4. UNIVARIATE AND MULTIVARIATE DIFFERENTIATION

Assume you all know how to calculate the derivative of a single variable function, i.e., given $f$, calculate $\frac{df(x)}{dx}$, denoted also $f'(\cdot)$. Important to know the difference between $f'(\cdot)$, which is a function and $f'(x)$, which is a number, the function evaluated at a point.

I’ll try to be careful to use this notation from now on: $g(\cdot)$ is a RULE, represents a function.

Since $f'(\cdot)$ is a function, like any other, it may have a derivative; if it does, call it $f''(\cdot)$. Do example $f(x) = x^2$.

Lots of standard kinds of functions you have to be able to differentiate in your sleep. Equivalent of being able to spell. Brainless activity.
4.1. The fundamental idea: linear approximations to nonlinear functions

The most important thing to keep in mind about calculus is:

(Mantra2) When you do calculus, you are, ALWAYS, approximating a small change in a differentiable function by evaluating THE (unique) appropriate LINEAR function at the magnitude of the change. That is, for sufficiently small $dx$, $f(x + dx) - f(x) \approx f'(x)dx$.

$$df^x(\cdot) = f'(x)(\cdot)$$ is the linear function that approximates the difference between $f(x + \cdot)$ and $f(x)$.

Draw picture of the comparison: concave example.

- graph of $f(\cdot)$ with points $x^*$ and $x^* + dx^*$.
- look at difference in the value of the function at the two points.
- look at the graph of the linear function $df = f'(x^*)dx$:  

\[ df = f'(x^*)dx \]
• note that the difference between \( (f(x^* + dx^*) - f(x^*)) \) and \( f'(x^*)dx^* \) will be arbitrarily small provided that \( dx^* \) is sufficiently small.

• Explain the enormous practical importance of this.

When \( dx \) is bigger, don’t do so well at approximation. Could do better if you threw in the second derivative, i.e.,

\[
f(x + dx) - f(x) \approx \text{even better } f'(x)dx + \frac{1}{2}f''(x)dx^2.
\]

Note that in the above figure, \( f''(x^*) < 0 \). Note also that the linear approximation gives an overestimate of the change, so that adding the second term helps.

This is the germ of a crucial theorem called Taylor’s theorem, that we’ll see more of later.

4.2. Univariate Calculus

You’ve probably seen the expression \( dy = f'(x)dx \), known as the differential, before. What sort of object is this? What’s the relationship between the differential and the derivative? Answer is ridiculously simple. Recall from last time about linear functions from \( \mathbb{R}^1 \) to \( \mathbb{R}^1 \) and scalars; every such function is uniquely defined by a single scalar. E.g., I talked about the function \( p \), when I really meant the rule \( f(\cdot) \) defined by \( f(x) = px \). Well... The differential is to the derivative as the linear function is to the scalar/vector that defines it.

The relationship between the differential and the derivative is the same as the relationship between any linear function and the scalar/vector/matrix that defines it.
Defn: the differential of $f$ at $x$ is the linear function defined by the scalar $f'(x)$. In other words, the differential of $f$ at $x$ is the unique linear function whose coefficient (slope) is equal to the derivative of $f$ at $x$.

Restate: The most important thing to keep in mind about calculus is: when you do calculus on $f$ at $x$, you are, ALWAYS, approximating a small change in $f$, starting from $x$, by evaluating the differential of $f$ at $x$ at the magnitude of the change.

That is, $dy = f'(x)dx$ is the value of the approximation to the change in $f$ when you shift from $x$ to $x + dx$.

Economic example: have $\pi(p)$, i.e., profits are a function of prices (assuming optimal input choices for prices $p$.) If $p$ change is small enough, $d\pi \approx \pi'(p)dp$.

It is the ability to make computations like this—we call them comparative statics—that distinguish economists from apes. Example:

- Let $f(x) = x^2$
- set $x = 1$, $dx = 0.1$
- then $f(x) = 1$, $f'(x) = 2$, $f(x + dx) = 1.21$.
- The true difference is .21.
- the estimated difference is $f'(x)dx = 0.20$, not bad.