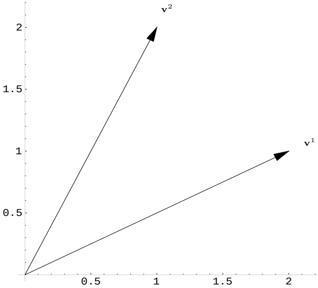
2nd Order Conditions for unconstrained opt (preview)

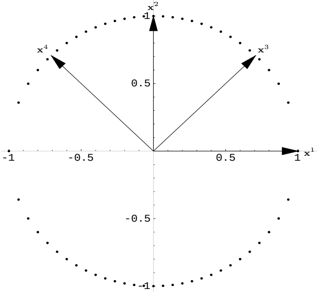
- If FOC conditions for optimum are satisfied,
  - could have, max, min, or neither.
- specifically, for $x, dx \in \mathbb{R}^n$,

$$f(\bar{x} + dx) - f(\bar{x}) \quad \approx \quad \bigtriangledown f(\bar{x})dx + \frac{1}{2}dx'\mathrm{H}\mathbf{f}(\bar{x})dx$$

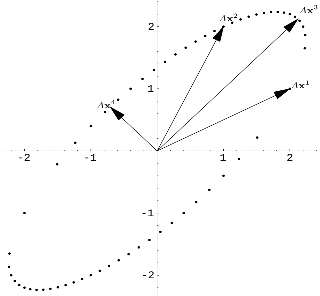
- 1st order *nec* condition for dx to be an unconstrained opt: $\bigtriangledown f(\bar{x}) = 0$.
- $\bar{x}$ is a local max, min or neither depending on $\mathrm{sgn}(dx'\mathrm{H}\mathbf{f}(\bar{x})dx)$
- it will be a local
  - *max* if $dx'\mathrm{H}\mathbf{f}(\bar{x})dx < 0$, $\forall dx$,
    - i.e., if $\mathrm{H}\mathbf{f}(\bar{x})$ is *a negative definite* matrix.
  - *min* if $dx'\mathrm{H}\mathbf{f}(\bar{x})dx > 0$, $\forall dx$,
    - i.e., if $\mathrm{H}\mathbf{f}(\bar{x})$ is *a positive definite* matrix.
  - *neither* if $\exists \mathbf{dy}, \mathbf{dz} \in \mathbb{R}^n$ s.t. $\mathbf{dy}'\mathrm{H}\mathbf{f}(\bar{x})\mathbf{dy} > 0 > \mathbf{dz}'\mathrm{H}\mathbf{f}(\bar{x})\mathbf{dz}$
    - i.e., if $\mathrm{H}\mathbf{f}(\bar{x})$ is *an indefinite* matrix.
- what follows is graphical intuition for what definiteness means
  - and its relationship to the *eigen-vectors* and *eigen-values* of H.



(a) The columns of A

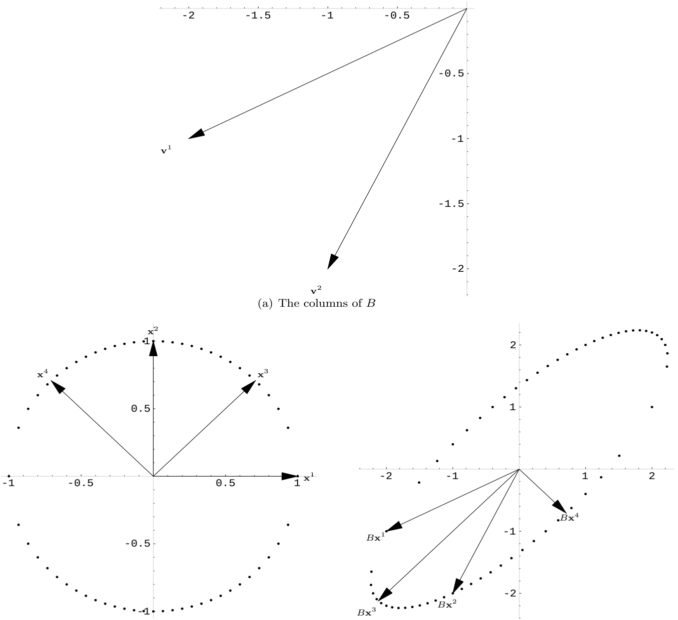


(b) Selected elements of the unit circle



(c) The image of the circle under A

FIGURE 1. What the matrix A does to the unit circle

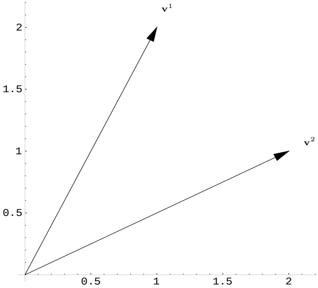


(a) The columns of B

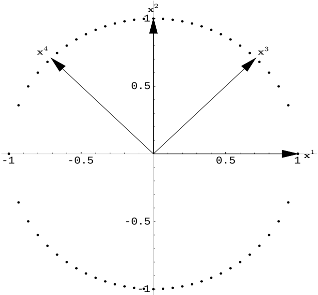
(b) Selected elements of the unit circle

(c) The image of the circle under B

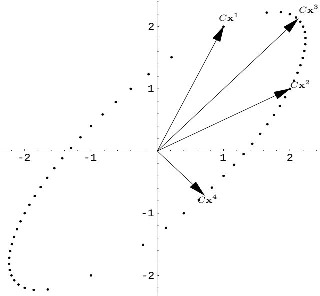
FIGURE 1. What the matrix B does to the unit circle



(a) The columns of C



(b) Selected elements of the unit circle



(c) The image of the circle under C

FIGURE 1. What the matrix C does to the unit circle

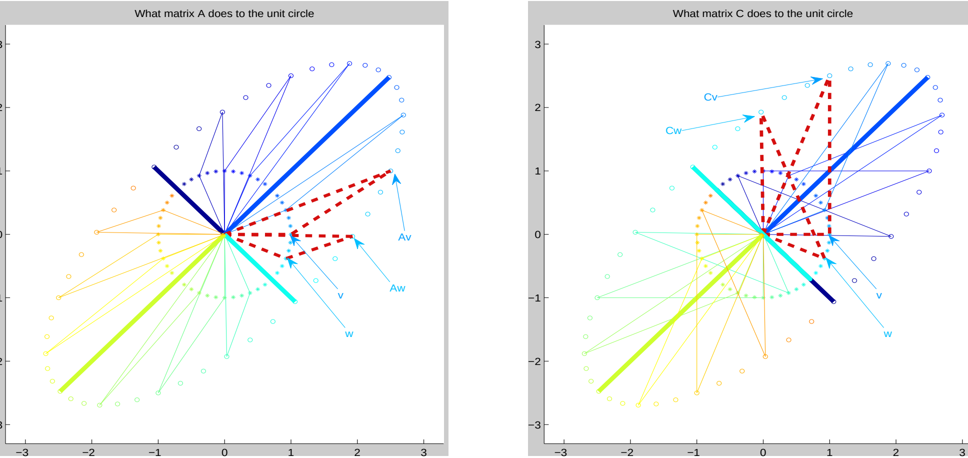
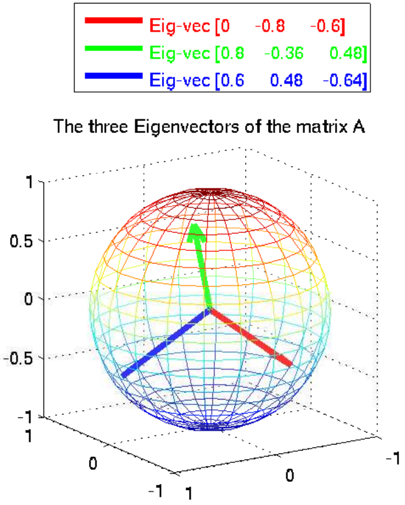
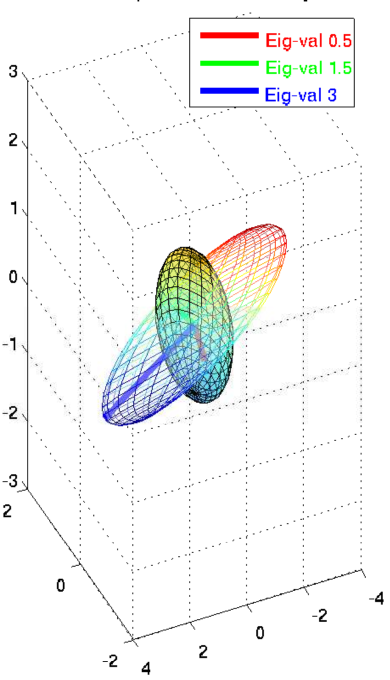


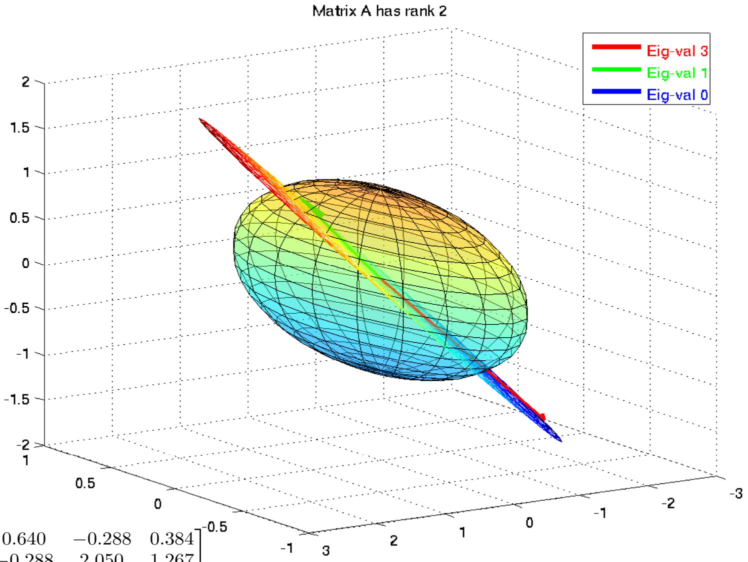
FIGURE 4. Relationship between orientation and determinant

A matrix is *definite* / *indefinite* **iff its rotations** *preserve* / *reverse* **orientation**

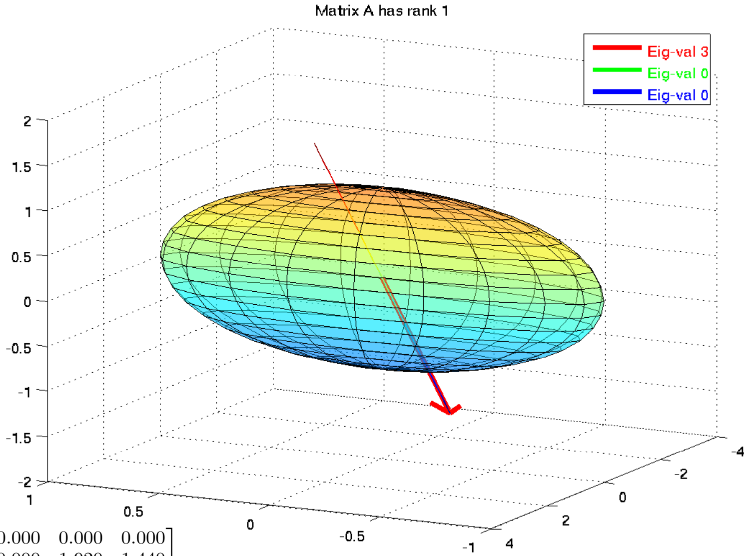




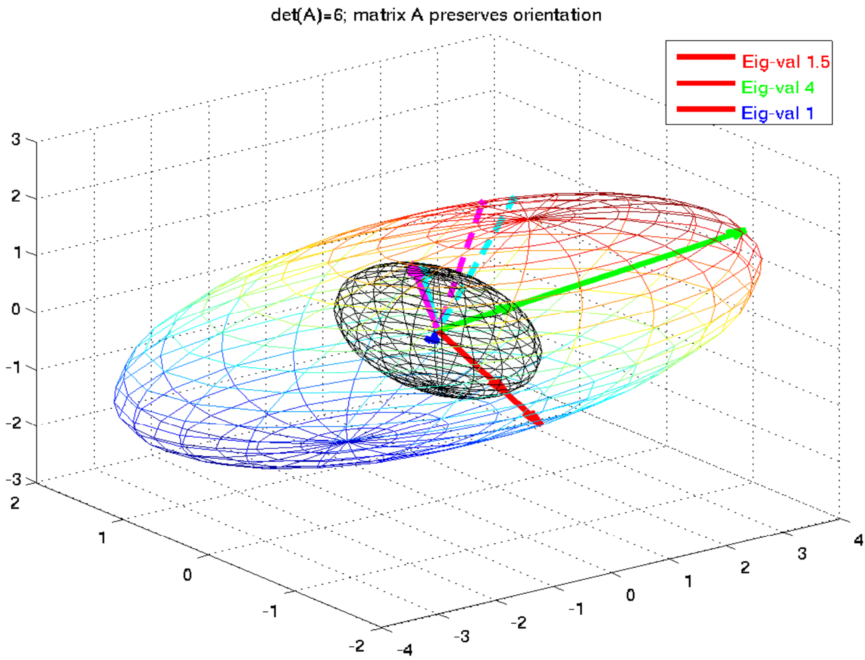
$$A = \begin{bmatrix} 2.040 & 0.432 & -0.576 \\ 0.432 & 1.205 & -0.940 \\ -0.576 & -0.940 & 1.754 \end{bmatrix}$$

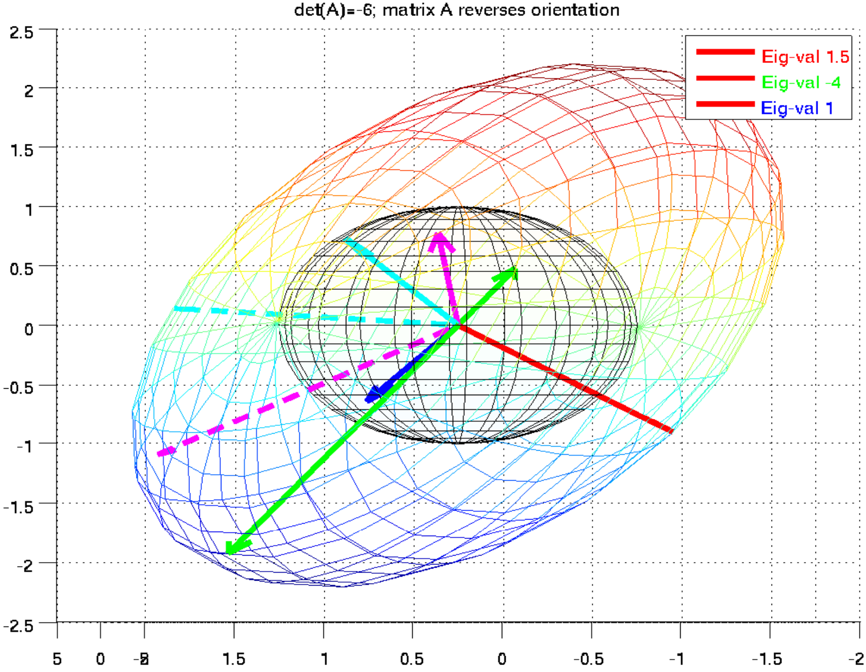


$$A = \begin{bmatrix} 0.640 & -0.288 & 0.384 \\ -0.288 & 2.050 & 1.267 \\ 0.384 & 1.267 & 1.310 \end{bmatrix}$$



$$A = \begin{bmatrix} 0.000 & 0.000 & 0.000 \\ 0.000 & 1.920 & 1.440 \\ 0.000 & 1.440 & 1.080 \end{bmatrix}$$


det(A)=6; matrix A preserves orientation
Eig-val 1.5
Eig-val 4
Eig-val 1
3
2
1
0
-1
-2
-3
2
1
0
-1
-2
-4
-3
-2
-1
0
1
2
3
4


det(A)=-6; matrix A reverses orientation
Eig-val 1.5
Eig-val -4
Eig-val 1
2.5
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
-2.5
5
0
1.5
1
0.5
0
-0.5
-1
-1.5
-2

1. A vector $v \in \mathbb{R}$ is an *eigenvector* of matrix $A(n \times n)$ if $\exists \lambda \in \mathbb{R}$ s.t. $Av = \lambda v$
2. For any *symmetric* $n \times n$ matrix, there exists a set of eigenvectors, each with unit length, that are pairwise orthogonal, i.e., for any two elements in the set, $\mathbf{v}^1$ and $\mathbf{v}^2$, then $\mathbf{v}^1 \cdot \mathbf{v}^2 = 0$.
3. For any *symmetric* $n \times n$ matrix, the matrix is *positive* (*negative*) definite if and only if all of its eigenvalues are positive (negative).
4. The *determinant* of a matrix is equal to the product of its eigenvectors
5. A *symmetric* $n \times n$ matrix, has exactly $2n$ unit eigenvectors iff all of the corresponding eigenvalues are distinct. If any two eigenvalues corresponding to distinct eigenvectors are equal, then there are an infinite number of unit eigenvectors
6. For any *symmetric* $n \times n$ matrix, the rank of the matrix is equal to the number of nonzero eigenvalues. More precisely, let A be a symmetric $n \times n$ matrix, let $V = \left\{ v^1, ..., v^n \right\}$ be a set of pair-wise orthogonal eigenvectors for A and let $\Lambda = \{\lambda_1, ..., \lambda_n\}$ be a set of corresponding eigenvalues, i.e., for each i, $Av^i = \lambda_i v^i$. Then the rank of A is equal to the number of nonzero elements of Λ.