7. Foundations of Comparative Statics (cont)

7.3. Implicit function Theorem (cont).

Implicit function Theorem: intermediate version: As with the other important concepts in the course, the implicit function theorem can be stated in various degrees of generality. We now go up one level and assume that $f$ has $n+1$ arguments.

**Theorem:** Given $f : \mathbb{R}^{n+1} \to \mathbb{R}^1$ continuously differentiable and $(\bar{\alpha}, \bar{x}) \in \mathbb{R}^n \times \mathbb{R}^1$ if $f_{n+1}(\bar{\alpha}, \bar{x}) \neq 0$, then there exist neighborhoods $U^\alpha$ of $\bar{\alpha}$ and $U^x$ of $\bar{x}$ and a continuously differentiable function $g : U^\alpha \to U^x$ such that for all $\alpha \in U^\alpha$,

\[
f(\alpha, g(\alpha)) = f(\bar{\alpha}, \bar{x}) \text{ i.e., } g \text{ puts us on the level set of } f \text{ containing } (\bar{\alpha}, \bar{x})
\]

\[
g_j(\alpha) = -f_j(\alpha, g(\alpha))/f_{n+1}(\alpha, g(\alpha)).
\]
In words, implicit function theorem says that if you have one equation in \( n + 1 \) unknowns, you can solve for \textit{any one} of the unknowns in terms of the other \( n \), provided that...

Proof again is a trivial exercise in differentiation: since

\[
f(\alpha, g(\alpha)) \equiv f(\bar{\alpha}, \bar{x})
\]

we can take the partial derivative of \( f \) w.r.t. \( \alpha_j \):

\[
\frac{df(\alpha, g(\alpha))}{d\alpha_j} = 0
\]

\[
= \frac{\partial f(\alpha, g(\alpha))}{\partial \alpha_j} + \frac{\partial f(\alpha, g(\alpha))}{\partial x} \frac{\partial g(\alpha)}{\partial \alpha_j}
\]

rearranging yields:

\[
\frac{\partial g(\alpha)}{\partial \alpha_j} = -\frac{\partial f(\alpha, g(\alpha))}{\partial x} \frac{\partial g(\alpha)}{\partial \alpha_j}
\]

An important feature to note is that the domain of \( f \) has one more dimension than the domain of \( g \). Reason is that it is the graph of \( g \), i.e., \( \{ \alpha, g(\alpha) : \alpha \in \text{etc} \} \) that recovers the level set. That is, the graph of a real valued function is a set that lives in a Euclidean space one dimension higher than the dimension of the domain of the function. In this case, a point \( (\alpha, g(\alpha)) \) is a point on the level set of \( f \).

Example: argued that the solution to \textit{any} economic system can be represented as the level set of some function. Here’s a simple economic model: \( S = S(t, p), D = D(y, p), S = D \), where \( p \) denotes market price, \( t \) denotes a tax rate paid by the producer and \( y \) denotes consumer income level. The solution to this model can be represented as the level set \( f(\alpha, x) \equiv 0 \), where \( f = S - D, \alpha = t, y \)
and \( x = p \). The level set of \( f \) corresponding to zero is the set of all (price, tax, income) triples such that the price clears the market for the corresponding values of the exogenous variables.

Explicitly we have the following relationship

\[
\alpha = (t, y) \\
x = p \\
f(\alpha, x) = S(t, p) - D(y, p) \\
g(\alpha) = p(t, y)
\]

\( p(t, y) \) tells us how \( p \) must change with params to keep us on the level set \( S(t, p) - D(y, p) = 0 \).

Solve for an initial equilibrium \((\bar{\alpha}, \bar{x}) = (\bar{t}, \bar{y}, \bar{p})\) and compute

\[
\frac{\partial p(\bar{t}, \bar{y})}{\partial t} = -\frac{\partial f(\bar{t}, \bar{y}, \bar{p})}{\partial p} = -\frac{\partial S(\bar{t}, \bar{p})}{\partial t} - \frac{\partial D(\bar{y}, \bar{p})}{\partial p}
\]

etc.

**Implicit function Theorem:** the most general version: The most general version says that if you have \( m \) equations in \( n + m \) unknowns, you can solve for any \( m \) of the unknowns in terms of the other \( n \), provided that the usual conditions are satisfied.

Interpretation: write your economic model in the form \( f(\alpha; x) = 0 \); solve for changes in \( x \in \mathbb{R}^m \) as a function of changes in the parameter vector \( \bar{\alpha} \in \mathbb{R}^n \). Alternatively, think of \( \alpha \) as a vector of exogenous variables for your model, and of \( x \) as a vector of endogenous variables. We are typically
interested in how the endogenous vector $x$ changes as the exogenous vector $\alpha$ changes. Write $f(\alpha, g(\alpha))$ and use implicit function theorem to find the slope of $g$ w.r.t. the components of $\alpha$.

Note that the relative size of $n$ and $m$ is completely immaterial. So long as you have $m$ equations and $m$ endogenous variables, you can have as many or as few exogenous variables as you want.

**The Implicit Function Theorem:**

**Theorem:** Given $f : \mathbb{R}^{n+m} \to \mathbb{R}^m$ continuously differentiable, let

$$J_{f_x}(\alpha, x) = \begin{bmatrix}
\frac{\partial f^1(\alpha; x)}{\partial x_1} & \frac{\partial f^1(\alpha; x)}{\partial x_2} & \cdots & \frac{\partial f^1(\alpha; x)}{\partial x_m} \\
\frac{\partial f^2(\alpha; x)}{\partial x_1} & \frac{\partial f^2(\alpha; x)}{\partial x_2} & \cdots & \frac{\partial f^2(\alpha; x)}{\partial x_m} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\partial f^m(\alpha; x)}{\partial x_1} & \frac{\partial f^m(\alpha; x)}{\partial x_2} & \cdots & \frac{\partial f^m(\alpha; x)}{\partial x_m}
\end{bmatrix}$$

where $f^1_{n+1}$ denotes the $n+1$'th partial derivative of the function $f^1$, which is, in turn, the first of the $m$ single-valued functions stacked on top of each other that make up the vector valued function $f$. Given $(\bar{\alpha}, \bar{x}) \in \mathbb{R}^n \times \mathbb{R}^m$, if the determinant of $J_{f_x}(\bar{\alpha}, \bar{x}))$ is not zero, then there exist neighborhoods $U^\alpha$ of $\bar{\alpha}$ and $U^x$ of $\bar{x}$ and a continuously differentiable function $g : U^\alpha \to U^x$ such that for all $\alpha \in U^\alpha$,

$$f(\alpha, g(\alpha)) = f(\bar{\alpha}, \bar{x})$$

i.e., $g$ puts us on the level set of $f$ containing $(\bar{\alpha}, \bar{x})$.
and

\[
\begin{bmatrix}
\frac{\partial g^1(\alpha)}{\partial \alpha_1} & \frac{\partial g^1(\alpha)}{\partial \alpha_2} & \cdots & \frac{\partial g^1(\alpha)}{\partial \alpha_n} \\
\frac{\partial g^2(\alpha)}{\partial \alpha_1} & \frac{\partial g^2(\alpha)}{\partial \alpha_2} & \cdots & \frac{\partial g^2(\alpha)}{\partial \alpha_n} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\partial g^m(\alpha)}{\partial \alpha_1} & \frac{\partial g^m(\alpha)}{\partial \alpha_2} & \cdots & \frac{\partial g^m(\alpha)}{\partial \alpha_n}
\end{bmatrix}
\]

\[
= - Jf_x(\alpha, g(\alpha))^{-1} \begin{bmatrix}
\frac{\partial f^1(\alpha, g(\alpha))}{\partial \alpha_1} & \frac{\partial f^1(\alpha, g(\alpha))}{\partial \alpha_2} & \cdots & \frac{\partial f^1(\alpha, g(\alpha))}{\partial \alpha_n} \\
\frac{\partial f^2(\alpha, g(\alpha))}{\partial \alpha_1} & \frac{\partial f^2(\alpha, g(\alpha))}{\partial \alpha_2} & \cdots & \frac{\partial f^2(\alpha, g(\alpha))}{\partial \alpha_n} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\partial f^m(\alpha, g(\alpha))}{\partial \alpha_1} & \frac{\partial f^m(\alpha, g(\alpha))}{\partial \alpha_2} & \cdots & \frac{\partial f^m(\alpha, g(\alpha))}{\partial \alpha_n}
\end{bmatrix}
\]

(1)

That is, we have an equality between an $m \times n$ matrix and an $m \times m$ matrix times an $m \times n$ matrix.

Observe that when $m = 1$, this just collapses to the old implicit function theorem.

As an example, consider a slightly more complex economic system, where $S^i = S^i(t, p_1 ... p_m)$, demand $D^i = D^i(y, p_1 ... p_m)$, etc:

\[
\alpha = (t, y)
\]

\[
x = p
\]

\[
f^i(\alpha, x) = S^i(t, p) - D^i(y, p)
\]

\[
g(\alpha) = p(t, y)
\]

$p(t, y)$ tells us how $p$ must change to keep us on the level set $S(t, p) - D(y, p) = 0$. 
Solve for an initial equilibrium \((\bar{\alpha}, \bar{x}) = (\bar{t}, \bar{y}, \bar{p})\) define

\[
J_f(\bar{t}, \bar{y}, \bar{p}) = \begin{bmatrix}
\frac{\partial f^1(\bar{t}, \bar{y}, \bar{p})}{\partial p_1} & \frac{\partial f^1(\bar{t}, \bar{y}, \bar{p})}{\partial p_2} & \cdots & \frac{\partial f^1(\bar{t}, \bar{y}, \bar{p})}{\partial p_m} \\
\frac{\partial f^2(\bar{t}, \bar{y}, \bar{p})}{\partial p_1} & \frac{\partial f^2(\bar{t}, \bar{y}, \bar{p})}{\partial p_2} & \cdots & \frac{\partial f^2(\bar{t}, \bar{y}, \bar{p})}{\partial p_m} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\partial f^m(\bar{t}, \bar{y}, \bar{p})}{\partial p_1} & \frac{\partial f^m(\bar{t}, \bar{y}, \bar{p})}{\partial p_2} & \cdots & \frac{\partial f^m(\bar{t}, \bar{y}, \bar{p})}{\partial p_m}
\end{bmatrix}
\]

and compute

\[
\begin{bmatrix}
\frac{\partial g^1(\alpha)}{\partial \alpha_1} \\
\frac{\partial g^2(\alpha)}{\partial \alpha_1} \\
\vdots \\
\frac{\partial g^n(\alpha)}{\partial \alpha_1}
\end{bmatrix}
= - J_f(\bar{t}, \bar{y}, \bar{p})^{-1}
\begin{bmatrix}
\frac{\partial f^1(\bar{t}, \bar{y}, \bar{p})}{\partial t} \\
\frac{\partial f^2(\bar{t}, \bar{y}, \bar{p})}{\partial t} \\
\vdots \\
\frac{\partial f^m(\bar{t}, \bar{y}, \bar{p})}{\partial t}
\end{bmatrix}
\]

etc..

7.4. Manipulating first order conditions using the Implicit Function Theorem.

Varian says “all of comparative statics is a straightforward application of the implicit function theorem.” Let’s see how.
Consider the standard utility maximization problem: maximize \( u(z) \), subject to \( p \cdot z = y \), where \( z, p \in \mathbb{R}^2 \). Set up the Lagrangian \( L(\bar{p}, \bar{y}; z, \lambda) = u(z) + \lambda(y - p \cdot z) \). I.e., initially, \( p \) and \( y \) are parameters.

- Solve for the first order conditions of the Lagrangian. That is, solve for \((\bar{z}, \bar{\lambda})\) such that \( \nabla L(\bar{p}, \bar{y}; \bar{z}, \bar{\lambda}) = 0 \). (Here the gradient consists of the partial derivatives w.r.t. the \( z \)'s and \( \lambda \), not w.r.t. to the parameters \( p \) and \( y \). Note that since the constraint must hold with equality, we must have \( \frac{\partial L}{\partial \lambda} = 0 \).)
- Observe that what we’ve done is find one point on a particular level set of \( \nabla L \).
- Now consider variations in the exogenous variables \( p \) and \( y \) and think about the level set \( \{ (z, \lambda, p, y) : \nabla L(\cdot, \cdot, \cdot, \cdot) = 0 \} \).
- Specifically, observe that \( \nabla L : \mathbb{R}^6 \to \mathbb{R}^3 \), so that we can solve for first three variables in terms of the last three.
- To convert this problem to the format of the general implicit function theorem, what’s \( f, x, \alpha, g \)?

\[
- f = \nabla L.
- \text{look at the level set } \nabla L = 0.
- x = (z_1, z_2, \lambda)
- \alpha = (p_1, p_2, y)
- g(\alpha) \equiv g(p_1, p_2, y) \equiv ((z_1(p_1, p_2, y), z_2(p_1, p_2, y), \lambda(p_1, p_2, y))
\]

Now mindlessly apply the implicit function theorem to compute \( \frac{\partial z_i(\cdot)}{\partial p_j} \), etc. That is:

\[
\begin{bmatrix}
\frac{\partial g^1(\cdot)}{\partial p_1} & \frac{\partial g^1(\cdot)}{\partial p_2} & \frac{\partial g^1(\cdot)}{\partial y} \\
\frac{\partial g^2(\cdot)}{\partial p_1} & \frac{\partial g^2(\cdot)}{\partial p_2} & \frac{\partial g^2(\cdot)}{\partial y} \\
\frac{\partial g^3(\cdot)}{\partial p_1} & \frac{\partial g^3(\cdot)}{\partial p_2} & \frac{\partial g^3(\cdot)}{\partial y}
\end{bmatrix}
= -J(\nabla L)^{-1}(z, \lambda)
\begin{bmatrix}
\frac{\partial L_{z_1}(\cdot)}{\partial p_1} & \frac{\partial L_{z_1}(\cdot)}{\partial p_2} & \frac{\partial L_{z_1}(\cdot)}{\partial y} \\
\frac{\partial L_{z_2}(\cdot)}{\partial p_1} & \frac{\partial L_{z_2}(\cdot)}{\partial p_2} & \frac{\partial L_{z_2}(\cdot)}{\partial y} \\
\frac{\partial L_{z_3}(\cdot)}{\partial p_1} & \frac{\partial L_{z_3}(\cdot)}{\partial p_2} & \frac{\partial L_{z_3}(\cdot)}{\partial y}
\end{bmatrix}
\]
where

\[ J(\nabla L)_{(z, \lambda)} = \begin{bmatrix}
  L_{z_1 z_1}(\cdot) & L_{z_1 z_2}(\cdot) & L_{z_1 \lambda}(\cdot) \\
  L_{z_2 z_1}(\cdot) & L_{z_2 z_2}(\cdot) & L_{z_2 \lambda}(\cdot) \\
  L_{\lambda z_1}(\cdot) & L_{\lambda z_2}(\cdot) & L_{\lambda \lambda}(\cdot)
\end{bmatrix}. \]

For example, consider computing the Slutsky equation by differentiating the first order conditions of the Lagrangian. Recall that the lagrangian is: \( L(\bar{p}, \bar{y}; z, \lambda) = u(z) + \lambda(y - p \cdot z) \). So that the first order conditions are:

\[ \nabla L = \begin{bmatrix}
  L_{z_1} \\
  L_{z_2} \\
  L_{\lambda}
\end{bmatrix} = \begin{bmatrix}
  u_1 - \lambda p_1 \\
  u_2 - \lambda p_2 \\
  y - p_1 z_1 - p_2 z_2
\end{bmatrix} = 0. \]

Hence

\[ HL = J(\nabla L)_{(z, \lambda)} = \begin{bmatrix}
  u_{11} & u_{12} & -p_1 \\
  u_{21} & u_{22} & -p_2 \\
  -p_1 & -p_2 & 0
\end{bmatrix}. \]

and we have

\[
\begin{bmatrix}
  \frac{\partial z_1}{\partial p_1} & \frac{\partial z_1}{\partial p_2} & \frac{\partial z_1}{\partial y} \\
  \frac{\partial z_2}{\partial p_1} & \frac{\partial z_2}{\partial p_2} & \frac{\partial z_2}{\partial y} \\
  \frac{\partial \lambda}{\partial p_1} & \frac{\partial \lambda}{\partial p_2} & \frac{\partial \lambda}{\partial y}
\end{bmatrix} = -J(\nabla L)^{-1}_{(z, \lambda)} \begin{bmatrix}
  -\lambda & 0 & 0 \\
  0 & -\lambda & 0 \\
  -z_1 & -z_2 & 1
\end{bmatrix} = J(\nabla L)^{-1}_{(z, \lambda)} \begin{bmatrix}
  \lambda & 0 & 0 \\
  0 & \lambda & 0 \\
  z_1 & z_2 & -1
\end{bmatrix}. \]
If you like, rewrite this expression as three equations, each of which is of the form $Ax = b$, e.g.,

$$
\begin{bmatrix}
u_{11} & u_{12} & -p_1 \\
u_{21} & u_{22} & -p_2 \\
-p_1 & -p_2 & 0
\end{bmatrix}
\begin{bmatrix}
\frac{\partial z_1}{\partial p_1} \\
\frac{\partial z_2}{\partial p_1} \\
\frac{\partial \lambda}{\partial p_1}
\end{bmatrix}
= 
\begin{bmatrix}
\lambda \\
0 \\
z_1
\end{bmatrix}
$$

and now apply Cramer's Rule, e.g.,

$$
\frac{\partial z_1}{\partial p_1} = \det(J(\nabla L)_{(z,\lambda)})^{-1} \det
\begin{bmatrix}
-\lambda & u_{12} & -p_1 \\
0 & u_{22} & -p_2 \\
-z_1 & -p_2 & -0
\end{bmatrix}
$$

Here's a second example, which is conceptually identical to the preceding one. Consider a competitive firm, producing a single output $y$ from a vector of inputs $x \in \mathbb{R}^n$ using a technology $f$. Let $p$ denote the output price and $w$ denote the vector of input prices. Given input and output prices, the first order condition for profit maximization is

$$
\text{FOC}(p, w, x) = p\nabla f(x) - w = 0
$$

where (2) is of course an $n$-vector of equalities. To guarantee a local maximum we require that $Hf(x)$ is negative definite. Usefully, this implies that the determinant of $Hf(x)$ is nonzero, and this turns out to be the condition we need in order to be able to use the implicit function theorem.

Our task is to derive the input demand functions, i.e., $x(w)$. To find this we mindlessly apply the theorem to the level set (2) to obtain $Jx(w)$, i.e.,

$$
Jx(w) = -(J\text{FOC}_x(p, w, x))^{-1} \times J\text{FOC}_w(p, w, x)
$$

$$
= -(pHf(p, w, x))^{-1} \times (-I^n)
$$

$$
= (pHf(p, w, x))^{-1} \times I^n
$$
where $I^n$ is the $n$-dimensional identity function, $J_{FOC_x}$ denotes the Jacobian of FOC treating $(p, w)$ as parameters, and $J_{FOC_w}$ denotes the Jacobian of FOC treating $(p, x)$ as parameters.