Comparative Statics Overview (pt 1):

1. **Envelope theorem**: simplifies computation of change in *value function* (optimized level of objective function) as parameters change
   1. Some function is being optimized
      1. function depends on some parameters
      2. how does optimum value change with these parameters?
   2. Context: First order conditions of an optimization problem
      Examples:
      1. optimized utility as a function of income
      2. maximized profit as a function of price

      Two kinds of optimization:
      1. unconstrained optimization (intuition is fairly straightforward)
      2. constrained optimization (intuition is much more elusive)
   3. Our analysis of the lagrangian was a special case of the envelope theorem
      1. invoked the (constrained) envelope thm to prove $\lambda_j =$ shadow value of b_j
      2. use same approach to compute shadow value" of *any* parameter

Comparative Statics Overview (pt 2):

2. **Envelope theorem** (bonus): in very special cases you can short-circuit the implicit function theorem, and use the envelope theorem to get the same information that the implicit function theorem would give you, but much more easily

   Examples:

   1. input demand functions (Hotelling's Lemma)
   2. conditional input demand functions (Shephard's Lemma)

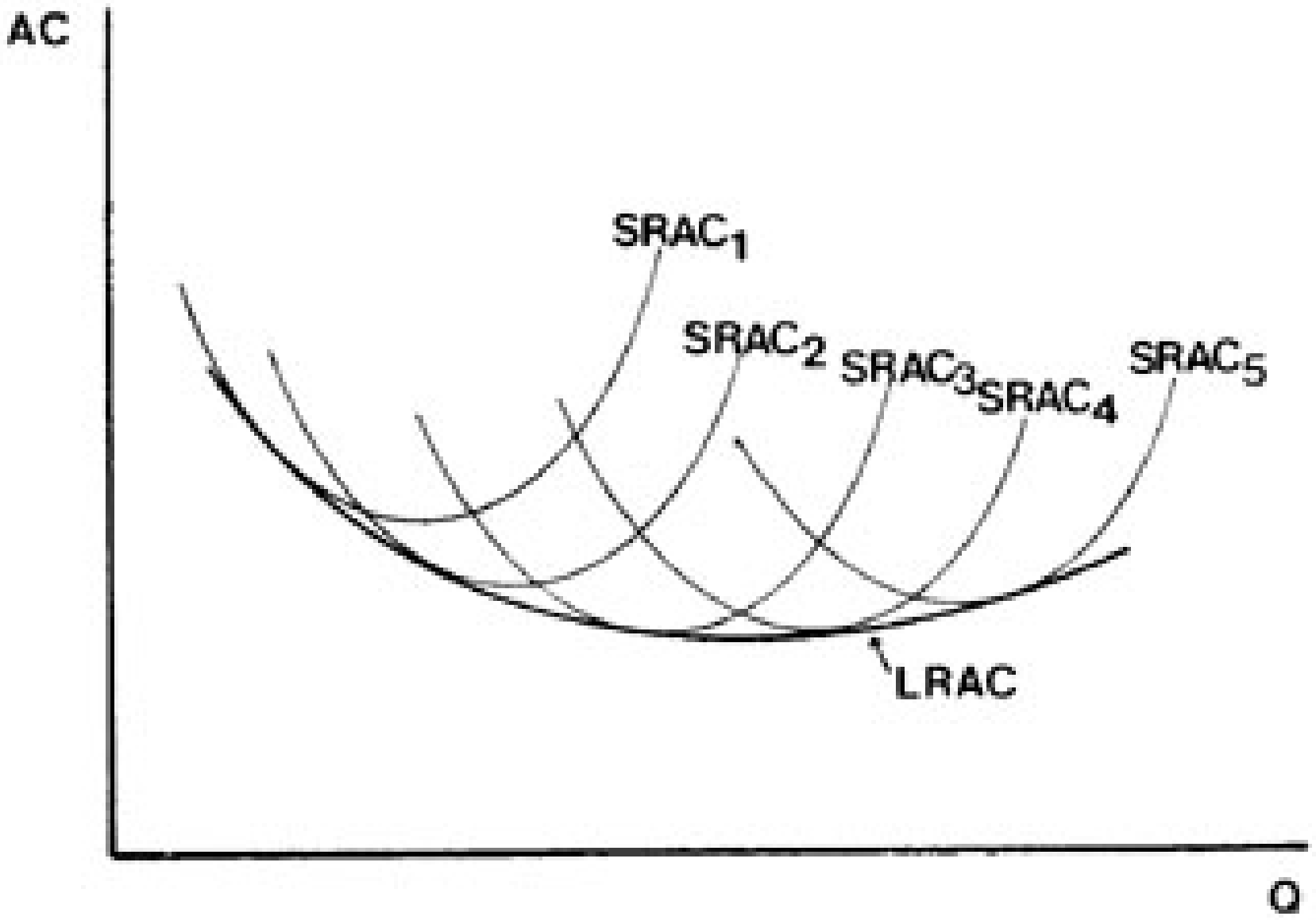
   These results are *very* special cases:

   1. not what the envelope theorem was "meant for"
   2. just happens to work cos some part of objective function is linear

Why is the envelope theorem called the envelope theorem?

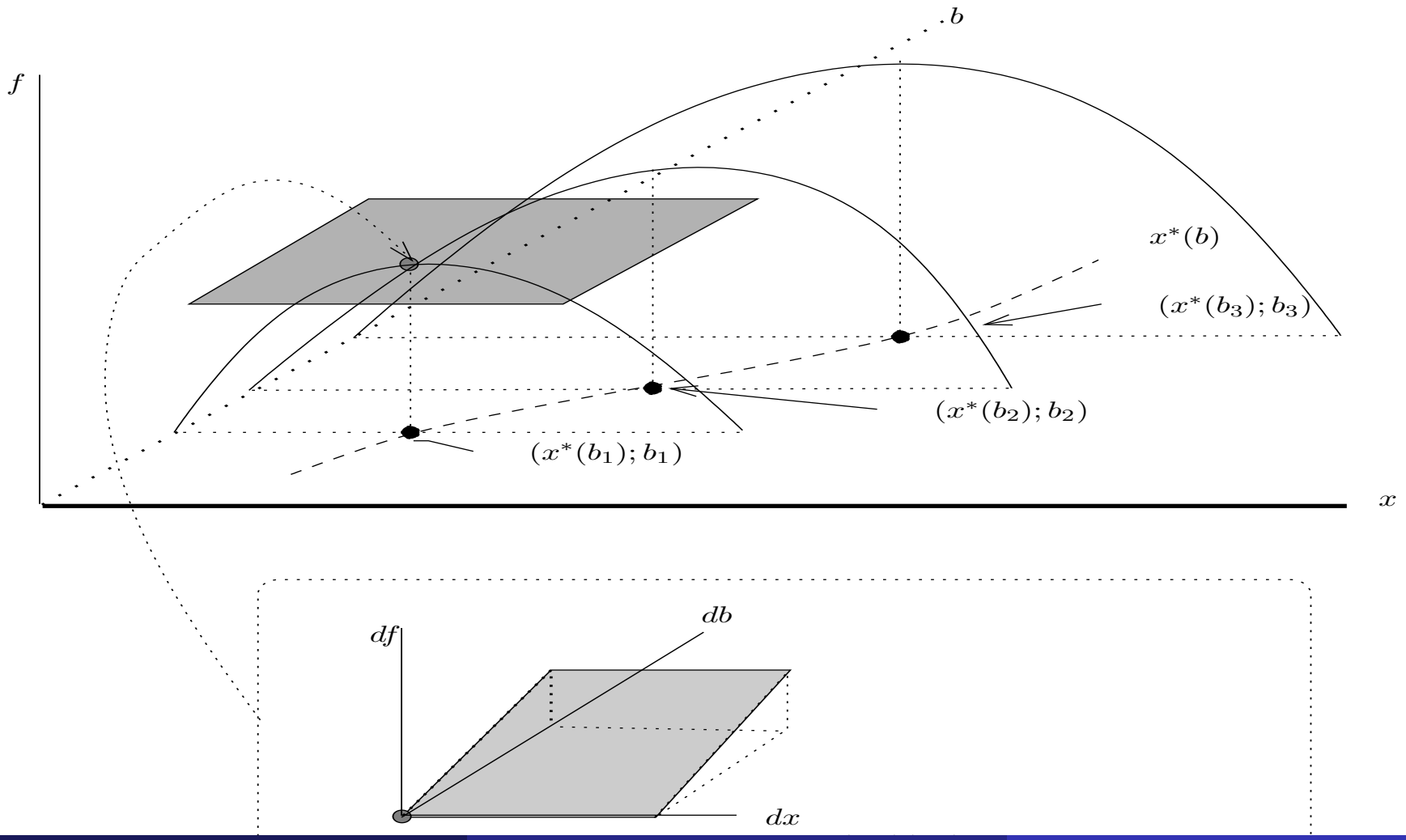
(From Wikipedia) The American Heritage Dictionary (3ed) gives one meaning of "envelope" to be: "A curve or surface that is tangent to every one of a family of curves or surfaces".

LRATC as lower envelope of SRATC's



$M(b)$ as the upper envelope of $f(x, b)$

$\forall b$, tangent line to $M(b) = f(x^*(b), b)$ lies on tangent plane to f, eval at $(x^*(b), b)$.



Comparative Statics Overview (pt 3):

3. **Implicit function theorem**: used to compute relationship between endogenous and exogenous variables.
   1. Context: First order conditions of an optimization problem. Examples
      - e.g., demands as a function of income
      - e.g., input demands as a function of output
   2. Context: Some kind of system of equilibrium equations. Examples:
      - market equilibrium prices as a function of economy endowments
      - Cournot quantities as a function of demand parameter, # of firms

Two envelope Theorems: (summary)

(UET) Unconstrained envelope theorem:

if $\mathbf{x}^*(\boldsymbol{\alpha})$ solves $\max_{\mathbf{x}} f(\mathbf{x};\boldsymbol{\alpha})$ and $\mathbf{x}^*(\cdot)$ is diff^{able} at $\boldsymbol{\alpha}$, then $\frac{df(\mathbf{x}^*(\boldsymbol{\alpha});\boldsymbol{\alpha})}{d\alpha_k} = \frac{\partial f(\mathbf{x}^*(\boldsymbol{\alpha});\boldsymbol{\alpha})}{\partial \alpha_k}$.

- *total* derivative of f w.r.t. α_k equals *partial* derivative of f w.r.t. α_k.
  - i.e., reoptimization in $\mathbf{x}$ direction has no (first-order) impact on $\frac{df(\mathbf{x}^*(\boldsymbol{\alpha});\boldsymbol{\alpha})}{d\alpha_k}$

(CET) Constrained envelope theorem:

If $(\mathbf{x}^*(\boldsymbol{\alpha}),\boldsymbol{\lambda}^*(\boldsymbol{\alpha}))$ solves $\max_{\mathbf{x}} f(\mathbf{x};\boldsymbol{\alpha})$ s.t. $h^j(\mathbf{x};\boldsymbol{\alpha}) \geq 0$, $j = 1,\ldots m$, and $\mathbf{x}^*(\cdot)$ is differentiable at $\boldsymbol{\alpha}$, then

$$\frac{df(\mathbf{x}^*(\boldsymbol{\alpha});\boldsymbol{\alpha})}{d\alpha_k} = \frac{\partial f(\mathbf{x}^*(\boldsymbol{\alpha});\boldsymbol{\alpha})}{\partial \alpha_k} + \sum_{j=1}^{m} \lambda_j^*(\boldsymbol{\alpha}) \frac{\partial h^j(\mathbf{x}^*(\boldsymbol{\alpha});\boldsymbol{\alpha})}{\partial \alpha_k}$$

For NPP, we wrote $g^j(\mathbf{x}) \leq b^j$ or $b^j - g^j(\mathbf{x}) \geq 0$; now have $h^j(\mathbf{x};\boldsymbol{\alpha}) = \alpha_{0j} - g^j(\mathbf{x};\boldsymbol{\alpha})$.

In words: *total* deriv of f w.r.t. α_k equals *partial* deriv of f w.r.t. α_k plus $\boldsymbol{\lambda}^*$-weighted sum of *partial* derivs of h^j's w.r.t. α_k

- only difference between UET and CET is the addition of constraints
  - again, reoptimization in $\mathbf{x}$ direction has no (first-order) impact on $\frac{df(\mathbf{x}^*(\boldsymbol{\alpha});\boldsymbol{\alpha})}{d\alpha_k}$

Envelope Theorem: unconstrained version

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Assume f is $\mathbb{C}^2$:

if $\mathbf{x}^*(\boldsymbol{\alpha})$ solves $\max_{\mathbf{x}} f(\mathbf{x};\boldsymbol{\alpha})$ and $\mathbf{x}^*(\cdot)$ is diff^{able} at $\boldsymbol{\alpha}$, then $\frac{df(\mathbf{x}^*(\boldsymbol{\alpha});\boldsymbol{\alpha})}{d\alpha_k} = \frac{\partial f(\mathbf{x}^*(\boldsymbol{\alpha});\boldsymbol{\alpha})}{\partial \alpha_k}$.

i.e., *total* derivative of f w.r.t. α_k equals *partial* derivative of f w.r.t. α_k.

(a sufficient condition for $\mathbf{x}(\cdot)$ to be diff^{able} at $\mathbf{x}$ is that it's the unique maximizer.)

$$\frac{df(\mathbf{x}^*(\boldsymbol{\alpha});\boldsymbol{\alpha})}{d\alpha_k} = \frac{\partial f(\mathbf{x}^*(\boldsymbol{\alpha});\boldsymbol{\alpha})}{\partial \alpha_k} + \sum_{i=1}^{n} \underbrace{\frac{\partial f(\mathbf{x}^*(\boldsymbol{\alpha});\boldsymbol{\alpha})}{\partial x_i}}_{=0} \frac{dx_i(\boldsymbol{\alpha})}{d\alpha_k}$$

which, since necessarily, $\nabla_{\mathbf{x}} f(\mathbf{x}^*(\boldsymbol{\alpha});\boldsymbol{\alpha}) = 0$, back to slide 15

$$= \frac{\partial f(\mathbf{x}^*(\boldsymbol{\alpha});\boldsymbol{\alpha})}{\partial \alpha_k}$$

Envelope Theorem: graphical intuition

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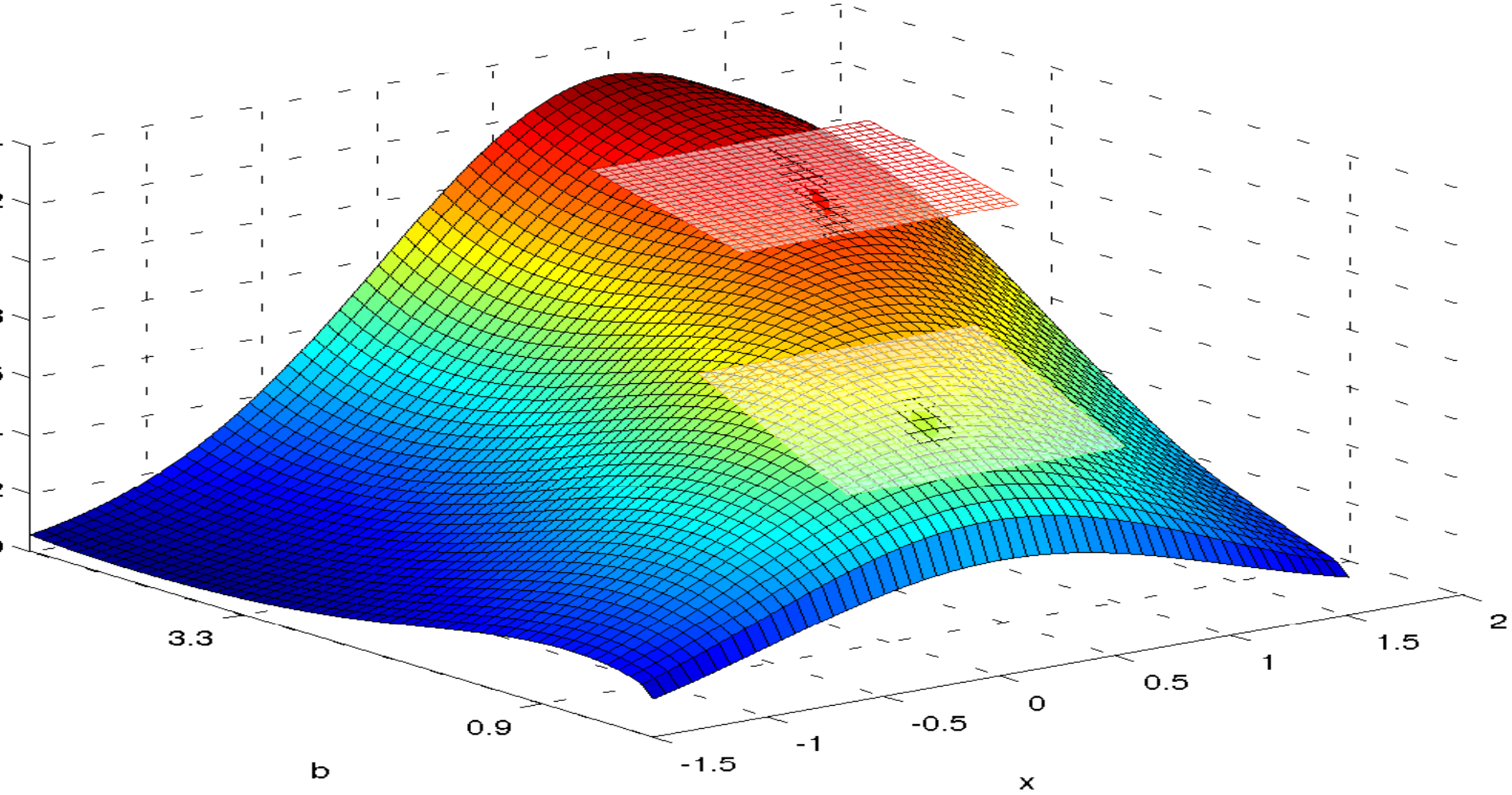
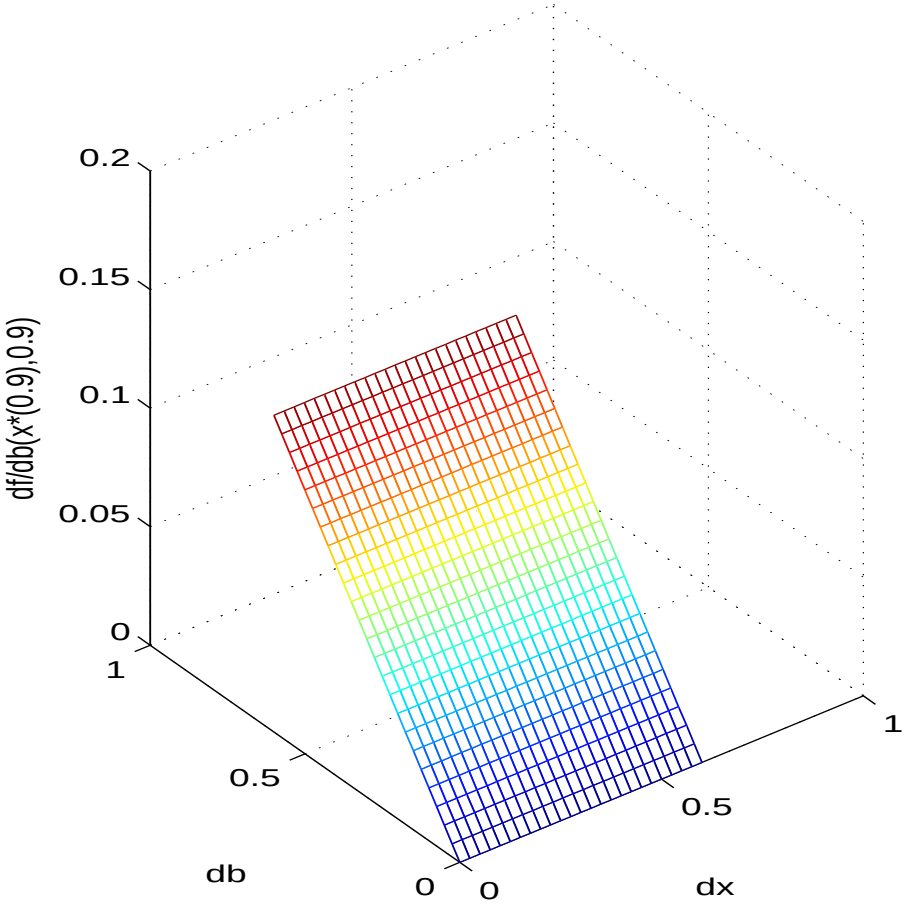
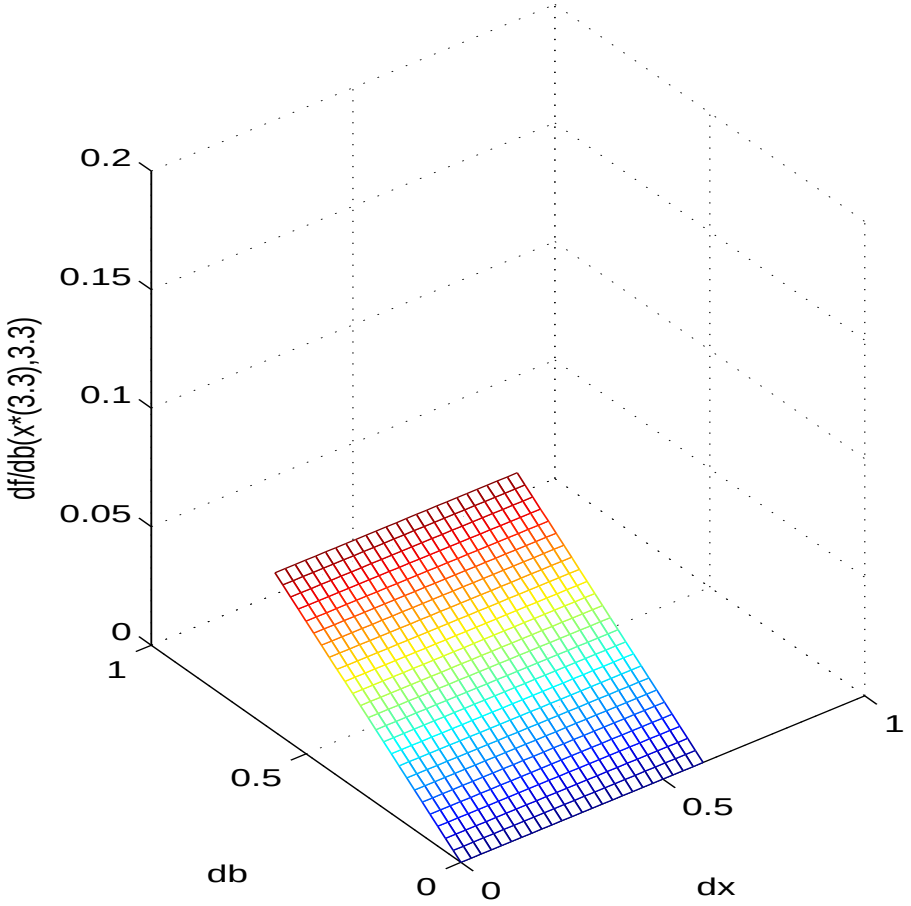


Envelope Theorem: tangent planes





Height of differential determined exclusively by magnitude of *db*

Envelope Theorem: tangent planes (alt view)

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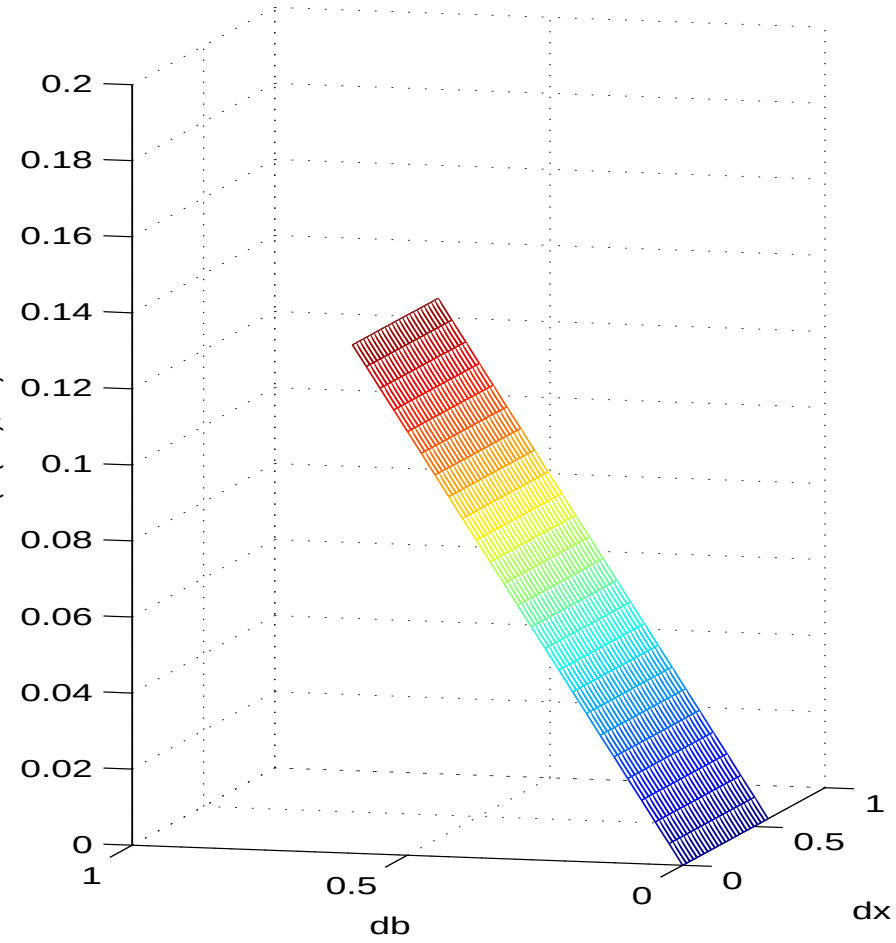
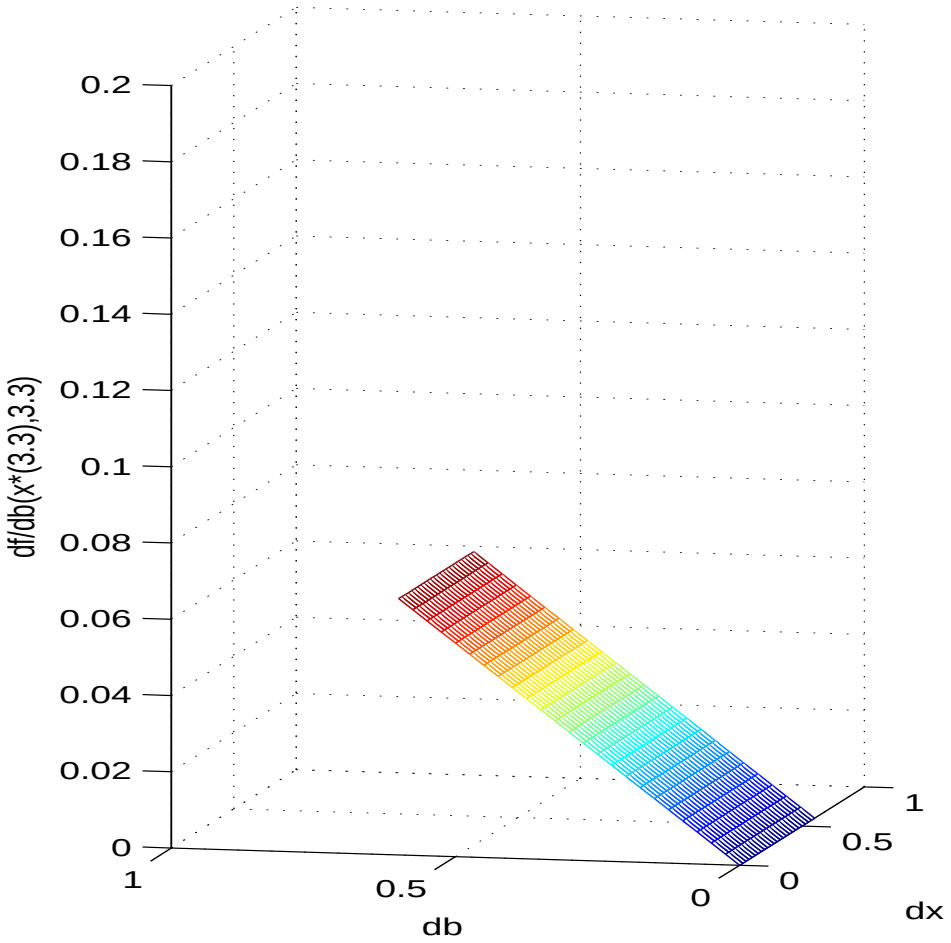
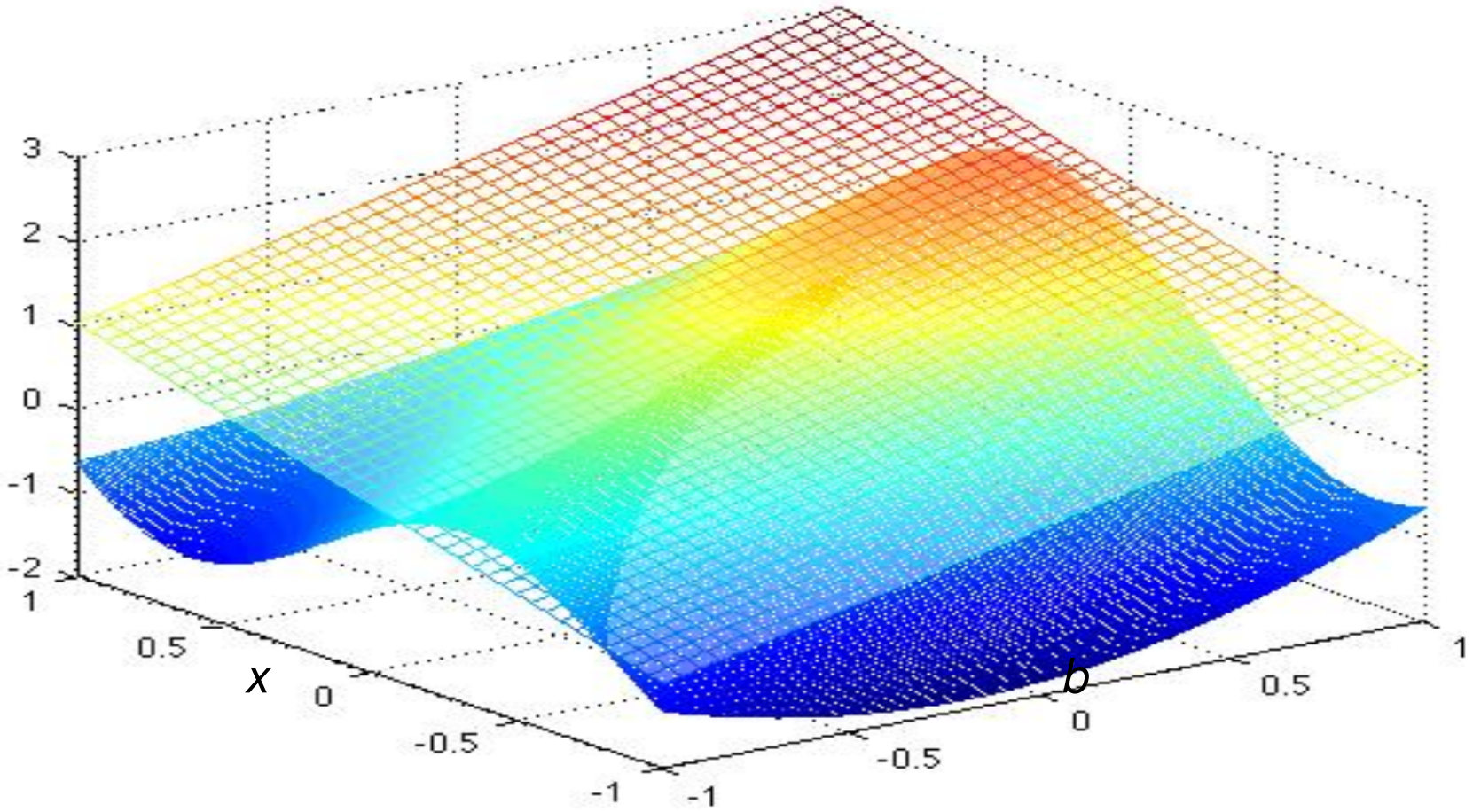




Tangent plane when *x* *not* optimized given *b*



In this case, movements in the *x* direction *do* affect height of differential

Unconstrained Envelope Th'm Eg: Hotelling's Lemma

Hotelling's lemma:

Let $\pi(\mathbf{x}; p, \mathbf{w}) = pf(\mathbf{x}) - \mathbf{w} \cdot \mathbf{x}$, given output price p and input price vector $\mathbf{w}$.

Let $\mathbf{x}^*(p, \mathbf{w})$ be the solution to the unconstrained max, given $p, \mathbf{w}$.

The i'th input demand function is $x_i^*(p, \mathbf{w}) = -\frac{\partial \pi(\mathbf{x}^*(p,\mathbf{w}); p, \mathbf{w})}{\partial w_i}$

Again math is straightforward: by the unconstrained envelope theorem,

$$\frac{d\pi(\mathbf{x}^*(p, \mathbf{w}); p, \mathbf{w})}{dw_i} = \frac{\partial \pi(\mathbf{x}^*(p, \mathbf{w}); p, \mathbf{w})}{\partial w_i} = -x_i^*(p, \mathbf{w})$$

Note that this is a "bonus" result:

- goal here is *not*: how does profit change when w_i changes?
- care about r.h.s. of equation not l.h.s

Why is this result so useful?

- given a functional form for f, get expr for $x_i^*(p, \mathbf{w})$ in terms of primitives

Numerical example of Hotelling's lemma

Recall $\pi(\mathbf{x}; p, \mathbf{w}) = pf(\mathbf{x}) - \mathbf{w} \cdot \mathbf{x}$
Let $\bigtriangledown f = (1/x_1, 1/x_2)$ and $\bigtriangledown_{\mathbf{x}} \pi = (p/x_1 - w_1, p/x_2 - w_2)$

The unconstrained optimum is $\mathbf{x}^*(p, \mathbf{w}) = (p/w_1, p/w_2)$.

Now compare the partial deriv, $\frac{\partial \pi(\mathbf{x}^*(p,\mathbf{w}); p, \mathbf{w})}{\partial w_i}$, to the total deriv, $\frac{d\pi(\mathbf{x}^*(p,\mathbf{w}); p, \mathbf{w})}{dw_i}$,

- ◇ in order to illustrate that the envelope theorem actually works

$$\frac{\partial \pi(\mathbf{x}^*(p,\mathbf{w}); p, \mathbf{w})}{\partial w_i} = -x_i^*(p, \mathbf{w}) = -\frac{p}{w_i} \quad \text{from FOC}$$

$$\frac{d\pi(\mathbf{x}^*(p,\mathbf{w}); p, \mathbf{w})}{dw_i} = \frac{\partial \pi(\mathbf{x}^*(p,\mathbf{w}); p, \mathbf{w})}{\partial w_i} + \sum_{j=1}^{2} \frac{\partial \pi(\mathbf{x}_j^*(p,\mathbf{w}); p, \mathbf{w})}{\partial x_j} \frac{dx_j^*(p,\mathbf{w})}{dw_i}$$

which, since for $j \neq i$, $x_j^*(p, \mathbf{w})$ doesn't depend on w_i

$$\begin{aligned}
&= \frac{\partial \pi(\mathbf{x}^*(p,\mathbf{w}); p, \mathbf{w})}{\partial w_i} + \frac{\partial \pi(\mathbf{x}_i^*(p,\mathbf{w}); p, \mathbf{w})}{\partial x_i} \frac{dx_i^*(p,\mathbf{w})}{dw_i} \\
&= -x_i^*(p, \mathbf{w}) + \left(p/x_i^*(p,\mathbf{w}) - w_i\right) \times -p/w_i^2 \\
&= -x_i^*(p, \mathbf{w}) + \underbrace{\left(p/p/w_i - w_i\right)}_{0} \times -p/w_i^2 = -x_i^*(p, \mathbf{w})
\end{aligned}$$

Hotellings Lemma applied to $f(\mathbf{x}) = \sum_{i=1}^{2} \log x_i$:
Unconstrained input demand: $x_i^*(p, \mathbf{w}) = \frac{p}{w_i}$

Two envelope Theorems: (redux)

(UET) Unconstrained envelope theorem:

if $\mathbf{x}^*(\boldsymbol{\alpha})$ solves $\max_{\mathbf{x}} f(\mathbf{x};\boldsymbol{\alpha})$ and $\mathbf{x}^*(\cdot)$ is diff^{able} at $\boldsymbol{\alpha}$, then $\frac{df(\mathbf{x}^*(\boldsymbol{\alpha});\boldsymbol{\alpha})}{d\alpha_k} = \frac{\partial f(\mathbf{x}^*(\boldsymbol{\alpha});\boldsymbol{\alpha})}{\partial \alpha_k}$.

- *total* derivative of f w.r.t. α_k equals *partial* derivative of f w.r.t. α_k.
  - i.e., reoptimization in $\mathbf{x}$ direction has no (first-order) impact on $\frac{df(\mathbf{x}^*(\boldsymbol{\alpha});\boldsymbol{\alpha})}{d\alpha_k}$

(CET) Constrained envelope theorem:

If $(\mathbf{x}^*(\boldsymbol{\alpha}),\boldsymbol{\lambda}^*(\boldsymbol{\alpha}))$ solves $\max_{\mathbf{x}} f(\mathbf{x};\boldsymbol{\alpha})$ s.t. $h^j(\mathbf{x};\boldsymbol{\alpha}) \geq 0$, $j = 1,\ldots m$, and $\mathbf{x}^*(\cdot)$ is differentiable at $\boldsymbol{\alpha}$, then

$$\frac{df(\mathbf{x}^*(\boldsymbol{\alpha});\boldsymbol{\alpha})}{d\alpha_k} = \frac{\partial f(\mathbf{x}^*(\boldsymbol{\alpha});\boldsymbol{\alpha})}{\partial \alpha_k} + \sum_{j=1}^{m} \lambda_j^*(\boldsymbol{\alpha}) \frac{\partial h^j(\mathbf{x}^*(\boldsymbol{\alpha});\boldsymbol{\alpha})}{\partial \alpha_k}$$

For NPP, we wrote $g^j(\mathbf{x}) \leq b^j$ or $b^j - g^j(\mathbf{x}) \geq 0$; now have $h^j(\mathbf{x};\boldsymbol{\alpha}) = \alpha_{0j} - g^j(\mathbf{x};\boldsymbol{\alpha})$.

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- only difference between UET and CET is the addition of constraints
  - again, reoptimization in $\mathbf{x}$ direction has no (first-order) impact on $\frac{df(\mathbf{x}^*(\boldsymbol{\alpha});\boldsymbol{\alpha})}{d\alpha_k}$

Constrained vs unconstrained envelope theorems

Major intuition deficit for the constrained th'm relative to unconstrained one

- for the unconstrained theorem, we have our intuitive picture
  - $\mathbf{x}$ term doesn't enter expr for $\frac{df(\mathbf{x},\boldsymbol{\alpha})}{d\alpha_j}$ because $\bigtriangledown_{\mathbf{x}} f(\mathbf{x},\boldsymbol{\alpha}) = 0$. go to slide 7
    - differential is flat in the each of the x_i directions go to slide 10
    - to a first order approx, a movement in any x direction doesn't matter
- for the constrained theorem, tangent plane isn't flat in any direction
  - $\bigtriangledown_{\mathbf{x}} f(\mathbf{x},\boldsymbol{\alpha}) \neq 0$.
  - the flat differential property no longer holds
    - i.e., movement in x directions *does* make a first-order difference
  - how can it be that the theorem still works???
- Answer: when the lagrangian λ is set to just the right value...,
  - soln to constrained prob is unconstrained extremum of Lagrangian
  - view the Lagrangian as a function of $(\mathbf{x},\boldsymbol{\alpha})$.
    - necessarily $\bigtriangledown_{\mathbf{x}} L(\mathbf{x},\boldsymbol{\alpha}) = 0$.
  - apply the intuition that we have for the unconstrained theorem.

Convert constrained problem to unconstrained one

Consider the following trivial constrained problem:

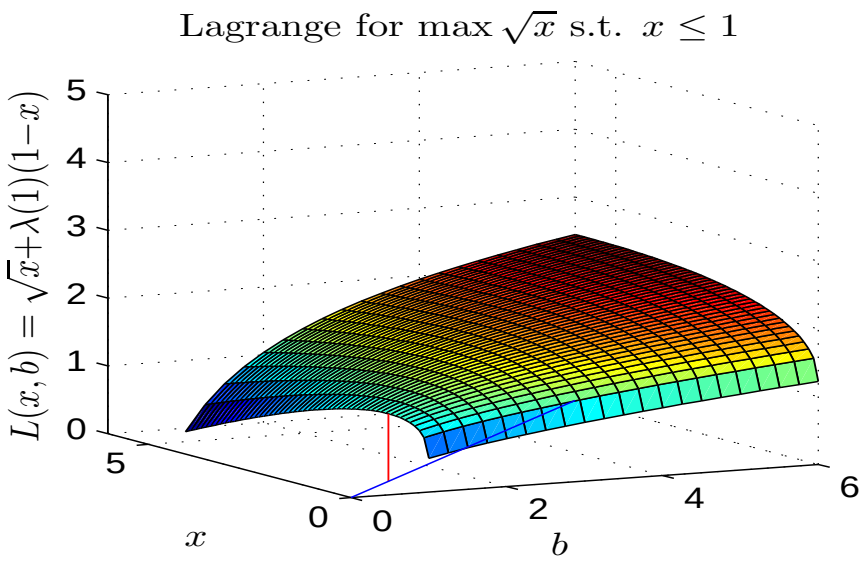
- $\max_{x \geq 0} f(x) = \sqrt{x}$ s.t. $g(x) \leq b$, where $g(x) = x$ and $b \geq 0$.
  - Form the lagrangian: $\mathcal{L}(x, \lambda; b) = \sqrt{x} + \lambda(b)(b - x)$.
  - Trivally, the soln is: $x^*(b) = b$, for all $b \geq 0$.
  - Solve for $\lambda(b)$:

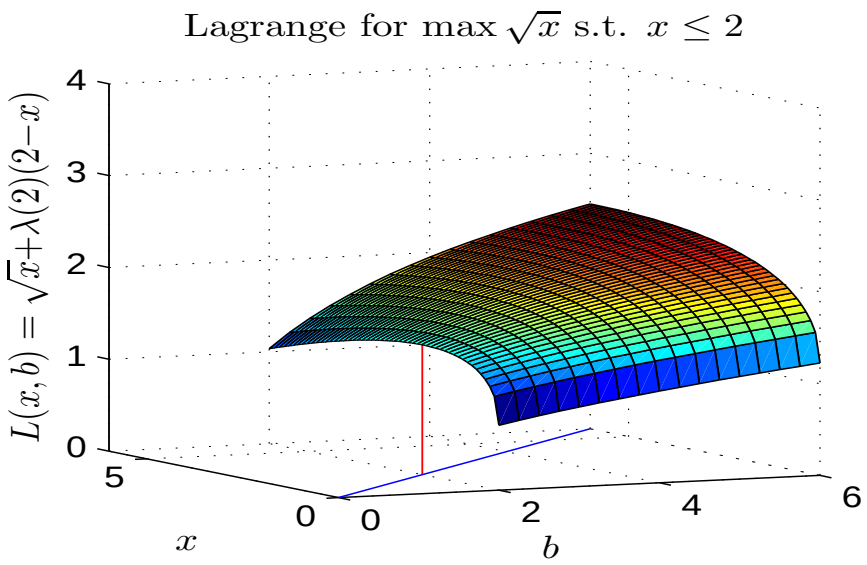
$$\lambda^*(b) = {f'(x^*(b))}/{g'(x^*(b))} = f'(x^*(b)) = {0.5}/{\sqrt{x^*(b)}} = {1}/{2\sqrt{b}}$$

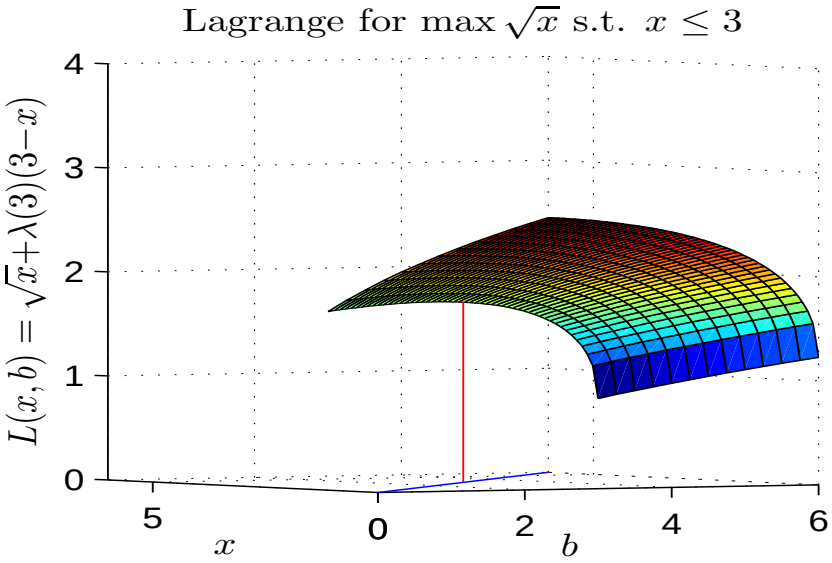
- Now we'll replace λ in Lagrangian with its soln value:
  - $\mathcal{L}(x; b) = \sqrt{x} + \lambda(b - x) = \sqrt{x} + {(b-x)}/{2\sqrt{b}}$.
  - Graph of $\mathcal{L}(x; b)$ looks like graph of $f(x, b)$ in slide #8
    - For each b, slope of $\mathcal{L}(\cdot; b)$ is flat in x direction at $x^*(b) = b$.
      - specifically: $\frac{\partial \mathcal{L}(x^*(b);b)}{\partial x} = \frac{1}{2\sqrt{x}} - \lambda^*(b) = \frac{1}{2\sqrt{b}} - \frac{1}{2\sqrt{b}} = 0$
      - So $\frac{d\mathcal{L}(x^*(b);b)}{db} = \frac{\partial \mathcal{L}(x^*(b);b)}{\partial b} = \lambda^*(b)$.

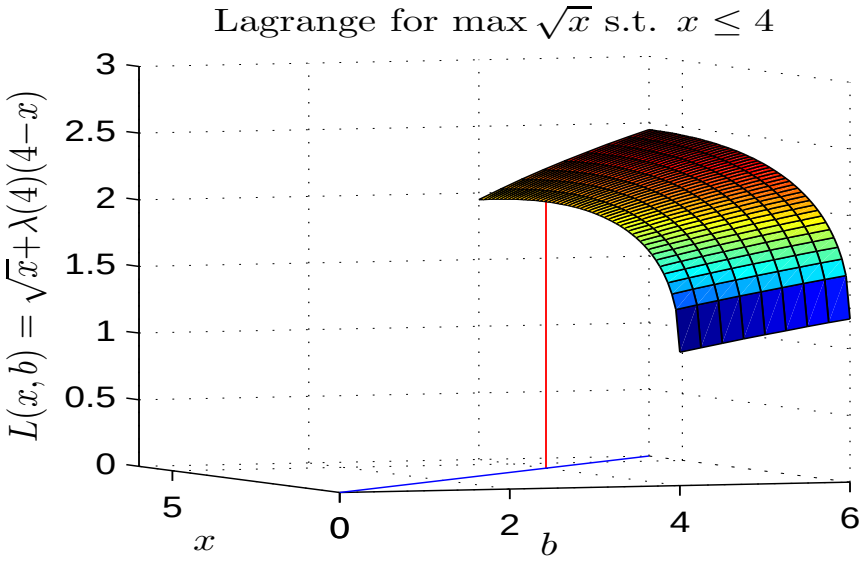
Graph of Lagrange in (x, b) space

go to slide 8



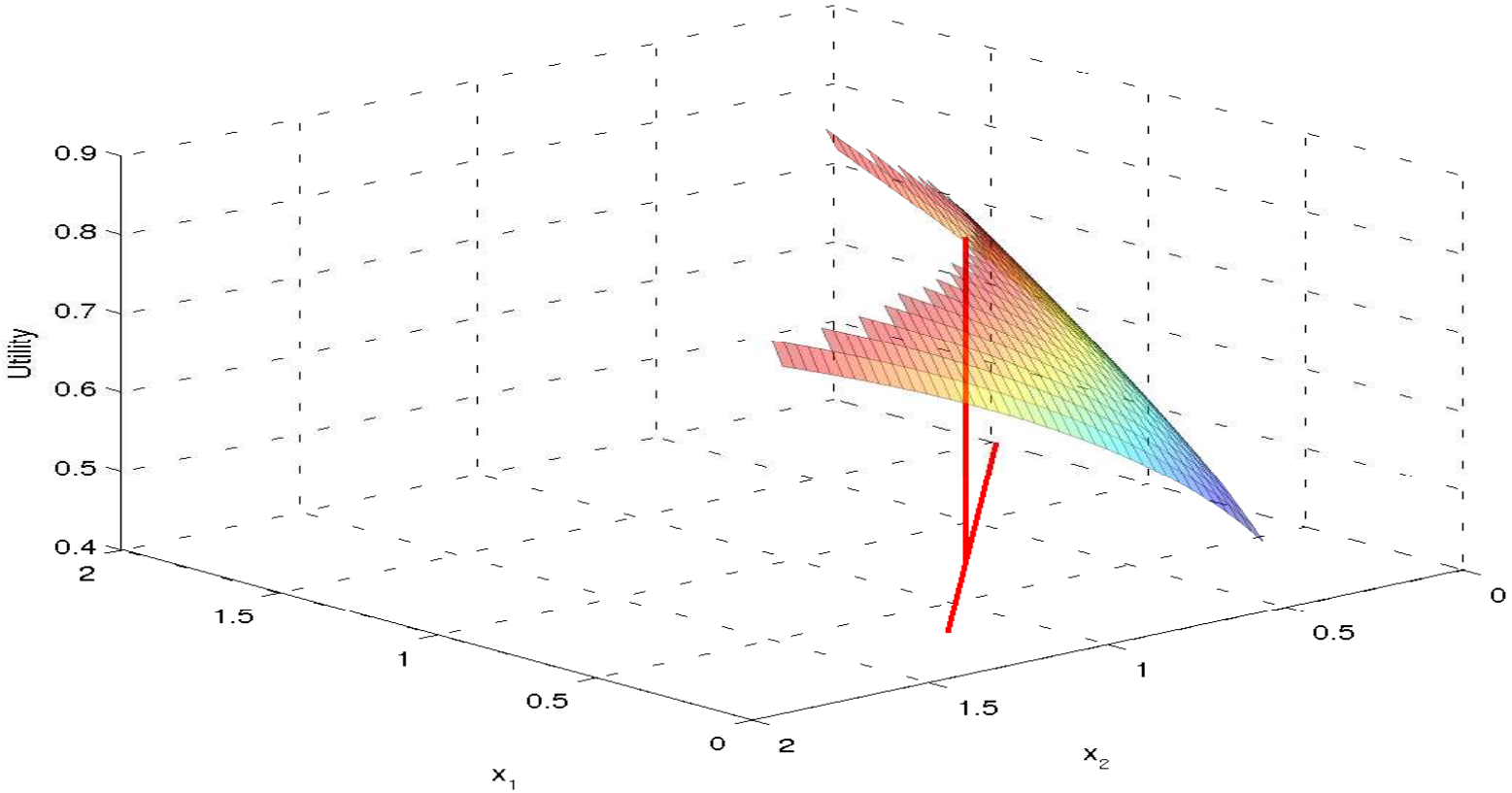






The 2-variable version: the consumer's optim problem

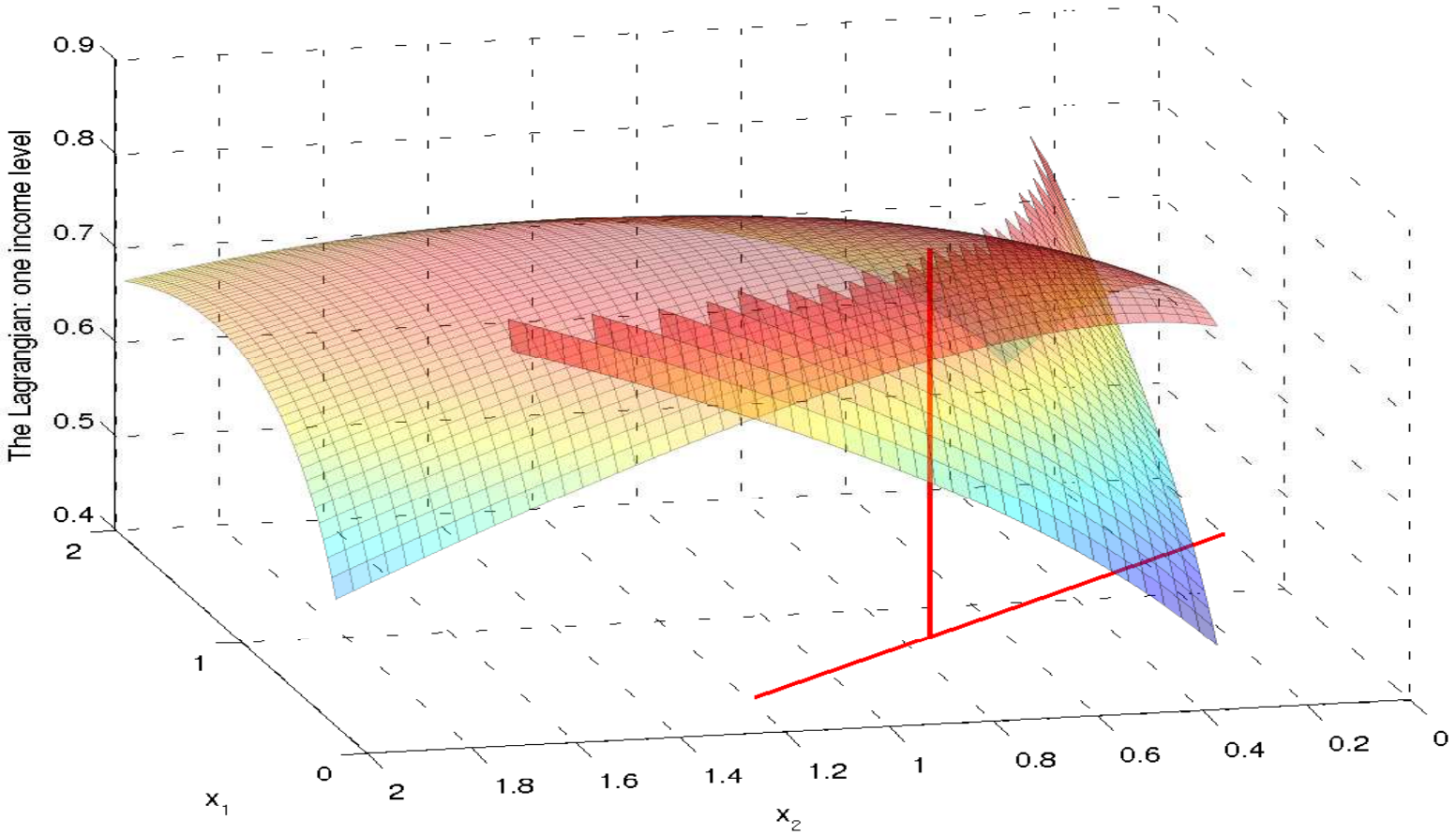
$$\max_{\mathbf{x}} u(\mathbf{x}) \quad = \quad (x_1 x_2)^{0.35} \quad \text{s.t.} \quad y - \mathbf{p} \cdot \mathbf{x} \quad \geq \quad 0$$



The objective function u is *not* flat in either direction x_i.

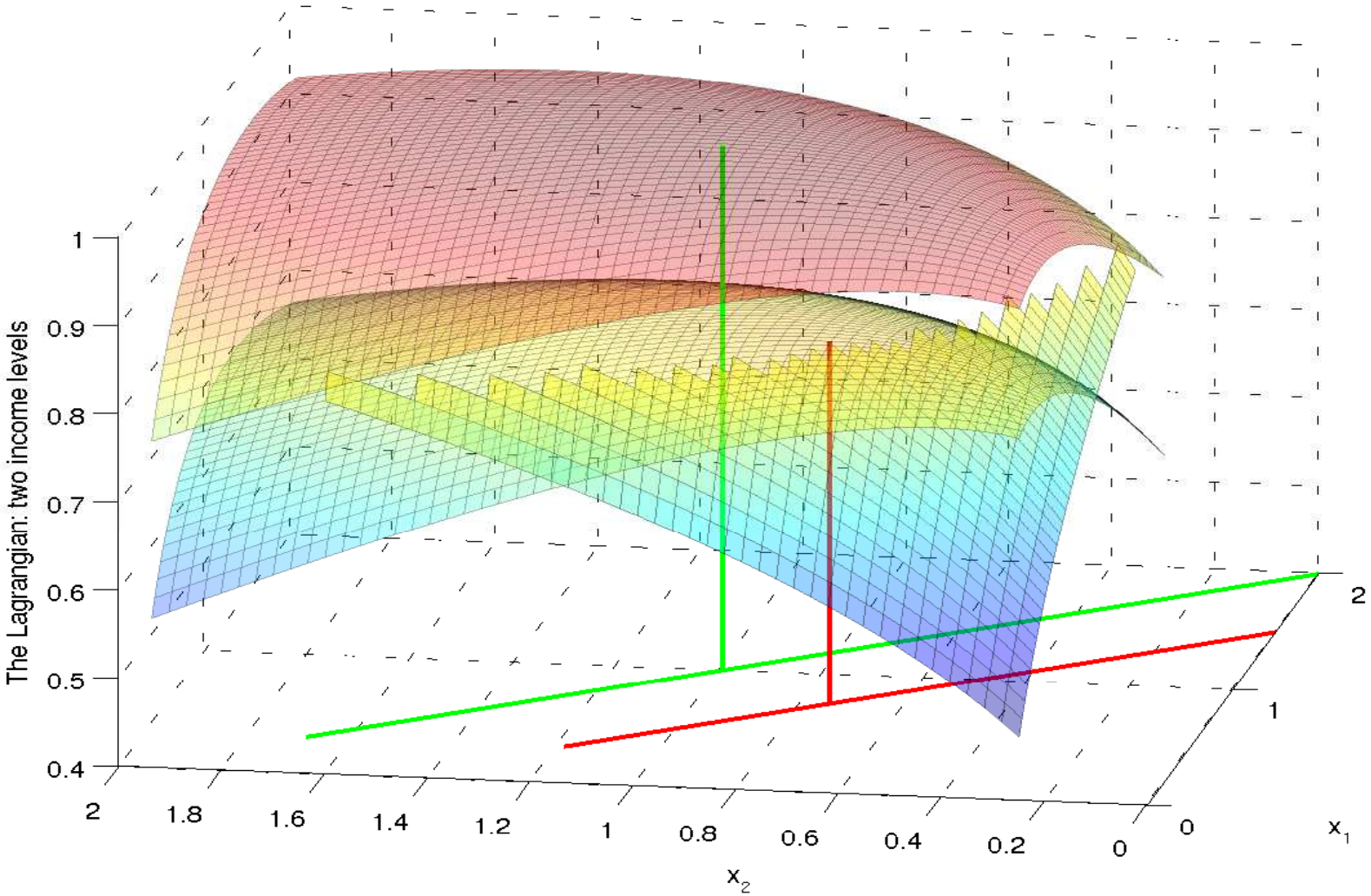
Unconstrained max of $\mathcal{L}$ at constrained max of f

$$\mathcal{L}(\mathbf{x}^*(y), \lambda^*(y); y) \quad = \quad u(\mathbf{x}^*(y)) \quad + \quad \lambda^*(y)\Big(y - \mathbf{p}\cdot\mathbf{x}^*(y)\Big)$$



But the Lagrangian function *is* flat w.r.t. both x_1 and x_2

Small dx leaves $\mathcal{L}$ unchanged (to 1st order approx)



$$\mathcal{L}\Big(\mathbf{x}^*(y),\lambda^*(y);y\Big) \quad \text{vs} \quad \mathcal{L}\Big(\mathbf{x}^*(y+\Delta y),\lambda^*(y+\Delta y);y+\Delta y\Big)$$

First order increase in $\mathcal{L}$ due *only* to Δy (adjustments in $\mathbf{x}$ barely matter)

Envelope Theorem: constrained version (NPP)

RECALL: In the NPP, $\bar{\lambda}^j$ is the "shadow value" of the j'th constraint.

$$M(b) = f(\bar{\mathbf{x}}(b)) = \mathcal{L}(\bar{\mathbf{x}}(b),\bar{\boldsymbol{\lambda}}(b),b) = f(\bar{\mathbf{x}}) + \bar{\boldsymbol{\lambda}}(b - g(\bar{\mathbf{x}})).$$

$$\frac{dM(b)}{db_j} = \frac{d\mathcal{L}(\bar{\mathbf{x}}(b),\bar{\boldsymbol{\lambda}}(b),b)}{db_j} = \sum_{i=1}^{n}\left\{f_i(\bar{\mathbf{x}}(b)) - \sum_{j=1}^{m}\bar{\lambda}_j(b)g_i^j(\bar{\mathbf{x}}(b))\right\}\frac{\partial x_i}{\partial b_j} + \sum_{j=1}^{m}\frac{\partial \lambda_j}{\partial b_j}(b_j - g^j(\bar{\mathbf{x}}(b))) + \bar{\lambda}_j(b)$$

- The $\frac{\partial x_i}{\partial b_j}$ and $\frac{\partial \lambda_j}{\partial b_j}$ terms all disappear: multiplied by zeros
- The only term remaining is $\bar{\lambda}_j(b)$.
- Conclude that $\frac{dM(b)}{db_j} = \bar{\lambda}_j(b) = \frac{\partial \mathcal{L}(\bar{\mathbf{x}}(b),\bar{\boldsymbol{\lambda}}(b),b)}{\partial b_j}$
  - Note that $\mathcal{L}$ depends on b^j in a particularly simple (linear) way
    - so b^j's disappears in the expression for $\frac{\partial M(b)}{\partial b_j}$.
    - moreover, b^j doesn't enter into the objective function f
  - more generally, there's a parameter vector $\boldsymbol{\alpha}$ that enters nonlinearly into $\mathcal{L}$
    - in general, $\boldsymbol{\alpha}$ will be an argument of *both* f and g^j's
    - we'll see, in general, $\frac{dM(\boldsymbol{\alpha})}{d\alpha_k} = \frac{\partial \mathcal{L}(\mathbf{x}^*,\boldsymbol{\lambda}^*;\boldsymbol{\alpha})}{\partial \alpha_k}$ but α_k won't disappear

Envelope Theorem: constrained version (general)

If $(\mathbf{x}^*(\boldsymbol{\alpha}), \boldsymbol{\lambda}^*(\boldsymbol{\alpha}))$ solves $\max_{\mathbf{x}} f(\mathbf{x};\boldsymbol{\alpha})$ s.t. $h^j(\mathbf{x};\boldsymbol{\alpha}) \geq 0$, $j = 1, \ldots m$, and $\mathbf{x}^*(\cdot)$ is differentiable at $\boldsymbol{\alpha}$, then

$$\frac{df(\mathbf{x}^*(\boldsymbol{\alpha});\boldsymbol{\alpha})}{d\alpha_k} = \frac{\partial f(\mathbf{x}^*(\boldsymbol{\alpha});\boldsymbol{\alpha})}{\partial \alpha_k} + \sum_{j=1}^{m} \lambda_j^*(\boldsymbol{\alpha}) \frac{\partial h^j(\mathbf{x}^*(\boldsymbol{\alpha});\boldsymbol{\alpha})}{\partial \alpha_k}$$

For NPP, we wrote $g^j(\mathbf{x}) \leq b^j$ or $b^j - g^j(\mathbf{x}) \geq 0$; now can have $h^j(\mathbf{x};\boldsymbol{\alpha}) = b^j - g^j(\mathbf{x};\boldsymbol{\alpha})$.

In words: *total* deriv of f w.r.t. α_k equals *partial* deriv of f w.r.t. α_k plus $\boldsymbol{\lambda}^*$-weighted sum of *partial* derivs of h^j's w.r.t. α_k (Prev slide a special case.) .

Proof: apply *unconstrained* Env thm to (generalized) Lagrangian;

$$\begin{aligned}
\mathcal{L}(\mathbf{x},\boldsymbol{\lambda};\boldsymbol{\alpha}) &= f(\mathbf{x};\boldsymbol{\alpha}) + \sum_{j=1}^{m} \lambda_j h^j(\mathbf{x};\boldsymbol{\alpha}) \\
\mathcal{L}(\mathbf{x}^*,\boldsymbol{\lambda}^*;\boldsymbol{\alpha}) &\equiv f(\mathbf{x}^*,\boldsymbol{\alpha}) \\
\frac{df(\mathbf{x}^*,\boldsymbol{\alpha})}{d\alpha_k} &\equiv \underbrace{\frac{d\mathcal{L}(\mathbf{x}^*,\boldsymbol{\lambda}^*;\boldsymbol{\alpha})}{d\alpha_k} = \frac{\partial \mathcal{L}(\mathbf{x}^*,\boldsymbol{\lambda}^*;\boldsymbol{\alpha})}{\partial \alpha_k}}_{\text{by the unconstrained envelope theorem}} \\
&= \frac{\partial f(\boldsymbol{\alpha},\mathbf{x}^*(\boldsymbol{\alpha}))}{\partial \alpha_k} + \sum_{j=1}^{m} \lambda_j^*(\boldsymbol{\alpha}) \frac{\partial h^j(\boldsymbol{\alpha},\mathbf{x}^*(\boldsymbol{\alpha}))}{\partial \alpha_k}
\end{aligned}$$

Constrained Envelope Theorem Example

E.g. of constrained envelope theorem (Shephard's lemma):
Let $\hat{c}(\mathbf{w}, \bar{q}) = \mathbf{w} \cdot \hat{\mathbf{x}}(\mathbf{w}, \bar{q})$ be the minimized level of costs given input prices $\mathbf{w}$ and output $\bar{q}$ (i.e., $\hat{\mathbf{x}}(\mathbf{w}, \bar{q})$ is optimal input mix given params). Then

The i'th conditional input demand function is $\hat{x}_i(\cdot) = \frac{\partial \hat{c}(\cdot,\cdot)}{\partial w_i}$.

Again, *not* asking: how does minimized cost change when w_i changes.

Proof:

$$\min_{\mathbf{x}} \ \hat{c}(\mathbf{x}; \mathbf{w}, \bar{q}) \qquad s.t. \qquad \bar{q} - f(\mathbf{x}) = 0$$

$$\max_{\mathbf{x}} \ -\hat{c}(\mathbf{x}; \mathbf{w}, \bar{q}) \qquad s.t. \qquad \bar{q} - f(\mathbf{x}) = 0$$

$$\mathcal{L}(\mathbf{x}, \lambda; \mathbf{w}, \bar{q}) = -\mathbf{w} \cdot \mathbf{x} + \lambda(\bar{q} - f(\mathbf{x}))$$

$$\mathcal{L}(\hat{\mathbf{x}}, \lambda^*; \mathbf{w}, \bar{q}) \equiv -\mathbf{w} \cdot \hat{\mathbf{x}} \quad \text{which we define to be} \quad -\hat{c}(\hat{\mathbf{x}}; \mathbf{w}, \bar{q})$$

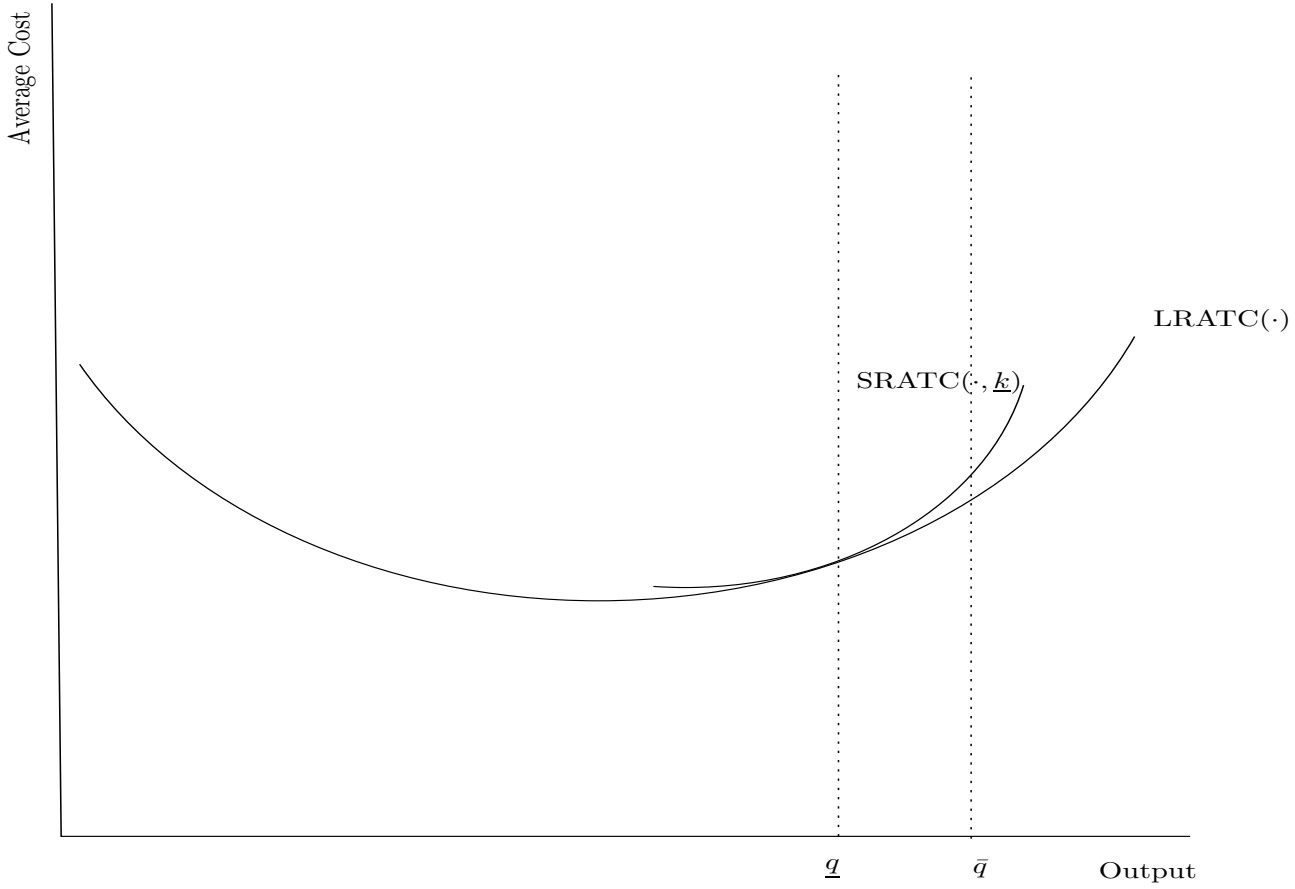
$$\frac{d\mathcal{L}(\hat{\mathbf{x}}, \lambda^*; \mathbf{w}, \bar{q})}{dw_i} = \frac{\partial \mathcal{L}(\hat{\mathbf{x}}, \lambda^*; \mathbf{w}, \bar{q})}{\partial w_i} = -\frac{\partial \hat{c}(\hat{\mathbf{x}}; \mathbf{w}, \bar{q})}{\partial w_i} = -\hat{x}_i(\mathbf{w})$$

Given a parameterized prodn function (see notes for CES),

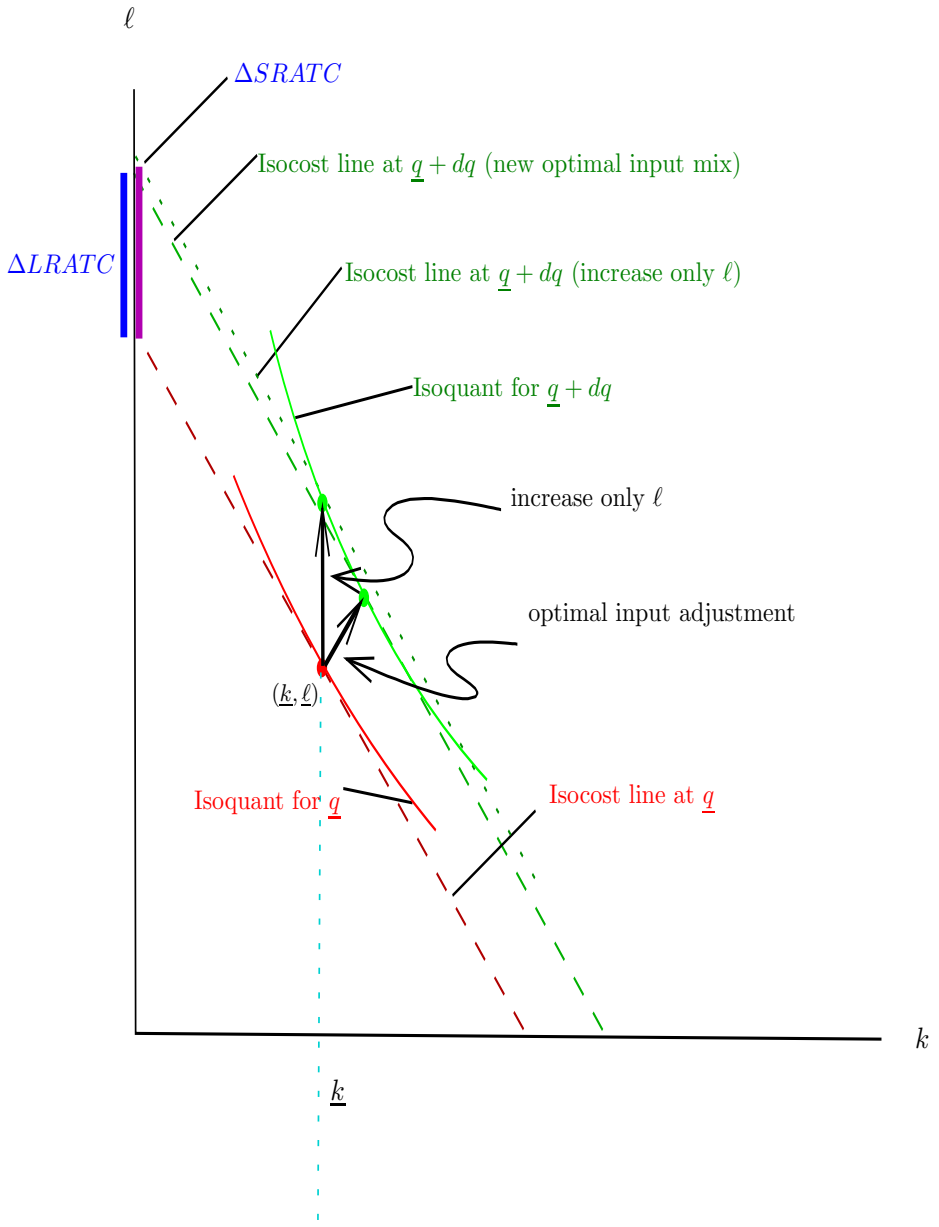
- ◇ env thm delivers an expression for $\hat{x}_i(\mathbf{w})$ in terms of primitives

The mother of all envelope theorems

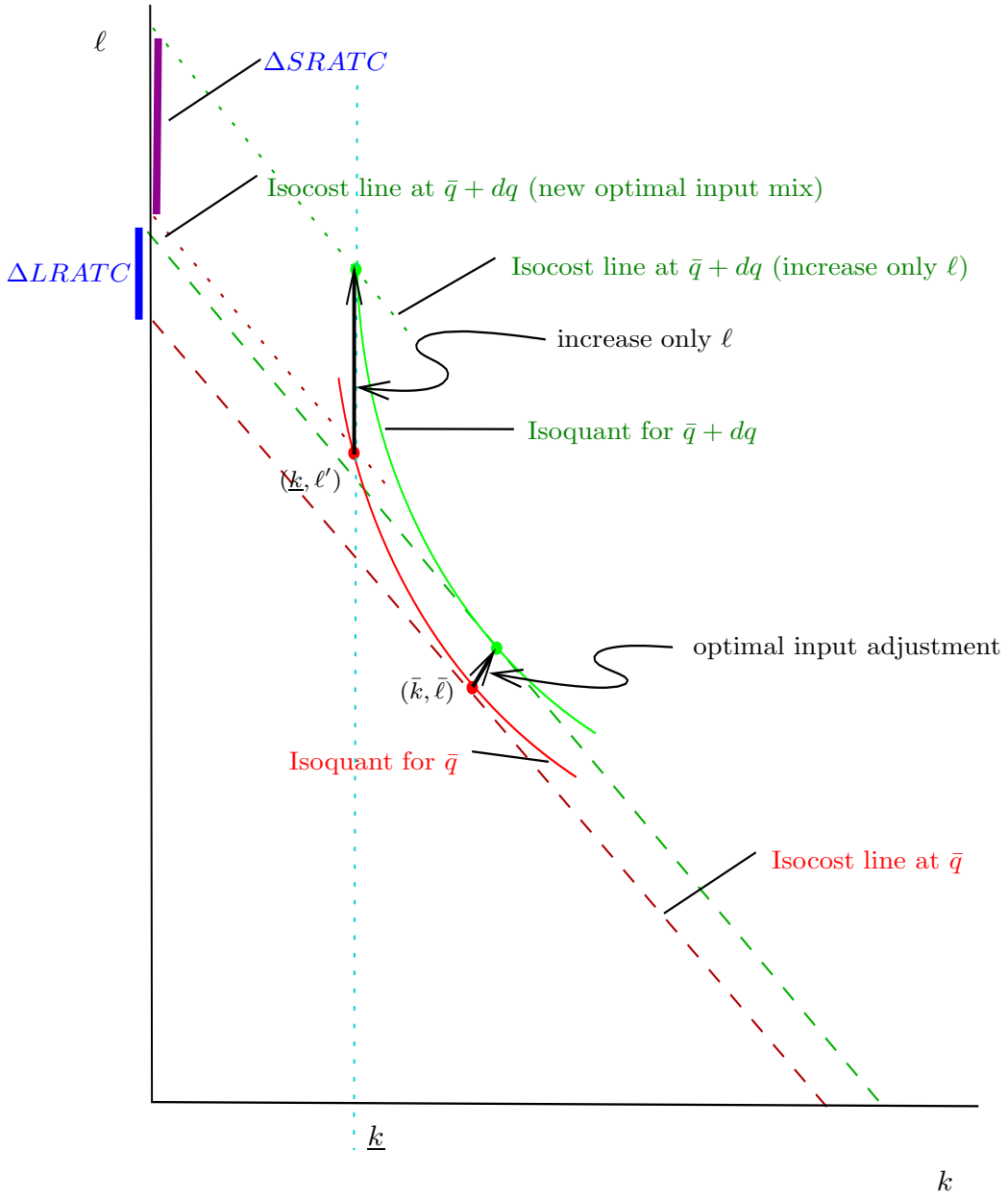
The LRATC curve (all inputs variable) is the *outer envelope* of the SRATC curves (some inputs fixed); when fixed inputs are at their optimal levels, the slopes of LRATC and SRATC curves are equal.



Δ Cost moving from $\underline{q}$ to $\underline{q} + dq$: ($\underline{k}$ is optimal for $\underline{q}$)



Δ Cost moving from $\bar{q}$ to $\bar{q} + dq$: ($\underline{k}$ is suboptimal for $\bar{q}$)



Minimizing LR and SR average total cost: the math

$q = f(\ell, k)$. Given q, $\ell(k;q)$ is determined: $(k, \ell(k;q))$ must be on q-isoquant

The long-run problem: $\quad \mathrm{ac}^{\mathrm{LR}}(k;\bar{q}) = \frac{w\ell(k;\bar{q})+rk}{\bar{q}}$

Unconstrained optimization: $\quad \max_k \; -\mathrm{ac}^{\mathrm{LR}}(k;\bar{q})$

By envelope theorem:

$$\frac{d\mathrm{ac}^{\mathrm{LR}}(k^{\mathrm{LR}}(\bar{q});\bar{q})}{dq} = \frac{\partial \mathrm{ac}^{\mathrm{LR}}(k^{\mathrm{LR}}(\bar{q});\bar{q})}{\partial q}$$

$$= \frac{-qw\frac{\partial \ell}{\partial q} + \left(w\ell(k^{\mathrm{LR}}(\bar{q});\bar{q}) + rk^{\mathrm{LR}}(\bar{q})\right)}{\bar{q}^2}$$

The short-run problem: $\quad \mathrm{ac}^{\mathrm{SR}}(k;\bar{q},\bar{k}) = \frac{w\ell(k;q)+r\bar{k}}{\bar{q}}, \quad k \le \bar{k}$

$$\mathcal{L}(k;\bar{q},\bar{k}) = -\frac{w\ell(k;\bar{q})+r\bar{k}}{\bar{q}} + \lambda(\bar{k}-k)$$

By envelope theorem:

$$\frac{d\mathrm{ac}^{\mathrm{SR}}(k^{\mathrm{SR}}(\bar{q});\bar{q})}{dq} = \frac{\partial \mathrm{ac}^{\mathrm{SR}}(k^{\mathrm{SR}}(\bar{q});\bar{q})}{\partial q} = \frac{d\mathcal{L}}{dq}$$

$$= \frac{-qw\frac{\partial \ell}{\partial q} + \left(w\ell(k^{\mathrm{SR}}(\bar{q});\bar{q}) + r\bar{k}\right)}{\bar{q}^2} + \lambda\frac{\partial k}{\partial q}$$

When $\bar{k} = k^{\mathrm{LR}}(\bar{q})$: $\quad \lambda = 0.$

$$\frac{d\mathrm{ac}^{\mathrm{SR}}(k^{\mathrm{SR}}(\bar{q});\bar{q})}{dq} = \frac{-qw\frac{\partial \ell}{\partial q} + \left(w\ell(k^{\mathrm{LR}}(\bar{q});\bar{q}) + rk^{\mathrm{LR}}(\bar{q})\right)}{\bar{q}^2} = \frac{d\mathrm{ac}^{\mathrm{LR}}(k^{\mathrm{LR}}(\bar{q});\bar{q})}{dq}$$