Suggested Solutions to Problem Set 5*

1. The rule for a single rotation is to choose the cropping age T when $Q'(T) = rQ(T)$.

(a) In this exercise, the change in growth is :

$$Q'(T) = 9(1-0.8e^{-0.87T})^2(-0.8e^{-0.87T})(-0.87) = 6.264(1-0.8e^{-0.87T})^2 e^{-0.87T}$$

The right hand side of the optimal condition is:

$$rQ(T) = 0.1\times 3(1-0.8e^{-0.87T})^3 = 0.3(1-0.8e^{-0.87T})^3$$

Our condition becomes:

$$Q'(T) = 6.264(1-0.8e^{-0.87T})^2 e^{-0.87T} = 0.3(1-0.8e^{-0.87T})^3 = rQ(T)$$

Dividing both sides by 0.3 and by $(1-0.8e^{-0.87T})^2$ gives:

$$20.88e^{-0.87T} = 1-0.8e^{-0.87T}$$

Collecting terms in *e* gives $e^{-0.87T} = \frac{1}{21.68} = 0.0461$. Taking the natural logarithm of each side yields: $\ln(0.0461) = -3.0764 = -0.87T \Rightarrow T = 3.536\ \textit{years}$ and the conclusion that a rhino's horn should be cut off after three and half years of growth.

(b.i) Factors that can reduce the rotation period include

- Higher interest rates
- Cropping more than once, i.e., regrowth
- Anticipated future sales restrictions (Why let something grow until nest year if you won't have a market in which to sell it.)
- Note: There is a possibility that the animal dies while under anesthetic during the horn-removal operation. This will lengthen the optimal rotation time. (That is, uncertainty will affect when something is cut.)

* Solutions provided by S. Marceau.

(b.ii) Factors that may explain more frequent dehorning in reality include

- Real higher interest rates (e.g., 30% rather than 10%)
- Open access
- Poaching

2. This is a renewable resource model. What is important are the biological growth function, the concept of steady state and the difference between the outcome under open access and the outcome that maximizes net benefits.

(a) At carrying capacity, the growth is zero:

$$g(S_{\max}) = 0 \Rightarrow 0.4S_{\max} - 0.00002(S_{\max})^2 = 0 \Rightarrow S_{\max} = \frac{0.4}{0.00002} = 20000$$

Maximum sustainable yield (MSY) is where g(S) is as large as possible. Find this by setting the derivative of the biological growth function equal to zero:

$$g'(S) = 0 \text{ at } MSY \Rightarrow 0.4 - 0.00004S_{msy} = 0 \Rightarrow S_{msy} = \frac{0.4}{0.00004} = 10000$$

The annual harvest at MSY is $g(S_{msy}) = 0.4 \times 10000 - 0.00002 \times 10000^2 = 2000$. These are shown on the graph.

(b) This describes an open access situation. Deer will be hunted until "profits" are driven to zero:

$$\Pi = MB \cdot x - C(x,S) - lx = 0 \Rightarrow 12x - \frac{60000x}{S} - 4x = 0 \Rightarrow S_{oa} = \frac{60000}{8} = 7500$$

Revenues will depend on the deer catch, i.e., the harvest level:

$$x_{oa} = g(S_{oa}) = 0.4 \times 7500 - 0.00002 \times 7500^2 = 1875$$

Revenues are $lx = 4 \times 1875 = 7500$.

(c) The population of deer on government-owned land is 7500 deer. This is less than the maximum sustainable yield $\left(S_{oa} = 7500 < S_{msy} = 10000\right)$.

(d) If the interest rate is zero, then the condition for maximizing net over time is: $l g'(S) = \frac{\partial C(x,S)}{\partial x}$. If r >0 , then the condition is $r l = g'(S) l - \frac{\partial C(x,S)}{\partial x}$.

Assume r = 0. Each of the required components can be found:

$$l = p - \frac{\partial C(x,S)}{\partial x} = 12 - \frac{60000}{S}$$

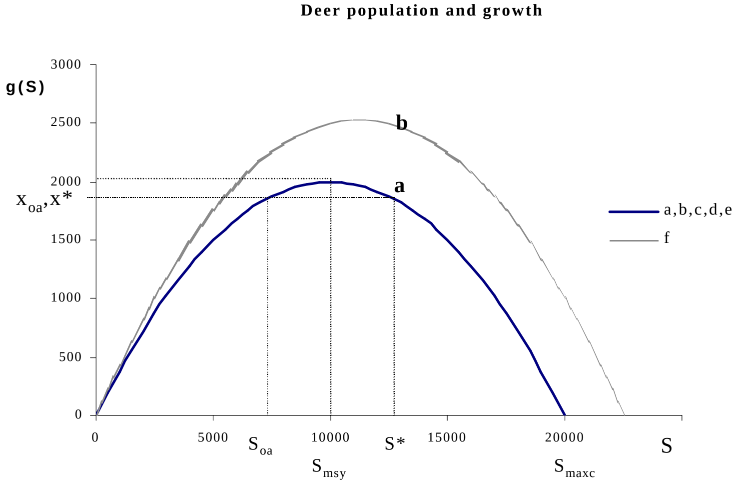
$$g'(S) = 0.4 - 0.00004S \Rightarrow l g'(S) = \left(12 - \frac{60000}{S}\right)\left(0.4 - 0.00004S\right) = 7.2 - 0.00048S - \frac{2400}{S}$$

$$\frac{\partial C(x,S)}{\partial S} = -\frac{60000x}{S} = -\frac{60000g(S)}{S} = -60000\left(\frac{0.4}{S} - 0.00002\right) = -\frac{2400}{S} + 1.2$$

Equate the last two to get $7.2 - 0.00048S - \frac{2400}{S} = -\frac{2400}{S} + 1.2 \Rightarrow S^* = \frac{6}{0.00048} = 12500$

Harvest each year will be $x^* = g(12500) = 0.4 \times 12500 - 0.00002 \times 12500^2 = 1875$.

(e) The deer population on private land is S = 12500, which is larger than the population on public land. The farmer takes into account the effect of growth on subsequent costs of harvesting. A lower stock ensures lower costs of hunted a given amount of deer. This stock effect is only relevant when you consider how your harvest today affects the amount of the stock you will have in the future. Open access harvesters don't consider the future. They only look at their current costs, which results in them



(b) In the steady state, we must have $x = g(S)$. Under open access, we know that hunters will earn zero profits. So, we know that $\Pi = px - C(x,S) - lx = 0$. We can write the two constraints as (i) $g(S) - x = 0$ and (ii) $px - C(x,S) - lx = 0$.

(c) Since the government sets the fee, the harvest level and the stock level, the control variables are *l*, *x* and *S*.

(d) $L\{l, x, S, l_1, l_2\} = lx + l_1\big(g(S) - x\big) + l_2\big(px - C(x,S) - lx\big)$

(e) We have the first order conditions when we take the first derivative of the Langrangian with respect to each the control variables and the multipliers and set those derivatives equal to zero as follows:

$$\frac{\partial L}{\partial l} = x - l_2 x = 0 \qquad (1)$$

$$\frac{\partial L}{\partial x} = l - l_1 + l_2\left(p - \frac{\partial C(x,S)}{\partial x} - l\right) = 0 \qquad (2)$$

$$\frac{\partial L}{\partial S} = l_1 g'(S) - l_2 \frac{\partial C(x,S)}{\partial S} = 0 \qquad (3)$$

$$\frac{\partial L}{\partial S} = g(S) - x = 0 \qquad (4)$$

$$\frac{\partial L}{\partial S} = px - C(x.S) - lx = 0 \qquad (5)$$

(f) Equation 1 implies $\lambda_2 = 1$. Substituting this into equation 2 gives:

$$l - l_1 + p - \frac{\partial C(x,S)}{\partial x} - l = 0 \; and \; p - l_1 = \frac{\partial C(x,S)}{\partial x} \qquad (6)$$

Substituting it into equation 3 gives:

$$l_1 g'(S) = \frac{\partial C(x,S)}{\partial S} \; and \; l_1 = \frac{\partial C(x,S)}{\partial S}\frac{1}{g'(S)} \qquad (7)$$

If we put (6) and (7) together, we find $p - \frac{\partial C(x,S)}{\partial S}\frac{1}{g'(S)} = \frac{\partial C(x,S)}{\partial x}$ which can be re-written as $\left[p - \frac{\partial C(x,S)}{\partial x}\right] g'(S) = \frac{\partial C(x,S)}{\partial S}$. According to equation 4, we could also replace *x* with *g(S)* and we would have an equation strictly in terms of *S*.

(g) The remaining equation is the exact same equation we have for the optimal use of the resource when the interest rate is zero. So, we know that S = 12 500 and x = 1875. How do we get *l*, the fee? From equation 5, we can get $l = p - \frac{C(x,S)}{x}$. Using the cost function given in question 2 and the value for *S* implies:

$$l = p - \frac{60000}{S} = 12 - \frac{60000}{12500} = 12 - 4.8 = 7.2$$

The fee that maximizes annual revenues is l = 7.20.