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Time-consistent policies ☆

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Abstract

In many cases the optimal open-loop policy to influence agents who solve dynamic problems is time inconsistent. We show how to construct a time-consistent open-loop policy rule. We also consider an additional restriction under which the time-consistent open-loop policy is stationary. We use examples to illustrate the properties of these tax rules.

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1. Introduction

When non-strategic agents with rational expectations solve dynamic optimization problems, and a government (or some other "leader") attempts to influence the agents' decisions, the government's optimal program is often time inconsistent. However, the possibility that the optimal program is time consistent is more general than is widely believed. We extend results in [6] by developing a simple means of testing whether a given open-loop policy rule, such as a linear income tax, is time consistent. This approach also identifies the (possibly non-linear) form of the policy that ensures time consistency, for a wide class of utility and production functions.

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Alternatively, given a particular functional form for the government's policy rule, we can select utility and production functions for which that policy rule is time consistent. The condition for consistency of the tax policies is intuitive: the effect of the tax on the agent's present discounted value of future utility must be independent of the level of capital (wealth). We then identify a *stationary* time-consistent policy rule that is subgame perfect.

2. The model

Our model is standard in the literature on optimal taxation in continuous time (see [1,6]). A representative agent chooses a consumption trajectory $c(t)$ in order to maximize the present discounted value of utility. The agent's wealth (capital stock) is $k(t)$ which yields the instantaneous output $f(k(t))$. The tax rule is $g(k,t)$, so after-tax income is $f(k(t)) - g(k,t)$ and investment is $\frac{dk}{dt} = f(k(t)) - g(k,t) - c(t)$. The government pays for the flow of a public good, $G(t)$, using taxes, without borrowing, so $G(t) = g(k,t)$. The utility of consumption is $U(c)$ and the utility derived from the public good is $V(G)$; both functions are concave. The agent's optimization problem is

$$\max_{\{c(t)\}} \int_0^\infty e^{-\rho t}[U(c(t)) + V(G(t))]\,dt$$
$$\text{s.t.}\ \frac{dk}{dt} = f(k(t)) - g(k,t) - c(t), \quad \text{with } k_0,\ g(k,t), G(t) \text{ given.} \tag{1}$$

The agent behaves as if *aggregate* tax collection, and thus the provision of the public good, is given. In view of the (assumed) separability of the instantaneous payoff, we can ignore the term $V(G(t))$ in studying the agent's control problem.

We adopt the following assumptions.

Assumption 1. The levels of consumption and the provision of the public good in the open-loop equilibrium are strictly positive.

Assumption 2. The feasible set of open-loop policy rules does not enable the government to achieve the first best outcome.

Assumption 3. The government's tax policy is multiplicatively separable: $g(k,t) = b(k)\tau(t)$ for some functions $b(k)$ and $\tau(t)$.

Assumption 1 rules out uninteresting complications caused by corner solutions. Assumption 2 states that the tax rule does not give the government enough leverage to achieve the first best outcome, and thus eliminates a trivial reason for time consistency. This assumption rules out a poll tax. Assumption 3 allows us to concentrate on interesting special cases: a linear income tax ($b(k) = f(k)$); a linear capital tax ($b(k) = k$); and a non-linear income or capital tax ($b(k) \neq k, b(k) \neq f(k)$). (A subsequent footnote explains how our major result changes when we drop

Assumption 3.) Fixing the function $b(k)$ does not restrict the government's ability to raise tax revenue for a given level of k, because the government is able to choose $\tau(t)$.

We use the following:

Definition 1. Conditional on fixed $b(k)$, a tax policy $b(k)\tau(t)$ is time consistent if and only if the trajectory $\tau(t)$ that is optimal at time $t = 0$ remains optimal at every $t \geqslant 0$ along the equilibrium trajectory.

If the agent believes that the government will adhere to the announced policy $b(k)\tau(t)$ and behaves optimally given this belief, then the government has no incentive to deviate from the time-dependent part of the policy, $\tau(t)$.

The qualifier "conditional on fixed $b(k)$" in Definition 1 means that the policy is "conditionally time consistent". Most discussions of time consistency implicitly contain this kind of conditionality. For example, Xie finds the time-consistent policy conditional on the use of a linear income tax $(b(k) = f(k))$. Since a major point of our paper is to show that we can always find a time-consistent policy by the appropriate choice of $b(k)$, it is important that we make this conditionality explicit. Hereafter we use the terms "time consistent" and "conditionally time consistent" interchangeably. A time-consistent policy is not necessarily subgame perfect.

3. The time-consistent tax policy

We assume that the necessary conditions to the agent's control problem provide a solution to that problem. Given the concavity of $U(c)$, the necessary conditions are sufficient if $f(k) - \tau(t)b(k)$ is concave. The function $\tau(t)b(k)$ is endogenous; we can check concavity after finding a candidate solution.

Ignoring the function $V(G(t))$, we write the current value of the agent's payoff as the function $J(k,t)$. This function solves the Bellman equation (where subscripts denote partial derivatives):

$$\rho J(k,t) = \max_c \{U(c) + J_k(k,t)(f(k) - b(k)\tau(t) - c)\} + J_t(k,t). \tag{2}$$

The first order condition to Eq. (2) implies that optimal consumption at a point in time depends only on the shadow value of capital $J_k(k,t)$.

The standard approach to finding the government's open loop tax policy is to maximize $\int_0^\infty e^{-\rho t}[U(c(t)) + V(G(t))]\,dt$ with respect to the tax policy $\{\tau(t)\}_{t=0}^{t=\infty}$, imposing the necessary conditions to the agent's optimization problem (1).[1] One necessary condition is the equation of motion for the costate variable for the state k. Denote this costate variable as $q(t)$. Assuming differentiability of the value function, we have $J_k \equiv q$. Xie uses the following:

[1] Simaan and Cruz [5] were among the first to use this method of solving Stackelberg differential games. In addition to the examples cited in [6], applications include [2–4].

Definition 2. The costate variable q is "uncontrollable" if and only if its value at time t is independent of current and future government actions $\{\tau(s)\}_{s=t}^{\infty}$.

Since consumption at a point in time depends only on the costate variable, consumption is uncontrollable if and only if the costate variable is uncontrollable.

Xie's Proposition 1 states that the open-loop linear income tax ($g(k,t) = f(k)\tau(t)$) is time consistent only if q is uncontrollable. We have a slightly more general result. (Appendix A contains proofs that are not included in the text.)

Lemma 1. *Suppose that Assumptions* 1–3 *hold. The open-loop tax policy is time consistent if and only if consumption is uncontrollable.*

Using this lemma we obtain:

Proposition 1. *Under Assumptions* 1–3, *the government's open-loop policy is time consistent if and only if the agent's value function is additively separable in the state variable and time*: $J(k,t) = W(k) + Z(t)$ *for some functions* $W(k)$ *and* $Z(t)$.

Proposition 1 implies that time consistency of an open-loop policy requires the function $b(k)$ to be proportional to the reciprocal of the shadow value of capital:

Corollary 1. *Suppose that the agent's value function is additively separable in the state variable and time. Then there exists a constant* α *such that*

$$W_k(k)b(k) = \alpha. \tag{3}$$

Proof. From Proposition 1, the optimal consumption rule is a function only of k. Substituting this optimal rule, $c = c^*(k)$, into Eq. (2) and using $J(k,t) = W(k) + Z(t)$ from Proposition 1, we write the agent's maximized Bellman equation as

$$\rho(W(k) + Z(t)) = U(c^*(k)) + W_k(k)[f(k) - \tau(t)b(k) - c^*(k)] + Z_t(t). \tag{4}$$

The additive separability of $J(k,t)$ requires that the right side of Eq. (4) must also be additively separable for any admissible $\tau(t)$. This requirement implies Eq. (3) with α equal to a constant. □

The left side of Eq. (4) is the present discounted value of future utility, expressed as an annuity with discount rate ρ. The reduction in the value of this annuity (i.e., the reduction in the value of the agent's program), caused by the tax, is $W_k(k)[\tau(t)b(k)]$. Eq. (3) means that this effect of the tax is independent of the value of k. The agent views a time-consistent tax like a lump sum reduction in the dollar value of future utility, equal to $\frac{\alpha\tau(t)}{\rho}$.

The previous results lead to the following necessary and sufficient condition for time consistency.

Proposition 2. *Under Assumptions* 1–3, *the government's open-loop policy is time consistent if and only if*

$$U'\left(\left[\frac{\rho - f_k(k)}{b_k(k)}\right]b(k) + f(k)\right)b(k) = \alpha. \tag{5}$$

The proof of this proposition shows that the agent's consumption rule is

$$c^*(k) = \left(\frac{\rho - f_k(k)}{b_k(k)}\right)b(k) + f(k). \tag{6}$$

We refer to Eq. (5) as the *consistency constraint*, since the government's optimal program is time consistent if and only if it holds.[2] Proposition 2 extends Xie's Proposition 3 in two respects. First, it shows that the possibility of time consistency is very general. Given primitive functions U and f, we can construct b to obtain a time-consistent tax. Xie restricts $b(k) \equiv f(k)$, i.e. he assumes that the government must use a linear income tax. Second, our Proposition 2 is a necessary and sufficient condition, rather than only a sufficient condition.

Given the utility and production functions, the consistency constraint is an ordinary differential equation (ODE). The solution to this ODE depends on two parameters, α and a constant of integration that determines the boundary condition to Eq. (5). Denote this constant of integration by γ. The set of time-consistent rules is the two-parameter family of functions $b(k; \alpha, \gamma)$; when there is no ambiguity, we suppress the arguments α and γ.

4. Stationarity

If the function $b(k)$ satisfies the time consistency constraint, then for values of k *along the optimal trajectory* the government has no incentive to revise the optimal time-dependent component of the tax $\tau(t)$ announced at time 0. If for some reason the state k departs from the equilibrium path, the government might want to change the original open-loop policy $\tau(t)$. In that case, the optimal open-loop policy is not subgame perfect.

However, if for all initial conditions the optimal function $\tau(t)$ announced at time 0 is a constant that is independent of the initial condition (i.e., if the policy is stationary) then the policy is obviously subgame perfect. The following proposition provides a restriction involving the primitive functions U, V, and f and the tax policy b that is necessary and sufficient for the optimal τ to be a constant. We assume that the steady state is independent of the initial condition.

[2] We mentioned that Assumption 3 is unnecessarily restrictive. Additive separability of the value function only requires $g(k,t) = x(k) + b(k)\tau(t)$ for some function $x(k)$ (rather than $g(k,t) = b(k)\tau(t)$ as Assumption 3 maintains). We can repeat the steps used to derive Eq. (5) to obtain the consistency constraint for the more general tax rule.

Proposition 3. *Suppose that $b(k)$ satisfies Eq.* (5) (*so that the tax is time consistent*) *and that $c = c^*(k)$ satisfies Eq.* (6) (*so that the agent behaves optimally*). *The government takes $b(k)$ and $c^*(k)$ as given and chooses the time-dependent component of the tax $\tau(t)$, in order to solve*

$$\max_{\{\tau(t)\}} \int_0^\infty e^{-\rho t}(U(c^*(k)) + V(\tau b(k)))\, dt$$
$$s.t. \frac{dk}{dt} = f(k(t)) - \tau(t)b(k) - c^*(k), \quad k_0 \text{ given.} \tag{7}$$

The optimal trajectory $\tau^*(t)$ is a constant τ (*independent of the initial condition*) *if and only if there exists a τ such that the following equation holds identically in k*:

$$(\rho + \eta)V' = (f' + \eta)U' - (\rho - f' + \tau b')\tau b V'' \tag{8}$$

where

$$\eta = -\left(\frac{U'b'}{U''b} + f'\right). \tag{9}$$

We refer to Eq. (8) as the stationarity constraint.

We showed above that given the primitive functions U and f, the set of time-consistent policy rules $b(k)$ is a two-parameter family of functions that depend on α and γ. If the government is required to use a time-consistent policy, then at time 0 it is able to choose α and γ and the open-loop trajectory $\tau(t)$ to maximize its payoff. If we also impose the requirement that the policy is stationary, then the policy rule $b(k)$ must satisfy Eq. (8). In general, there is no guarantee that there exist functions $b(k)$ and $c^*(k)$ that satisfy Eqs. (5), (6), and (8). The next section uses examples to show that such functions exist in some cases.

It might appear that when imposing the stationarity constraint we obtain an additional degree of freedom, the parameter τ. That is, it might appear that in selecting a stationary time-consistent policy the government is able to choose three parameters, α, γ and τ. This interpretation is incorrect. Without loss of generality, we can normalize by setting $\tau = 1$. The government has only two free parameters, α and γ.

5. Examples with logarithmic utility

We use an example with logarithmic utility to illustrate the relation between the time-consistent policy and the production function $f(k)$. We then show how the stationarity requirement reduces the set of time-consistent policies.

For $U(c) = \ln c$, Eq. (5) can be written as

$$\frac{db}{dk} = \alpha b \frac{-\rho + \frac{df}{dk}}{\alpha f - b}. \tag{10}$$

Eq. (10) illustrates that (given the utility function) we can treat either the production function $f(k)$ or the tax function $b(k)$ as primitive; using (10) to solve for the other function, we obtain a time-consistent tax.

Substituting Eq. (10) into Eq. (6) we obtain

$$c = \frac{\rho - \frac{df}{dk}}{\alpha(-\rho + \frac{df}{dk})}(\alpha f - b) + f = \frac{b}{\alpha}. \tag{11}$$

Thus, under logarithmic utility the use of a time-consistent tax implies that $c = \frac{b}{\alpha}$.[3] The government chooses the constant α. The relation between b and c does not depend on the production function $f(k)$. The time-consistency constraint does not restrict the ratio between public and private expenditures, since the government is able to choose the function $\tau(t)$.

Suppose that we take the production function as primitive[4] and moreover we assume that production is linear: $f(k) = Ak$, $A > 0$. In this case, inspection of Eq. (10) confirms that the affine tax is a particular solution; i.e. the linear tax is time consistent. The linear wealth tax in this case is $b = \alpha\rho K$, and the corresponding income tax is $b = \frac{\alpha\rho}{A}Ak$. The general solution to the ODE gives $b(k)$ as an implicit function of k. We can invert that implicit function to write k as an explicit function of b:

$$k = \frac{b}{\alpha\rho} + \gamma b^{\frac{A}{-\rho+A}}. \tag{12}$$

Any tax rule that solves this implicit equation is time consistent. Some of these taxes may give the regulator a higher payoff than the linear tax. Provided that $\tau(t) > 0$, convexity of $b(k)$ insures that the necessary conditions to the agent's problem are sufficient to give an optimum. Convexity of $b(k)$ holds if and only if $\gamma \leqslant 0$.[5]

We now consider the stationary time-consistent policy for the case where $U(c) = \ln c$ and $V(G) = \ln(G) = \ln(b)$. The last equality uses the budget constraint $G = \tau b(k)$ and the normalization $\tau = 1$. Using the definition of η (Eq. (9)) and Eq. (11) we have

$$\eta = \frac{c\frac{db}{dk}}{b} - \frac{df}{dk} = \frac{\frac{db}{dk} - \alpha\frac{df}{dk}}{\alpha}.$$

[3] In the case where $U(c) = \frac{c^{1-\sigma}}{1-\sigma}$, $0 < \sigma < 1$, we can show that if the tax rule satisfies the consistency condition, then the consumption rule is $c = \alpha^{-\frac{1}{\sigma}} b^{\frac{1}{\sigma}}$. This result reduces to Eq. (11) as $\sigma \to 1$ (i.e. for logarithmic utility). Suppose, in addition, that $f(k) = k^{\theta}$. In the case where $\theta = \sigma$ it is straightforward to show that $b(k) = f(k)$ satisfies the consistency condition, as Xie's Proposition 2 states. For $\theta \neq \sigma$ we can solve the consistency condition numerically to obtain time-consistent tax rules.

[4] As noted above, we can also take the tax function as primitive. An earlier version of this paper shows that the linear income tax ($b(k) \equiv f(k)$) is time consistent if and only if the production function is affine; the linear capital tax ($b(k) \equiv k$) is time consistent if and only if $f(k) = k\mu \ln k + k\gamma$, with $\mu \equiv \frac{1-\alpha\rho}{\alpha}$ and $\alpha \geqslant \frac{1}{\rho}$.

[5] For $\gamma = 0$, $b(k)$ is obviously monotonically increasing. For $\gamma < 0$, $b(k)$ is monotonically increasing for all k if $A < \rho$. If $\gamma < 0$ and in addition $A > \rho$, then $b(k)$ is increasing provided that k is less than a critical value. This critical value can easily be obtained from Eq. (12).

Using this expression and Eq. (11), Eq. (8) simplifies to

$$\frac{db}{dk} = \frac{db}{dk}\frac{c-b}{\alpha c} = \frac{db}{dk}\frac{b-\alpha b}{\alpha b} = \frac{db}{dk}\frac{1-\alpha}{\alpha}. \tag{13}$$

Since $\frac{db}{dk} \neq 0$ Eq. (13) implies that $\alpha = \frac{1}{2}$.

For logarithmic utility, the time-consistent stationary tax is a solution to Eq. (10) with $\alpha = \frac{1}{2}$. The stationarity constraint removes one degree of freedom from the government, by pinning down the value of α. The government still has one degree of freedom: it chooses the boundary condition to the ODE (10); i.e., the government chooses the parameter γ.

We noted that the time-consistency constraint does not restrict the equilibrium ratio of public and private expenditures. The stationarity constraint, however, implies that consumption of the private good is twice the level of consumption of the public good. This result does not depend on the production function or on the particular time-consistent stationary tax that the government uses.

If we specialize further by choosing linear production, $f(k) = Ak$, the stationary time-consistent tax is a solution to Eq. (12) with $\alpha = \frac{1}{2}$. Suppose, in addition, we assume that government uses an affine tax, i.e. a tax of the form $b = \beta k + \phi$. Substituting this expression into Eq. (10) and equating coefficients of powers of k, we conclude that $\phi = 0$ and $\beta = \frac{\rho}{2}$. Since income equals $f(k) = Ak$, the unique affine income tax is $\frac{\rho}{2A}$. This result reproduces Eq. (23) in Xie. The result also shows (for logarithmic utility and linear production) that any affine tax is linear: it never involves a lump sum tax/subsidy.

The linear tax does not enable the government to achieve the first best outcome. Under the linear tax $c = 2b = \rho k$, and under the first best outcome $c = b = \frac{\rho}{2}k$. Consequently we cannot rule out the possibility that the government has a higher payoff if it uses one of the nonlinear taxes that solve Eq. (12) with $\alpha = \frac{1}{2}$.

Appendix. Proofs

Proof of Lemma 1. The argument that establishes Xie's Proposition 1 also demonstrates the "only if" part of the claim in the more general case where $g(k,t) = b(k)\tau(t)$, e.g. where $b(k) \neq f(k)$; we do not repeat the argument here. To establish the "if" part, note that in the case where consumption is not controllable, the government has a control problem with one state variable, k, the initial value of which is predetermined. (When $q_t = q(k_t)$, i.e., when $q(t)$ is independent of $\tau(s)$, $s \geqslant t$, the initial value of q is not free.) The solution to this kind of control problem satisfies the dynamic programming Principle of Optimality, and is thus time consistent. □

The proof of Proposition 1 requires an intermediate result. A perturbation of a "reference" tax policy is

$$\tau(t) = \tau^*(t) + ah(t),$$

where $\tau^*(t)$ is the reference tax policy, $h(t)$ is a continuously differentiable function of time which represents a perturbation, and a is a scaling parameter for perturbation. When $a = 0$ we obtain the reference policy. The value function of the consumer is now parameterized by a, given the perturbation function $h(t)$ and the policy function $\tau^*(t)$. We write this value function as $\tilde{J}(k_t, \{\tau(s)\}_{s=t}^{\infty})$ to emphasize that the agent's payoff depends on the future trajectory of taxes. For a fixed tax trajectory $\{\tau(s)\}_{s=t}^{\infty}$ the only exogenous time-dependent change arises from the change in the minimum value of the time dummy, $s = t$. Thus, $\tilde{J}(k_t, \{\tau(s)\}_{s=t}^{\infty}) \equiv J(k_t, t)$. In other words, the second argument in the function $J(k_t, t)$ "summarizes" the effect, on the agent's payoff, of the future sequence of taxes, $\{\tau(s)\}_{s=t}^{\infty}$. Using this notation we obtain:

Lemma A.1. *Suppose Assumptions* 1–3 *hold. The open-loop optimal tax policy is time consistent if and only if*

$$\frac{\partial}{\partial a}\frac{\partial \tilde{J}(k_t, \{\tau(s)\}_{s=t}^{\infty})}{\partial k} = 0 \tag{A.1}$$

for all admissible $\tau^*(t)$ *and* $h(t)$.

Proof. By the first order condition to the Bellman equation (2), consumption equals $U'^{-1}(J_k)$, i.e. consumption depends only on the shadow value of capital. By the identity $\tilde{J}(k_t, \{\tau(s)\}_{s=t}^{\infty}) \equiv J(k_t, t)$, the shadow value of the state, J_k (and thus consumption) is uncontrollable if and only if Eq. (A.1) holds for all admissible $\tau^*(t)$ and $h(t)$. By Lemma 1, Eq. (A.1) is therefore necessary and sufficient for time consistency. □

Proof of Proposition 1. We first establish the "only if" part of the proposition. From Lemma A.1, time consistency implies Eq. (A.1), which implies that $\frac{\partial \tilde{J}(k_t, \{\tau(s)\}_{s=t}^{\infty})}{\partial K} = \psi(k)$ for some function $\psi(\cdot)$. Taking the integral of both sides,

$$\tilde{J} = \int^{k} \psi(k)\, dk + Z(t), \tag{A.2}$$

where $Z(t)$ is the constant of integration. The identity $\tilde{J}(k_t, \{\tau(s)\}_{s=t}^{\infty}) \equiv J(k_t, t)$ completes the demonstration.

To establish the "if" part of the proposition we can simply note that Eq. (A.2) implies Eq. (A.1), and then invoke Lemma A.1. □

Proof of Proposition 2. The proof of the sufficient part is an adaptation of [6, Proposition 3, p. 419]. Suppose that Eq. (5) is satisfied. It suffices to show that under this condition, the consumption path is independent of the time-dependent part of tax policy.

Consider the consumption plan given by Eq. (6). We can verify that this consumption plan satisfies the first order conditions for the agent's control problem and the transversality condition. (This verification uses the same steps as Xie's proof.) Moreover this consumption plan is independent of the time-dependent part of tax policy, $\tau(t)$ and therefore is uncontrollable.

The necessary part follows from Corollary 1. Using (3) in (4) we obtain an equation for $W(k)$:

$$\rho W(k) = U(c^*(k)) + W_k(k)[f(k) - c^*(k)]. \tag{A.3}$$

Eqs. (3) and (A.3) hold identically. We differentiate them with respect to k to obtain

$$W_{kk}b + W_k b_k = 0, \tag{A.4}$$

$$\rho W_k = W_{kk}(f - c) + W_k f_k. \tag{A.5}$$

Using (A.4) and (A.5) we can solve for the optimal consumption rule $c^*(k)$ to obtain Eq. (6). Finally, using the first order condition $U'(c) = W_k$ and Eqs. (3) and (6) we obtain Eq. (5). □

We use a lemma to establish Proposition 3. Denote $\tau^{\mathrm{o}}(t;k_0)$ as the open-loop representation of the solution to the problem (7), and denote $\tau^{\mathrm{fb}}(k)$ as the feedback representation of the solution to this problem. We assume that the steady state to this problem, k^{ss}, is independent of the initial condition. The proof of Proposition 3 uses the following lemma:

Lemma A.2. *Given a time-consistent policy rule $b(k)$, the optimal $\tau^{\mathrm{o}}(t)$, is a constant (independent of the initial condition) for all initial conditions $k_0 \neq k^{\mathrm{ss}}$ if and only if the feedback form of the policy, $\tau^{\mathrm{fb}}(k)$ is independent of k.*

Proof. Taking as given the parameters α and γ, Eqs. (6) and (5) determine the functions $c(k;\alpha,\gamma)$ and $b(k;\alpha,\gamma)$. Given these functions, the government solves an autonomous control problem. The value of its program is a function $S(k;\alpha,\gamma)$ that satisfies the Bellman equation

$$\rho S(k) = \max_{\tau} \{U(c(k)) + V(\tau b(k)) + S_k(k)(f(k) - c(k) - \tau b(k))\}. \tag{A.6}$$

The solution to the government's stationary control problem is a policy rule $\tau^{\mathrm{fb}}(k)$. Using this function and the agent's control rule, we can solve the equation for $\frac{dk}{dt}$ to obtain $k^{\mathrm{e}}(t;k_0)$, the equilibrium value of the state ("e" denotes equilibrium) at time t. Substituting $k^{\mathrm{e}}(t;k_0)$ into the government's policy rule, we obtain the open-loop representation of the policy, $\tau^{\mathrm{o}}(t,k_0) \equiv \tau^{\mathrm{fb}}(k^{\mathrm{e}}(t;k_0))$. Since $k_0 \neq k^{\mathrm{ss}}$, $\frac{dk^{\mathrm{e}}}{dt} \neq 0$ along the optimal trajectory. Therefore $\tau^{\mathrm{o}}(t,k_0)$ is a constant if and only if $\tau^{\mathrm{fb}}(k)$ is independent of k. □

Proof of Proposition 3. In view of Lemma A.2 we need to show that the feedback form of the government's control rule, $\tau^{\mathrm{fb}}(k)$ is independent of k. We begin with some notation, for the purpose of simplifying the derivations. Use Eq. (6) to write

$$f - c = -\frac{\rho - f'}{b'} b. \tag{A.7}$$

Define

$$\eta \equiv \frac{d}{dk}\left(\frac{\rho - f'}{b'} b\right). \tag{A.8}$$

With this notation we have

$$f' - c' = -\eta. \tag{A.9}$$

Differentiating both sides of Eq. (5) with respect to k, using definition (A.8), implies

$$U'b' + U''[\eta + f']b = 0.$$

Rearranging this expression gives Eq. (9).

We now proceed with the main argument. Using Eq. (A.7), we rewrite the government problem (A.6) as

$$\rho S = \max_{\tau}\left\{U + V - S'\left[\frac{\rho - f'}{b'} + \tau\right]b\right\}. \tag{A.10}$$

The first order condition to (A.10) is

$$V'(\tau b(k)) = S'(k). \tag{A.11}$$

Eq. (A.11) implicitly defines the optimal tax rule $\tau(k)$. Substituting this tax rule into Eq. (A.11) and taking the derivative of both sides, we obtain

$$S'' = V''(\tau b' + b\tau'). \tag{A.12}$$

Substituting the optimal tax rule $\tau(k)$ into (A.10) we obtain the maximized Bellman equation. We take the derivative with respect to k of both sides of the maximized Bellman equation to obtain

$$\rho S' = U'c' + V'\tau b' - S'(\eta + \tau b') - S''\left(\frac{\rho - f'}{b'} b + \tau b\right). \tag{A.13}$$

We then use Eqs. (A.9) to eliminate c', and Eqs. (A.11) and (A.12) to eliminate S' and S'' from Eq. (A.13). The resulting equation and the identity $\tau'(k) \equiv 0$ are both satisfied if and only if Eq. (8) holds identically in k. □

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