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5. Foundations of Comparative Statics

5.1. The envelope theorem for unconstrained maximization

In economics, we’re often interested in a function which has two arguments; the second is a function of the first. As we’ve discussed in a previous lecture (CALCULUS2), economists typically deal with this by invoking the (unfortunately named) concept of the total derivative: if \( f(b, x) = f(b, x(b)) \), then total derivative of \( f \) x.r.t \( b \) is \( df/db = f_b(b, x(b)) + f_x(b, x(b))x'(b) \), where \( b \) and \( x \) are here scalars. When \( b \) changes in this case, there is a change in \( f \) due to two factors: first \( b \) changes, also, \( x \) changes as \( b \) changes.

In this lecture, we’ll consider the case in which the second argument is a special kind of function of \( b \); \( x^*(\cdot) \) is the value of \( x \) that maximises (or minimizes) \( f \) for each value of \( b \). The function \( f(b, x^*(b)) \) is then called the value function.
Example \( \pi(p, q^*(p)) \); for each \( p \), pick the \( q \) that maximizes profits for that \( p \); call this function \( q^*(p) \).

Now ask how profit adjust as price changes and the producer adjusts quantity.

Other examples of value functions in economics are the expenditure function and the indirect utility function.

Answer is given by the envelope theorem which says that in this case, \( \frac{df}{db} = \frac{\partial f}{\partial b} \). (i.e., you’ve learnt to tell the difference between \( df \) and \( \partial f \); now you find that in this case, there isn’t any difference.)

The envelope theorem: Varian: Given \( f : \mathbb{R}^2 \to \mathbb{R} \) (differentiable) and a function \( x^* : \mathbb{R} \to \mathbb{R} \) (differentiable) defined by the condition that for each \( b \), \( x^*(b) \) maximizes \( f(b, \cdot) \). Then the total derivative of the function \( f(b, x^*(b)) \) with respect to \( b \) is \( \frac{df(\cdot, x^*(\cdot))}{db} = \frac{\partial f(\cdot, x^*(\cdot))}{\partial b} \).

Mathematical proof is trivial

\[
\frac{df(b, x^*(b))}{db} = f_b(b, x^*(b)) + f_x(b, x^*(b)) \frac{dx^*(b)}{db}
\]

Necessary condition for \( x \) to maximize \( f(b, \cdot) \) is that \( f_x(b, x) = 0 \); this is how \( x^*(b) \) is defined; hence \( f_x(b, x^*(b)) = 0 \) by definition of \( x^*(b) \).

The picture is much more important; note that in the picture, the golden rule is broken, for display purposes: the first component of the function is pictured on the horizontal axis.

- The line in the domain \( x^*(b) \) has the property that vertically above points on this line, the function \( f(b, \cdot) \) is maximized in the \( x \) direction.
When you evaluate $df$, the $dx$ term has no effect because it is multiplied by zero.

**FIGURE 1.** The envelope theorem and the differential

- Now as you move out along the line $x^*(b)$ there are in general two contributors to the change in $f$; $f$ changes because $b$ changes AND because $x$ changes.
- In this particular case, $f$ doesn’t change when $x$ changes, but it does change when $b$ changes.
- It’s worth noting that there is nothing special about moving out along the line $x^*(b)$: if you start out at $(b, x^*(b))$ and move in any direction whatsoever, the only thing that matters is the movement in the $b$ direction.
• Compare this picture to the general case in which we move out at the same angle, where
the change in $f$ due to $x$ is very substantial;
• In the general case, the $x$ effect really makes a difference.

5.2. The envelope theorem for constrained maximization

The point of the above version of the envelope theorem is that when you are maximizing unconstrainedly a function $f(b, \cdot)$, where $b$ is a parameter, then the rate of change in $f$ as $b$ changes does not depend on the rate at which $x^*(\cdot)$ moves with $b$. (In class, I did the theorem for the case in which $x^*$ was a scalar; it is also obviously true when $x^*$ is a vector.)

An analogous result holds when you solve the problem:

$$\text{maximize } f(b, x) \text{ subject to the constraint that } h(b, x) = 0. \quad (1)$$

Let $x^*(b)$ denote the solution to (1) and let $M(b)$ denote the maximized value of $f$ given $b$.

(Notice that there is a difference between (1) and the familiar specification, i.e., max $f(b, x)$ subject to $g(x) = b$. But the familiar specification is a special case of the current one. To see this, let $h(b, x) = b - g(x)$. The current specification allows for more general comparative statics than we have seen before. In our original specification i.e., max $f(b, x)$ subject to $g(x) = b$, we learnt how to do comparative statics w.r.t. $b$, but not with respect to the other parameters of $g(\cdot)$. When we write the problem in the current form, we can do comparative statics w.r.t. any parameter of either the objective or the constraint. For example, suppose that our problem is max $u(x)$ s.t. $p \cdot x = y$. Our analysis in the preceding section taught us how to do comparative statics w.r.t. $y$ but not w.r.t. the components of $p$. We are about to see how to do comparative statics w.r.t. these components as well.)
The constrained version of the envelope theorem says that \[ \frac{dM(b)}{db} = \frac{\partial f(b, x^*(b))}{\partial b} + \lambda^*(b) \frac{\partial h(b, x^*(b))}{\partial b}. \]

(Notice that for the special case in which \( f \) does not depend on \( b \), so that \( \frac{\partial f(b, x^*(b))}{\partial b} = 0 \), and \( h(b, x^*) = b - g(x^*) \), so that \( \frac{\partial h(b, x^*(b))}{\partial b} = 1 \), the expression \( \frac{dM(b)}{db} = \frac{\partial f(b, x^*(b))}{\partial b} + \lambda^*(b) \frac{\partial h(b, x^*(b))}{\partial b} \) reduces to simply \( \frac{dM(b)}{db} = \lambda^*(b) \), which is the old familiar result: \( \lambda^*(\cdot) \) measures the rate at which the objective function increases as the constraint is relaxed.) As in the unconstrained version of the envelope theorem, the total derivative of \( M \) w.r.t. \( b \) does involve the partial derivatives of \( f \) w.r.t. the elements of the \( x^* \) vector, but these terms disappear in the expression for \( \frac{dM(b)}{db} \).

The striking difference between the unconstrained and the constrained theorems is that in the former case, the movement in the \( x \) direction didn’t matter because the \( \frac{\partial f(b, x^*(b))}{\partial x_i} \)'s were zero. In the present case, the gradient of \( f \) isn’t zero, and yet the movement in the \( x \) direction still doesn’t matter. So while the unconstrained theorem is very easy to explain intuitively, the constrained theorem is by no means so.

Fig. 2 illustrates the theorem, for the simplest case in which there is only one constraint, \( h(b, x^*) = b - g(x^*) \). To present the result in its sharpest form, the figure depicts a linear optimization problem, i.e., the level sets of both the objective \( f \) and the constraint \( h \) are affine functions. In the upper panel, we start out at an optimum \( x^*(\tilde{b}) \) on the constraint set, i.e., the level set of \( h \) associated with \( \tilde{b} \). (Note that for this linear case, the optimum is not unique). In the right panel, the level sets of the objective function have a different slope from that of the constraint function, so we start out at an arbitrary (non-optimal) point \( \tilde{x}^* \) on the constraint set. Now consider what happens when \( b \) changes to \( b' \), and the constraint line moves outwards. It is critical to my story that the new constraint line is parallel to the old one: this must be the case because \( g \) is affine, so its gradient cannot change direction (or length) with \( x \). In the upper panel, the change in the objective function doesn’t depend on how \( x^* \) moves, provided that \( x^* \) moves to the new, parallel constraint line. In the right panel, different directions of movement from the old constraint
line to the new result in different changes in the objective function. This is a graphical depiction of the mathematical result that when you start out at a constrained optimum, and shift to the new, parallel constraint line, the change in the value of the objective depends on the partials of both $f$ and $h$ w.r.t. $b$ but not on the partials of either $f$ or $h$ w.r.t. the components of $x^*$. 

When either the objective or the constraint are nonlinear, i.e., when you add some curvature to the level sets, it's a little less easy to see from pictures what's going on. Indeed the important implication of the envelope theorem seems to be false, since differences in the direction of movement $\mathbf{dx}$ do make a difference. However, as the two panels of Fig. 3 illustrate, the difference is much smaller when you start out at an optimum $x^*(\hat{b})$ (as in the left panel of the figure) than when you start out at some arbitrary point $\mathbf{x}^*$ (as in the right panel of the figure). In both panels, you get to a higher level set when you move to the new constraint line in the direction $\mathbf{dx}$ of the gradient than if you move in a different direction, such as $\mathbf{dx}'$. In the left panel, however, the difference between the level sets you reach is only second order, i.e., due to the Hessian term, but in the right panel the difference is first order (i.e., due to the first order term in the Taylor expansion).

As we've noted, Fig. 3 seems to contradict the envelope theorem, which says that the only things that matter are (a) how far out the constraint moves when $b$ changes; (b) how rapidly $f$ increases
as you move out to the new constraint. What doesn’t matter, according to the envelope theorem, is the direction in which you move in order to get to the new constraint line. The reconciliation of this paradox (at least for the case when only one constraint is binding—when two or more are binding we have to tell a more complicated story) is that the envelope theorem is only telling you about the first order Taylor approximation to the change in $f$ when $b$ changes. In Fig. 2, there are no second order effects, i.e., the Hessian is zero. So the first term in the Taylor expansion tells the whole story: movements in the $\mathbf{x}^*$ direction don’t matter at all when you start out at an optimum. In the left panel of Fig. 3, there is no first order effect because we start out at an optimum, only second order effects. Since the envelope theorem just gives you a first-order approximation to the true change in $f$ when $b$ changes, the differences we see in the left panel of Fig. 3 evaporate. In the right panel of Fig. 3, we don’t start from an optimum and there are both first and second order effects.

Now for the formalism. We’ll do the general NPP, i.e., with $m$ inequality constraints. Once again, we’ll see that the direction in which the solution vector $\mathbf{x}^*(\cdot)$ moves as $b$ moves has no effect on the total derivative of $f$ w.r.t. $b$.

The envelope theorem for constrained maximization: Consider $f : \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^1$ (differentiable) and
\[ h : \mathbb{R}^1 \times \mathbb{R}^n \to \mathbb{R}^m \text{ (differentiable). For } b \in \mathbb{R}, \text{ suppose that } (\mathbf{x}^*(b), \lambda^*(b)) \text{ satisfy the KKT conditions for the nonlinear programming problem:} \]

\[
\max_{\mathbf{x}} f(b, \mathbf{x}) \text{ such that } h^j(b, \mathbf{x}) \geq 0, \text{ for } j = 1, \ldots, m.
\]

Then the total derivative of the function \( f(b, \mathbf{x}^*(b)) \) with respect to \( b \) is

\[
\frac{df(b, \mathbf{x}^*(b))}{db} = \frac{\partial f(b, \mathbf{x}^*(b))}{\partial b} + \sum_{j=1}^{m} \lambda_j^*(b) \frac{\partial h^j(b, \mathbf{x}^*(b))}{\partial b}
\]

Please note that the KKT for this more general version of the NPP is that

\[
\nabla f(b, \mathbf{x}^*(b)) = -\sum_{j=1}^{m} \lambda_j^*(b) \nabla h^j(b, \mathbf{x}^*(b)), \text{ with } \lambda_j^*(b) \geq 0 \text{ for all } j, \text{ and } h^j(b, \mathbf{x}^*(b)) > 0 \implies \lambda_j^*(b) = 0.
\]

In words, in this setting the gradient of \( f \) belongs to the nonpositive cone defined by the gradients of the constraints that are satisfied with equality at \( \mathbf{x}^*(b) \). To reconcile this with our usual treatment, note that for the special case in which \( h^j(b, \mathbf{x}) = b^j - g^j(\mathbf{x}) \), we have \( \nabla h^j = -\nabla g^j \), so that in this special case, the nonnegative cone defined by any subset of the gradients of \( g^j \)'s is the nonpositive cone defined the corresponding subset of the gradients of \( h^j \)'s.

Proof: The Lagrangian for this problem is

\[
L(b, \mathbf{x}^*(b), \lambda^*(b)) = f(b, \mathbf{x}^*(b)) + \sum_{j=1}^{m} \lambda_j^*(b) h^j(b, \mathbf{x}^*(b))
\]

since \( \lambda_j^*(b) h^j(b, \mathbf{x}^*(b)) = 0 \) for all \( j \), we have

\[
M(b) \equiv f(b, \mathbf{x}^*(b)) \equiv L(b, \mathbf{x}^*(b), \lambda^*(b))
\]
It follows that 
\[
\frac{dM(b, \mathbf{x}^*(b))}{db} = \frac{df(b, \mathbf{x}^*(b))}{db} = \frac{dL(b, \mathbf{x}^*(b), \mathbf{\lambda}^*(b))}{db} \\
= \frac{\partial L(b, \mathbf{x}^*(b), \mathbf{\lambda}^*(b))}{\partial b} + \sum_{i=1}^{n} \frac{\partial L(b, \mathbf{x}^*(b), \mathbf{\lambda}^*(b))}{\partial x_i} \frac{dx_i^*(b)}{db} + \sum_{j=1}^{m} \frac{\partial L(b, \mathbf{x}^*(b), \mathbf{\lambda}^*(b))}{\partial \lambda_j} \frac{d\lambda_j^*(b)}{db} \\
= \frac{\partial f(b, \mathbf{x}^*(b))}{\partial b} + \sum_{j=1}^{m} \lambda_j^*(b) \frac{\partial h_j(b, \mathbf{x}^*(b))}{\partial b} \\
+ \sum_{i=1}^{n} \left( \frac{\partial f(b, \mathbf{x}^*(b))}{\partial x_i} + \sum_{j=1}^{m} \lambda_j^*(b) \frac{\partial h_j(b, \mathbf{x}^*(b))}{\partial x_i} \right) \frac{dx_i(b)}{db} + \sum_{j=1}^{m} h_j(b, \mathbf{x}^*(b)) \frac{d\lambda_j^*(b)}{db}
\]

Since $\mathbf{x}^*(b)$ satisfies the KKT conditions each of the $\frac{\partial L(b, \mathbf{x}^*(b), \mathbf{\lambda}^*(b))}{\partial x_i}$'s is zero. That is, for each $i$, the term in the large parentheses on the second line is zero. Moreover if $h(b, \mathbf{x}^*(b)) < 0$, then $\lambda_j^*(\cdot)$ is zero on a neighborhood of $b$ and hence $\frac{d\lambda_j^*(b)}{db}$ is zero; Hence for all $j$ $h_j(b, \mathbf{x}^*(b)) \frac{d\lambda_j^*(b)}{db}$ is zero.

Conclude that all terms on the second line are zero. This proves
\[
\frac{df(b, \mathbf{x}^*(b))}{db} = \frac{\partial f(b, \mathbf{x}^*(b))}{\partial b} + \sum_{j=1}^{m} \lambda_j^*(b) \frac{\partial h_j(b, \mathbf{x}^*(b))}{\partial b}
\]

5.3. Application of the envelope theorem for constrained maximization

Jacob Viner’s famous figure of the long-run and short-run average total cost functions provides a nice example of the envelope theorem. Consider a production function $q = f(\ell, k)$. The LRATC curve assigns to each $q$ the average total cost associated with the cost-minimizing combination of labor and capital. That is,
\[
\text{LRATC}(q) = \min_{\{\ell, k\}} \frac{w\ell + rk}{q} \quad \text{s.t.} \quad q \leq f(\ell, k)
\]
For each level of capital $k$, $\text{SRATC}^k(q)$ is the average cost of producing $q$ using $k$ and whatever is the required level of $\ell$. For each $q$, $\text{LRATC}(q) = \text{SRATC}^k(q)$ at the level $k$ that is optimal for that $q$. The graphs exhibit the well-known property that each SRATC curve is tangent to the LRATC curve at the point where they agree (see Fig. 4). Because of this relationship, the LRATC is referred to as the outer envelope of the SRATC curves. The “puzzle” here is that one would expect the short-run curve to be steeper than the long-run curve at $\bar{q}$, because when output increases from $q$, capital is held constant in the short-run, but varies in the long run: we would expect that by adjusting both inputs in response to an increase in $q$, the producer could reduce costs relative to the case in which he is required to hold capital constant.

The envelope theorem resolves the puzzle. To see the relationship between this theorem and Fig. 4 we have to move into input space. See Fig. 5 which, note, is very similar to Fig. 3. Along the long-run average cost curve, we have solved the cost-minimization problem, so that the constrained envelope theorem holds. For example, at the point $(\bar{q}, \text{LRATC}(\bar{q}))$ in Fig. 4, the input mix $(\bar{k}, \bar{\ell})$ is chosen to be at the point where the isoquant corresponding to $q$ is tangent to the iso-cost line. Now suppose as in the upper panel of Fig. 5, you move from $(\bar{k}, \bar{\ell})$ to the new isoquant line corresponding to $\bar{q} + dq$. To a first order approximation, the new isoquant will necessarily be parallel to the old (because, again, the gradient is constant in a first order approximation). As the figure illustrates, it doesn’t matter much which way you combine inputs in order to get to the new isoquant: to a first order approximation, all input mixes result in the same cost increment. In particular, if you produce the additional output entirely by increasing labor (as you do along the SRATC curve), then the increment in your cost is, to a first order approximation the same as if you had increased both inputs in the optimal proportions (as
Figure 5. The envelope theorem applied to Viner’s cost diagram.

you do along the LRATC curve). In other words, the slopes of your LRATC and SRATC curves are the same at $q$. 
Now suppose you are producing a higher level of output, $\bar{q}$. In the bottom panel, we compare the case in which we are producing with the original, now suboptimal level of capital $k$ and labor $\ell' > \bar{\ell}$, against the level of inputs $(\bar{k}, \bar{\ell})$ that would be optimal for this new higher level $\bar{q}$. This situation is represented by the lower panel of Fig. 5. As the figure indicates, when you move from the isoquant corresponding to $(\bar{k}, \bar{\ell})$ to the new isoquant line corresponding to $\bar{q} + dq$, it matters a lot whether you are starting from the optimal input mix $(\bar{k}, \bar{\ell})$ or starting from your original capital level $k$, which requires an input mix of $(k, \ell')$. There are two things to notice:

1. Notice that when you produce with $(\bar{k}, \bar{\ell})$ you are on a lower isocost line than when you produce with $(k, \ell')$. This observation corresponds to the fact that in the Average Cost diagram (Fig. 4) above the point $\bar{q}$ the level of the short run average cost curve is higher than the level of the long run average cost curve.

2. Now observe that if you are initially producing $\bar{q}$ and then you shift to producing $\bar{q} + dq$, if your starting point is the suboptimal mix $(k, \ell')$, and you are allowed only to increase $\ell$ because $k$ is fixed, then the increase in costs is much larger than if your starting point were the optimal mix $(\bar{k}, \bar{\ell})$, and you adjust both inputs. This observation corresponds to the fact that in the Average Cost diagram (Fig. 4) above the point $\bar{q}$ the slope of the short run average cost curve is steeper than the slope of the long run average cost curve.