Tremendously important application of eigenvalues and eigenvectors relates to difference equations.

- Consider a system of linear homogeneous first-order difference equations of the form \( x_t = Ax_{t-1} \).
- Think of this as an infinite system of equations, one for each time period.
- Each initial vector generates a sequence of vectors \( \{x_0, x_1, \ldots \} \) s.t. \( t > 0, x_t = Ax_{t-1} \): any such sequence is called a solution to the system.
- What are the properties of these solutions: do they converge to some specific vector?
- A steady state/equilibrium/stationary solution to an equation system is a vector \( x \) satisfying \( x = Ax \).
- Familiar result is that given a system of difference equations of the form \( x_t = Ax_{t-1} \) such that the eigenvalues of \( A \) are all distinct, the system has a unique steady state at \( 0 \) iff the
largest (in absolute value) eigenvalue of $A$ is less than unity. Trace out what happens: start with a circle, then ask what happens to the first ellipse, etc.

- Ignore the issue of unit eigenvalues.
- On the other hand, note that if one of the eigenvalues is greater than one in absolute value, then every starting vector that is not an eigenvector, eventually ends up pointing in a direction close to the direction of the eigenvector corresponding to the largest eigenvalue.
- Note the difference between behavior of difference equation systems depending on whether the matrix is positive or negative definite: monotonic behavior in the former case, big oscillations in the second case.
- Now in the case of difference equations we typically have to deal with asymmetric matrices, which give rise to much more interesting dynamics. Take the twisting matrix above, changing it a little so its eigenvalues are less than unity $F = \begin{bmatrix} \frac{4}{5} & \frac{1}{5} \\ \frac{-1}{5} & \frac{4}{5} \end{bmatrix}$. Note that the determinant of this matrix is less than one, so that the circle will spiral inwards: draw what happens to any starting point on the unit circle.
- Now consider a linear nonhomogeneous difference equations of the form $x_t = Ax_{t-1} + b$. Much richer dynamics. First solve for a steady state: $x^* = Ax^* + b$, ie., $x^* = (I - A)^{-1}b$, provided that the matrix $(I - A)$ is nonsingular. Now transform the nonhomogeneous system into a homogeneous one by subtracting the steady state equation from the original to obtain $(x_t - x^*) = A(x_{t-1} - x^*)$ which is an homogeneous system.
- Corresponding result for nonhomogeneous difference equations of the form $x_t = Ax_{t-1} + b$: if all of the eigenvalues of $A$ are distinct, and $(I - A)$ has full rank, then the system has a unique steady state at $(I - A)^{-1}b$, iff the largest (in absolute value) eigenvalue of $A$ is less than unity.