7. Implicit Function Theorem and the Envelope Theorem (cont) 1

7.5. Inverse function theorem

The inverse function theorem is a special case of the implicit function theorem. It applies to the case where \( n = m \), i.e., same number of \( \alpha \)'s as \( x \)'s and \( f(\alpha; x) = \alpha - \eta(x) \), i.e., for each \( i \), \( f^i = \alpha_i - \eta^i(x) \) or \( \alpha_i = \eta^i(x) \).

**Theorem:** Given \( \eta : \mathbb{R}^m \to \mathbb{R}^m \) and \( \bar{\alpha} \in \mathbb{R}^m \), If the determinant of \( J\eta(\bar{\alpha}) \) is not zero, then there exists a neighborhood of \( \bar{\alpha} \) and a differentiable function \( \eta^{-1} : \mathbb{R}^m \to \mathbb{R}^m \) such that on this neighborhood

\[
\eta^{-1}(x) = x
\]

and

\[
\begin{bmatrix}
\frac{\partial(\eta^{-1})^1(x)}{\partial x_1} & \frac{\partial(\eta^{-1})^1(x)}{\partial x_2} & \cdots & \frac{\partial(\eta^{-1})^1(x)}{\partial x_m} \\
\frac{\partial(\eta^{-1})^2(x)}{\partial x_1} & \frac{\partial(\eta^{-1})^2(x)}{\partial x_2} & \cdots & \frac{\partial(\eta^{-1})^2(x)}{\partial x_m} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\partial(\eta^{-1})^m(x)}{\partial x_1} & \frac{\partial(\eta^{-1})^m(x)}{\partial x_2} & \cdots & \frac{\partial(\eta^{-1})^m(x)}{\partial x_m}
\end{bmatrix} = (J\eta(\bar{\alpha}))^{-1}
\]
We’ll do it intuitively first, in one dimension. Suppose you have a function \( \eta : \mathbb{R}^1 \to \mathbb{R}^1 \) that is invertible, \( (x = \eta(\alpha)) \); i.e., every point in the range is associated with a unique point in the domain.

In this case, you can certainly write \( \alpha \) as a function of \( x \); that is, define the function \( \eta^{-1} : \mathbb{R}^1 \to \mathbb{R}^1 \) such that \( \eta^{-1}(x) \) picks out the \( \alpha \) value that \( \eta \) took to \( x \); mathematically, \( \eta^{-1} \) is defined by the condition that \( \eta^{-1}(\eta(\alpha)) = \alpha \)

- The closed form of the inverse may be very hard to compute, e.g., suppose \( x = \exp^{\alpha} \times \sqrt{\sin(\alpha)}/\cos(\alpha) \).
  
  Could in principle define \( \alpha \) as a function of \( x \), but it could get messy.
- Easier to use the inverse function theorem, which says that \( \partial \eta^{-1}(\cdot) / \partial x = (\partial \eta(\cdot) / \partial \alpha)^{-1} \).
Example: \( x = \eta(\alpha) = 1/\alpha \), so that \( \partial \eta(\cdot) / \partial \alpha = -1/\alpha^2 \);

- first we'll take the derivative of the inverse by hand: we have
  - \( \alpha = \eta^{-1}(x) = 1/x \),
  - \( \partial \eta^{-1}(\cdot) / \partial x = -1/x^2 \);
- now use the inverse function theorem:
  - \( \partial \eta(\cdot) / \partial \alpha = -1/\alpha^2 \);
  - applying the inverse function theorem \( \partial \eta^{-1}(\cdot) / \partial x = (\partial \eta(\cdot) / \partial \alpha)^{-1} = (-1/\alpha^2)^{-1} = -\alpha^2 \);
  - substitute \( x \) for \( \alpha \) to obtain \( \partial \eta^{-1}(\cdot) / \partial x = -1/x^2 \);

Now return to the formalism of the implicit function theorem. When \( m = n \), the expression from the last lecture becomes:

\[
\begin{bmatrix}
\frac{\partial g^1(\alpha)}{\partial \alpha_1} & \frac{\partial g^1(\alpha)}{\partial \alpha_2} & \cdots & \frac{\partial g^1(\alpha)}{\partial \alpha_n} \\
\frac{\partial g^2(\alpha)}{\partial \alpha_1} & \frac{\partial g^2(\alpha)}{\partial \alpha_2} & \cdots & \frac{\partial g^2(\alpha)}{\partial \alpha_n} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\partial g^n(\alpha)}{\partial \alpha_1} & \frac{\partial g^n(\alpha)}{\partial \alpha_2} & \cdots & \frac{\partial g^n(\alpha)}{\partial \alpha_n}
\end{bmatrix} = - \Gamma(\alpha, x)^{-1} = \begin{bmatrix}
\frac{\partial f^1(g(\alpha))}{\partial \alpha_1} & \frac{\partial f^1(g(\alpha))}{\partial \alpha_2} & \cdots & \frac{\partial f^1(g(\alpha))}{\partial \alpha_n} \\
\frac{\partial f^2(g(\alpha))}{\partial \alpha_1} & \frac{\partial f^2(g(\alpha))}{\partial \alpha_2} & \cdots & \frac{\partial f^2(g(\alpha))}{\partial \alpha_n} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\partial f^n(g(\alpha))}{\partial \alpha_1} & \frac{\partial f^n(g(\alpha))}{\partial \alpha_2} & \cdots & \frac{\partial f^n(g(\alpha))}{\partial \alpha_n}
\end{bmatrix}
\]

where

\[
\Gamma(\alpha, x) = J^\alpha \eta(\alpha, x) = \begin{bmatrix}
\frac{\partial f^1(\alpha, x)}{\partial x_1} & \frac{\partial f^1(\alpha, x)}{\partial x_2} & \cdots & \frac{\partial f^1(\alpha, x)}{\partial x_n} \\
\frac{\partial f^2(\alpha, x)}{\partial x_1} & \frac{\partial f^2(\alpha, x)}{\partial x_2} & \cdots & \frac{\partial f^2(\alpha, x)}{\partial x_n} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\partial f^n(\alpha, x)}{\partial x_1} & \frac{\partial f^n(\alpha, x)}{\partial x_2} & \cdots & \frac{\partial f^n(\alpha, x)}{\partial x_n}
\end{bmatrix}
\]

When \( m = n \), we can exchange \( \alpha \) and \( x \) as follows:

\[
\begin{bmatrix}
\frac{\partial s^1(x)}{\partial x_1} & \frac{\partial s^1(x)}{\partial x_2} & \cdots & \frac{\partial s^1(x)}{\partial x_n} \\
\frac{\partial s^2(x)}{\partial x_1} & \frac{\partial s^2(x)}{\partial x_2} & \cdots & \frac{\partial s^2(x)}{\partial x_n} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\partial s^n(x)}{\partial x_1} & \frac{\partial s^n(x)}{\partial x_2} & \cdots & \frac{\partial s^n(x)}{\partial x_n}
\end{bmatrix} = - \Gamma(x, \alpha)^{-1} = \begin{bmatrix}
\frac{\partial f^1(x, g(x))}{\partial x_1} & \frac{\partial f^1(x, g(x))}{\partial x_2} & \cdots & \frac{\partial f^1(x, g(x))}{\partial x_n} \\
\frac{\partial f^2(x, g(x))}{\partial x_1} & \frac{\partial f^2(x, g(x))}{\partial x_2} & \cdots & \frac{\partial f^2(x, g(x))}{\partial x_n} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\partial f^n(x, g(x))}{\partial x_1} & \frac{\partial f^n(x, g(x))}{\partial x_2} & \cdots & \frac{\partial f^n(x, g(x))}{\partial x_n}
\end{bmatrix}
\]
where

$$-\Gamma(x, \alpha) = -J^o \eta(x, \alpha) = -\begin{bmatrix} \frac{\partial f^1(x, \alpha)}{\partial \alpha_1} & \frac{\partial f^1(x, \alpha)}{\partial \alpha_2} & \ldots & \frac{\partial f^1(x, \alpha)}{\partial \alpha_n} \\ \frac{\partial f^2(x, \alpha)}{\partial \alpha_1} & \frac{\partial f^2(x, \alpha)}{\partial \alpha_2} & \ldots & \frac{\partial f^2(x, \alpha)}{\partial \alpha_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f^n(x, \alpha)}{\partial \alpha_1} & \frac{\partial f^n(x, \alpha)}{\partial \alpha_2} & \ldots & \frac{\partial f^n(x, \alpha)}{\partial \alpha_n} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\partial \eta^1(x, \alpha)}{\partial \alpha_1} & \frac{\partial \eta^1(x, \alpha)}{\partial \alpha_2} & \ldots & \frac{\partial \eta^1(x, \alpha)}{\partial \alpha_n} \\ \frac{\partial \eta^2(x, \alpha)}{\partial \alpha_1} & \frac{\partial \eta^2(x, \alpha)}{\partial \alpha_2} & \ldots & \frac{\partial \eta^2(x, \alpha)}{\partial \alpha_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial \eta^n(x, \alpha)}{\partial \alpha_1} & \frac{\partial \eta^n(x, \alpha)}{\partial \alpha_2} & \ldots & \frac{\partial \eta^n(x, \alpha)}{\partial \alpha_n} \end{bmatrix}$$

Notice how the two minus signs cancel each other out. Notice also that all we have done here is to switch the endog and the exog variables.

We’re no longer going to worry about whether the function is globally invertible, and just talk about when it’s locally invertible, i.e., when i.e., $\Gamma(x, \alpha)$ defined above is an invertible matrix. In the special case where $f(x; \alpha) = x - \eta(\alpha)$, the second matrix in equation (??) becomes the identity matrix and we have, locally, $\alpha = g(x)$, where

$$\begin{bmatrix} \frac{\partial g^1(x)}{\partial x_1} & \frac{\partial g^1(x)}{\partial x_2} & \ldots & \frac{\partial g^1(x)}{\partial x_n} \\ \frac{\partial g^2(x)}{\partial x_1} & \frac{\partial g^2(x)}{\partial x_2} & \ldots & \frac{\partial g^2(x)}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial g^n(x)}{\partial x_1} & \frac{\partial g^n(x)}{\partial x_2} & \ldots & \frac{\partial g^n(x)}{\partial x_n} \end{bmatrix} = -\Gamma(x, \alpha)^{-1} = \left(J^o \eta(x, \alpha)\right)^{-1} \quad (1)$$

When $m = n = 1$, this reduces, of course, to $\alpha = (\eta'(x))^{-1}$.

Here’s an economic example of how the inverse and implicit function theorems might by combined: consider a joint production function $\zeta : x \rightarrow q$, where inputs $x$ and outputs $q$ are elements of $\mathbb{R}^n$. First let’s construct the cost function $C(q) = w \cdot x(q)$, where $x$ is chosen to minimize costs given input prices $w$. That is, $x(q)$ solves the NPP

$$\min_x \ w \cdot x \quad \text{s.t.} \quad q = \zeta(x)$$
To solve this, set up the Lagrangian:

\[ L(q, w; x, \lambda) = -w \cdot x + \sum_{i=1}^{n} \lambda^i (q^i - \zeta^i(x)) \]

The first order conditions for \( L \) give us \( x(q) \). Moreover, \( C(q) = L(x(q), \lambda; w) \). Finally, the marginal cost function is given by \( MC(q) = \left[ \frac{\partial C(q)}{\partial q_1}, \ldots, \frac{\partial C(q)}{\partial q_n} \right] \), and

\[
\frac{\partial C(q)}{\partial q_i} = \sum_{j=1}^{n} w_j \frac{\partial x_j(q)}{\partial q_i}
\]

where, applying the implicit function theorem

\[
\left[ \frac{\partial x_1(q)}{\partial q_1} \quad \frac{\partial x_1(q)}{\partial q_2} \quad \cdots \quad \frac{\partial x_1(q)}{\partial q_n} \\
\frac{\partial x_2(q)}{\partial q_1} \quad \frac{\partial x_2(q)}{\partial q_2} \quad \cdots \quad \frac{\partial x_2(q)}{\partial q_n} \\
\vdots \quad \vdots \quad \cdots \quad \vdots \\
\frac{\partial x_n(q)}{\partial q_1} \quad \frac{\partial x_n(q)}{\partial q_2} \quad \cdots \quad \frac{\partial x_n(q)}{\partial q_n} \right] \cdot \left[ \begin{array}{c} \frac{\partial L_{x_1}(q, w; x(q), \lambda(q))}{\partial q_1} \\
\vdots \\
\frac{\partial L_{x_n}(q, w; x(q), \lambda(q))}{\partial q_1} \end{array} \right] = \left( \Gamma(q, w; x(q), \lambda(q)) \right)^{-1} \left[ \begin{array}{c} \frac{\partial L_{x_1}(q, w; x(q), \lambda(q))}{\partial q_1} \\
\vdots \\
\frac{\partial L_{x_n}(q, w; x(q), \lambda(q))}{\partial q_1} \\
\frac{\partial L_{x_1}(q, w; x(q), \lambda(q))}{\partial q_2} \\
\vdots \\
\frac{\partial L_{x_n}(q, w; x(q), \lambda(q))}{\partial q_2} \\
\vdots \\
\vdots \\
\frac{\partial L_{x_1}(q, w; x(q), \lambda(q))}{\partial q_n} \\
\vdots \\
\frac{\partial L_{x_n}(q, w; x(q), \lambda(q))}{\partial q_n} \end{array} \right]
\]

and

\[
\Gamma(q, w; x(q), \lambda(q)) = \left[ \begin{array}{cccc} \frac{\partial L_{x_1}(q, w; x(q), \lambda(q))}{\partial x_1} & \ldots & \frac{\partial L_{x_1}(q, w; x(q), \lambda(q))}{\partial x_n} \\
\vdots & \ddots & \vdots \\
\frac{\partial L_{x_n}(q, w; x(q), \lambda(q))}{\partial x_1} & \ldots & \frac{\partial L_{x_n}(q, w; x(q), \lambda(q))}{\partial x_n} \\
\frac{\partial L_{\lambda_1}(q, w; x(q), \lambda(q))}{\partial x_1} & \ldots & \frac{\partial L_{\lambda_1}(q, w; x(q), \lambda(q))}{\partial x_n} \\
\vdots & \ddots & \vdots \\
\frac{\partial L_{\lambda_n}(q, w; x(q), \lambda(q))}{\partial x_1} & \ldots & \frac{\partial L_{\lambda_n}(q, w; x(q), \lambda(q))}{\partial x_n} \end{array} \right]
\]

Now suppose the firm using \( \zeta \) is a competitive firm, so that it sets \( MC(q) = p \). Ultimately, we want to know how the vector of factors with \( p \). Set \( h(p; q) = p - MC(q) \) and note that \( h \) is identically zero. I.e., \( h(p; q) \) is of the form \( f(\alpha; x) \). So here's a case in which the economics gives us \( p \) in terms
of \( q \), i.e., the exogenous variable in terms of the endogenous variable. We of course want to know the reverse relationship, i.e., how \( q \) changes as prices change. To get this, we apply the inverse function theorem to obtain the matrix of \( \frac{dq_i}{dp_j} \)'s.

\[
\begin{bmatrix}
\frac{\partial q_1(x)}{\partial p_1} & \frac{\partial q_1(x)}{\partial p_2} & \cdots & \frac{\partial q_1(x)}{\partial p_n} \\
\frac{\partial q_2(x)}{\partial p_1} & \frac{\partial q_2(x)}{\partial p_2} & \cdots & \frac{\partial q_2(x)}{\partial p_n} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\partial q_n(x)}{\partial p_1} & \frac{\partial q_n(x)}{\partial p_2} & \cdots & \frac{\partial q_n(x)}{\partial p_n}
\end{bmatrix}
\begin{bmatrix}
\frac{\partial MC^1(x)}{\partial q_1} & \frac{\partial MC^1(x)}{\partial q_2} & \cdots & \frac{\partial MC^1(x)}{\partial q_n} \\
\frac{\partial MC^2(x)}{\partial q_1} & \frac{\partial MC^2(x)}{\partial q_2} & \cdots & \frac{\partial MC^2(x)}{\partial q_n} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\partial MC^n(x)}{\partial q_1} & \frac{\partial MC^n(x)}{\partial q_2} & \cdots & \frac{\partial MC^n(x)}{\partial q_n}
\end{bmatrix}^{-1}
\]