vectors V is called a *vector space* if for all $\mathbf{x}^1, \mathbf{x}^2 \in V$, for all $\alpha_1, \alpha_2 \in \mathbb{R}$, $\alpha_1 \mathbf{x}^1 + \alpha_2 \mathbf{x}^2 \in V$.

We'll begin by considering the various vector spaces that belong to $\mathbb{R}^2$.

Example: the set of vectors represented by a line. Note that a line can (and should) be thought of as just another set of vectors. Think about all of the lines you can draw in $\mathbb{R}^2$. Which lines are vector spaces, according to the above definition?

- verbal defn excludes any straight line that doesn't go thru the origin.
- verbal defn excludes any line that "curves."
- verbal defn excludes any straight line thru the origin that "stops."

What's left is rays thru the origin. By a "ray through the origin" I mean a straight line that passes through the origin and goes on forever, in either direction. Every ray through the origin is a vector space. Mathematically, any ray through the origin is defined as the set of all scalar multiples of a given vector. It's called a "one-dimensional vector space", because it is "constructed" from a single vector. Note that neither the nonnegative or the positive cones defined by a single vector are vector spaces.

Turns out that there's exactly two more vector spaces in $\mathbb{R}^2$:

- The set consisting of the entire plane is a vector space, i.e., $\mathbb{R}^2$. This is a two dimensional vector space.
- The set consisting of zero alone is a vector space. This is a zero dimensional vector space.

See how our math definition matches up with our verbal definition, by considering the examples above.

- look at a line that curves: take any vector in the *set* consisting of the points in the line. Let $\mathbf{v}^1, \mathbf{v}^2$ be any vectors in the line. Take zero times the first and 2 times the second. Is it in the set? No. Conclude that the curved line is not a vector space.
- look at a straight line that doesn't pass through the origin. Let $\mathbf{v}^1, \mathbf{v}^2$ be any vectors in the line. Take zero times both. In the set? No. Conclude that a straight line that doesn't pass through the origin is not a vector space.
- take any straight line thru the origin that "stops." Let $\mathbf{v}^1, \mathbf{v}^2$ be any vectors in the line. Take zero times the first and a large enough multiple times the second. Is it in the set? No. Conclude that a line thru the origin that "stops" is not a vector space.

Now consider zero: satisfies the above definition. Similarly the whole plane. But an arbitary positive cone wouldn't.

What are the vector spaces in $\mathbb{R}^3$.

- The zero vector (with three components).
- Lines thru the origin
- Planes that contain the origin
- The whole space $\mathbb{R}^3$ itself
- There aren't any others.

A vector space that lives inside another vector space is called a *vector subspace* of the original vector space.

- The vector space consisting of zero alone is a vector subspace of *every* vector space
- Lines thru the origin are vector subspaces of $\mathbb{R}^2$, $\mathbb{R}^3$ or $\mathbb{R}^{\text{whatever}}$, depending on the number of components of the vector that defines the line.
- A plane in 3D that contains the origin is a vector subspace of $\mathbb{R}^3$.

1.7. **Spanning, Dimension, Basis**

Intuitively, a set of vectors in $\mathbb{R}^n$ is said to *span* a vector space if you can build the entire vector space out of the original set of vectors. (We'll always assume that the set of vectors are all conformable, i.e., have the same number of elements.)

Defn: Given a vector space V, a set of vectors $S = \{\mathbf{v}^1, \mathbf{v}^2, ...\mathbf{v}^i, ...\mathbf{v}^n\}$ spans V if any element of V can be written as a linear combination of the elements of S.

Note that a given set of vectors can span many different spaces, some of which are subsets of others.

Example: Consider a set of vectors$\{\mathbf{v}^1, \mathbf{v}^2\} \subset \mathbb{R}^2$. Assume that $\mathbf{v}^1$ and $\mathbf{v}^2$. are linearly independent. The different vector spaces that this set spans are:

- $\{(0, 0)\}$
- *any* line through (0,0) (not necessarily one including either $\mathbf{v}^1$ or $\mathbf{v}^2$).
- $\mathbb{R}^2$

Moreover *every* set of vectors spans *some* vector space. Proof: the vector space consisting of $\{\mathbf{0}\} \subset \mathbb{R}^n$ is always spanned by any given set of vectors in $\mathbb{R}^n$.

Defn: The *span* of a set of vectors $S = \{\mathbf{v}^1, \mathbf{v}^2, ..., \mathbf{v}^n\}$ is the (unique) vector space V satisfying the property: if W and V are both spanned by S, then $W \subset V$.

That is, *the span* of a set of vectors is the biggest vector space that the set spans: all other spaces that the set spans are subspaces of *the span* of the set.

Fact: (Takes a little bit of work to prove.) If V is *the span* of a set of vectors $S = \{\mathbf{v}^1, \mathbf{v}^2, ..., \mathbf{v}^n\}$, then $S \subset V$.

More examples of vector spaces and the vectors that span them:

- Take a line through the origin in $\mathbb{R}^n$. Call this vector space V.
  - What sets of vectors span this vector space? Answer: *any element* of V except one spans it. The exception is the vector $\mathbf{0}$, i.e., given any element $\mathbf{v} \in V$ except zero, *all* of the elements of V can be written as scalar multiples of $\mathbf{v}$.
  - Vectors that belong to the same line through the origin (i.e., the same vector subspace of $\mathbb{R}^n$) are called *collinear*, i.e., $\mathbf{v}^1$ and $\mathbf{v}^2$ are collinear if there exists $\alpha \in \mathbb{R}$ such that $\mathbf{v}^2 = \alpha\mathbf{v}^1$.
- Take the vector space consisting of zero, i.e., $\mathbf{0} \in \mathbb{R}^n$. What spans it? Ans: *any* subset of $\mathbb{R}^n$, including the vector $\mathbf{0}$.
- Take the whole of $\mathbb{R}^2$: what sets of vectors span it? Answer: any pair of vectors that *don't both belong to* the same ray through the origin.
- Take the horizontal plane in $\mathbb{R}^3$. What sets of vectors span it? How many vectors do you need?
- Take an arbitrary plane in $\mathbb{R}^3$ that passes through the origin. Not quite so easy to see that it is a vector space, but it is.
- Note that if a set of vectors spans a space and you throw in any number of additional vectors (with the same number of components) then the augmented set still spans the space. For example, if $\{\mathbf{v}^1, \mathbf{v}^2\} \subset \mathbb{R}^n$ spans $V \subset \mathbb{R}^n$ then $\{\mathbf{v}^1, \mathbf{v}^2, ..., \mathbf{v}^n\} \subset \mathbb{R}^n$ does too. To see this, that $\mathbf{v}' \in V$ and write it as a linear combination of $\mathbf{v}^1$ and $\mathbf{v}^2$ plus zero times all the other vectors.

Since spanning sets of vectors may contain many vectors that aren't necessary to span the space, it is useful to identify spanning sets of vectors that don't contain any "excess fat."

Definition: Given a vector subspace W of a vector space V, a set of vectors $S = \{\mathbf{v}^1, \mathbf{v}^2, ..., \mathbf{v}^n\} \subset V$, S is a *minimal spanning set* for W if S spans W and for $i = 1, ...n$, $S \backslash \mathbf{v}^i$ does not span W.

For example,

- Any single nonzero element $\mathbf{v}$ in $\mathbb{R}^n$ is a minimal spanning set of some vector space W; it spans the vector space consisting of the line that contains $\mathbf{v}$.

- Let W be the horizontal plane in $\mathbb{R}^2$; any pair of noncollinear vectors whose third components are zero spans W.

Examples of spanning sets that aren't minimal:

- Take a line through the origin, and a bunch of vectors belonging to the line, i.e., a bunch of collinear vectors. Can toss all but one out.
- Same thing for a plane in $\mathbb{R}^3$: can toss all but two out.

Minimal spanning sets are often confused with basis sets, but they are not the same thing.

Defn: Given a vector space V, a set of vectors $S = \{\mathbf{v}^1, \mathbf{v}^2, ..., \mathbf{v}^n\}$ is a *basis* for V if $S \subset V$ and S is a minimal spanning set for V.

Note the difference between a minimal spanning set and a basis: a basis for V is required to belong to V. Take $\{\mathbf{v}^1, \mathbf{v}^2\} \subset \mathbb{R}^2$ which are linearly independent so this set spans $\mathbb{R}^2$. For $\mathbf{v}^3 \in \mathbb{R}^2$ such that $\mathbf{v}^3$ is not collinear with $\mathbf{v}^i$, for $i = 1, 2$. Then $\{\mathbf{v}^1, \mathbf{v}^2\}$ is a minimal spanning set for the subspace $W = \{\alpha\mathbf{v}^3 : \alpha \in \mathbb{R}\}$. But $\{\mathbf{v}^1, \mathbf{v}^2\}$ is not a basis for W.

Theorem: if a set of vectors forms a basis for some vector space, then the set must be a linear independent set.

"Proof:" Suppose that the set $\{\mathbf{v}^1, \mathbf{v}^2, \mathbf{v}^3\}$ spans the vector space V, but that $\mathbf{v}^3$ can be written as a linear combination of $\mathbf{v}^1$ and $\mathbf{v}^2$, specifically, $\mathbf{v}_3 = \alpha_1\mathbf{v}^1 = \alpha_2\mathbf{v}^2$. Then you don't need $\mathbf{v}^3$ to construct V. Whenever you thought $\mathbf{v}^3$ was essential for constructing some vector $\mathbf{v} \in V$, replace $\mathbf{v}^3$ with $= \alpha_1\mathbf{v}^1 = \alpha_2\mathbf{v}^2$. and now you've written $\mathbf{v}$ as a linear combination of just $\mathbf{v}^1$ and $\mathbf{v}^2$.

Theorem (deep result, nontrivial to prove): Any two bases for a given vector space must have the same number of elements.

Examples:

- Lines thru the origin
- Planes in $\mathbb{R}^3$

Because of the preceding theorem, it makes sense to talk about the size of a vector space: size is the number of elements in any basis.

Defn: The *dimension* of a vector space V is the number of elements in any basis for V.

- A line through the origin is a one-dimensional vector space (or a one-dimensional vector subspace of $\mathbb{R}^2$).
- A plane in 3D is a two-dimensional subspace of $\mathbb{R}^3$.
- By convention,the vector space consisting of zero is considered to be a zero dimensional vector space. In order to be consistent with our definition—i.e., the dimension of a vector space is the number of elements in any basis set—a basis set for the space $\{\mathbf{0}\}$ must have zero elements, i.e., it must be the empty set.

Note well that we are talking now of a rather different usage of the word "dimension" from the familiar usage. One often talks about 3 dimensional vectors, etc. A "3 dimensional" vector space is something totally different from what some people (not me) call a "3 dimensional vector." Could have a 3 dimensional vector space that is a subspace of $\mathbb{R}^{17}$: its bases would all consist of three "17-dimensional" vectors. For this reason, I'll try never to use the term "3-dimensional vector", saying instead, "3-component vector". I'll reserve the term "dimension" to refer to the size of a vector space.

Fact: the dimension of a vector space spanned by a given set of m vectors cannot have a dimension higher than m.

Fact: Exactly one vector space in $\mathbb{R}^n$ has a unique set of basis vectors. For every other vector space, there are infinitely many sets of basis vectors. The exception is the zero-dimensional vector space. For any other vector space, take a basis, multiply every element by $\alpha \in \mathbb{R}$ and you have another one.

So far, we've been talking about vector subspaces of $\mathbb{R}^n$. There are lots of other vector spaces. For example, the set of all *sequences in* $\mathbb{R}$, i.e., the set of all mappings from $\mathbb{N}$ to $\mathbb{R}$ is a vector space. We'll call this space $\mathbb{R}^\infty$. What's a basis for this space? Obviously, it has to be a subset of the space, i.e., each element in the basis has to be a sequence in $\mathbb{R}$ itself. There are lots of such bases. One is the set $\{e^k\}_{k=1}^\infty$, where *each* e^k is itself a sequence, defined by $e_r^k = \begin{cases} 1 & \text{if } k = r \\ 0 & \text{otherwise} \end{cases}$. To see that this is indeed a basis for $\mathbb{R}^\infty$, take any sequence $\{\mathbf{x}_n\}_{n=1}^\infty$ and observe that it can be written as $\sum_{k=1}^\infty \mathbf{x}_k e^k$.

Consider a *repeating sequence* of the form $\{x_1, x_2, ..., x_n, x_1, x_2, ..., x_n, ...\}$. Let V^n denote the set all such sequences, i.e., for $n = 5$, $V^5 = \{\{x_n\} : x_n = a, b, c, d, e, a, b, c, d, e, ...,\}$ for some $a \in \mathbb{R}$, $b \in \mathbb{R}$, $c \in \mathbb{R}$, $e \in \mathbb{R}$, $e \in \mathbb{R}\}$. What's the dimension of V^n. What's a basis for V^n?

1.8. **Matrices and Rank**

A matrix is nothing other than a bunch of column vectors stacked beside each other, or a bunch of row vectors stacked on top of each other. All of the above stuff about vector spaces and the sets that span them can be restated in terms of matrices: What's the difference between a set of vectors that is a basis for V and the matrix consisting of those vectors stacked beside each other? Of which are there "more of:" basis sets or matrices consisting of basis sets? (Obviously, "more" is a slippery concept here, but we're not being very formal at the moment.)

- Can we say that a *matrix* spans a vector space? In fact any matrix spans two vector spaces: the columns span one, and the rows span another. If the matrix is $m \times n$, then the rows span a subspace of $\mathbb{R}^n$ i.e., we have m rows each with n elements) and the columns span a subspace of $\mathbb{R}^m$
- We don't talk about the dimension of a matrix (well some people do, but here the word dimension means something different), but we do talk about its *rank*
  - The *row rank* of a matrix is the dimension of the (unique) vector space which is the span of the *rows* of the matrix.
  - The *column rank* of a matrix is the dimension of the (unique) vector space which is the span of the *columns* of the matrix.
  - (Deep fact if not pure magic:) The dimension of the vector space spanned by the rows is the same as the dimension of the vector space spanned by the columns. Matrix could have 14 rows and 348 columns. The rows span some subspace of $\mathbb{R}^{348}$. The columns span some subspace of $\mathbb{R}^{14}$. The dimensions of these entirely different subspaces (e.g., might be 7) are the same.

– For this reason, we can talk about *the rank* of a matrix, which is equal to both the row and the column rank.
– The rank of a matrix cannot exceed the minimum of the number of the matrix's rows and the number of its columns
– A matrix is said to be of *full rank* if its rank is equal to the minimum of the number of rows and the number of columns, that is, if it is as large as it can be.
– Under what circumstances do the rows or the colums of a matrix form a basis for some vector space? For an $m \times n$ matrix, with more columns than rows, i.e., $n > m$, the rows *maybe* (but not necessarily will) form a basis for a vector space, but the columns can't:
  * The rows will form a basis for an m-dimensional subspace of $\mathbb{R}^n$ if and only if the rows are linearly independent
  * The columns can't form a basis, because there are n of them and any space that they span must be a subspace of $\mathbb{R}^m$: you need at most m vectors to do this and $n > m$, so that the columns *must* be a linear dependent set
  * consider, for example, the schematic 2×3 matrix $A = \begin{bmatrix} x & x & x \\ x & x & x \end{bmatrix}$.
    · the two rows form a basis for a 2-dimensional subspace of $\mathbb{R}^3$ iff they are linear independent
    · the three columns can't be a basis for $\mathbb{R}^2$ because there are too many of them.
– Take a square matrix, M, which is $n \times n$. There are vector subspaces R and C of $\mathbb{R}^n$ such that R is the span of the rows of the matrix and C is the span of the columns. What property of the matrix tells me whether or not R and C have dimension n or less? Ans again: linear independence.
– Suppose that a square matrix M forms a basis for some vector space. Which vector space will it be: Ans: it must be $\mathbb{R}^n$. If M has less than full rank, then the vector spaces R and C spanned respectively by the rows and the columns, both with dimension less than n, will *in general* be *different* vector spaces.
  * For example, consider $A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$.
    (1) The rows span the line in $\mathbb{R}^2$ given by $\{\alpha(1,2) : \alpha \in \mathbb{R}\}$.
    (2) The cols span the line in $\mathbb{R}^2$ given by $\{\alpha(1,3) : \alpha \in \mathbb{R}\}$.
  * Under what conditions will the two spaces $R = C$? Ans: if the matrix is symmetric.