Atemporal Uncertainty

How to evaluate?
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How to evaluate?
Time & Uncertainty Aggregation

Atemporal Uncertainty

How to evaluate?

von Neumann & Morgenstern (1944):

'vNM axioms' (weak order, continuity, independence)

exists \( u \) such that

\[ U \equiv E_p u = \sum_i p_i u(x^i) = \int_X u \, dp \]

represents preferences
‘Intertemporal Certainty’

How to evaluate?
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Additive Separability

$$x_1 \quad x_2 \quad x_3 \quad \ldots \ldots \quad x_T$$

$$t=1 \quad t=2 \quad t=3 \quad \ldots \ldots \quad t=T$$
‘Intertemporal Certainty’

How to evaluate?

Koopmans (1960), ..., (Wakker 1988):

Additive Separability (also: Certainty Additivity, Coordinate Independence,...)

\[ u_1(x_1) - u_2(x_2) - u_3(x_3) \ldots \ldots \ u_T(x_T) \rightarrow \sum_t \]

exist \( u_1, \ldots, u_T \) such that \( U \equiv \sum_t u_t(x_t) \) represents preferences

\[ t=1 \quad t=2 \quad t=3 \quad \ldots \ldots \quad t=T \]
‘Real World’: intertemporal uncertainty
‘Real World’: intertemporal uncertainty

\[ x_1^1 \quad x_2^1 \quad x_3^1 \quad \ldots \quad x_T^1 \]

\[ x_1^2 \quad x_2^2 \quad x_3^2 \quad \ldots \quad x_T^2 \]

\[ x_1^3 \quad x_2^3 \quad x_3^3 \quad \ldots \quad x_T^3 \]

\[ x_1^4 \quad x_2^4 \quad x_3^4 \quad \ldots \quad x_T^4 \]

\[ x_1^5 \quad x_2^5 \quad x_3^5 \quad \ldots \quad x_T^5 \]

\[ t=1 \quad t=2 \quad t=3 \quad \ldots \quad t=T \]
‘Real World’: intertemporal uncertainty

\[ x_1^1, x_2^1, x_3^1, \ldots, x_T^1 \]
\[ x_1^2, x_2^2, x_3^2, \ldots, x_T^2 \]
\[ x_1^3, x_2^3, x_3^3, \ldots, x_T^3 \]
\[ x_1^4, x_2^4, x_3^4, \ldots, x_T^4 \]
\[ x_1^5, x_2^5, x_3^5, \ldots, x_T^5 \]

\[ t=1, t=2, t=3, \ldots, t=T \]
Time & Uncertainty Aggregation

\[
\begin{align*}
  &u_1(x_1^1) = u_2(x_2^1) = u_3(x_3^1) = \ldots = u_T(x_T^1) \\
  &u_1(x_1^2) = u_2(x_2^2) = u_3(x_3^2) = \ldots = u_T(x_T^2) \\
  &u_1(x_1^3) = u_2(x_2^3) = u_3(x_3^3) = \ldots = u_T(x_T^3) \\
  &u_1(x_1^4) = u_2(x_2^4) = u_3(x_3^4) = \ldots = u_T(x_T^4) \\
  &u_1(x_1^5) = u_2(x_2^5) = u_3(x_3^5) = \ldots = u_T(x_T^5)
\end{align*}
\]
Standard Model: intertemporally additive expected utility

\[ U = \mathbb{E}_p \sum_t u_t(x_t) \]
Reconsider: an axiomatic reasoning
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\[ x_1^1 \quad x_2^1 \quad x_3^1 \quad \ldots \quad x_T^1 \]
\[ x_1^2 \quad x_2^2 \quad x_3^2 \quad \ldots \quad x_T^2 \]
\[ x_1^3 \quad x_2^3 \quad x_3^3 \quad \ldots \quad x_T^3 \]
\[ x_1^4 \quad x_2^4 \quad x_3^4 \quad \ldots \quad x_T^4 \]
\[ x_1^5 \quad x_2^5 \quad x_3^5 \quad \ldots \quad x_T^5 \]

\[ t=1 \quad t=2 \quad t=3 \quad \ldots \quad t=T \]
Reconsider: an axiomatic reasoning

\[
\exists u_i \text{s.th. } u_1(x_1^1) \quad u_2(x_1^2) \quad u_3(x_1^3) \quad \ldots \quad u_T(x_1^T)
\]

\[
\exists u_i \text{s.th. } u_1(x_2^1) \quad u_2(x_2^2) \quad u_3(x_2^3) \quad \ldots \quad u_T(x_2^T)
\]

\[
\exists u_i \text{s.th. } u_1(x_3^1) \quad u_2(x_3^2) \quad u_3(x_3^3) \quad \ldots \quad u_T(x_3^T)
\]

\[
\exists u_i \text{s.th. } u_1(x_4^1) \quad u_2(x_4^2) \quad u_3(x_4^3) \quad \ldots \quad u_T(x_4^T)
\]

\[
\exists u_i \text{s.th. } u_1(x_5^1) \quad u_2(x_5^2) \quad u_3(x_5^3) \quad \ldots \quad u_T(x_5^T)
\]

\[
E_p \quad E_p \quad E_p \quad \ldots \quad E_p
\]

\[
t=1 \quad t=2 \quad t=3 \quad \ldots \quad t=T
\]
Reconsider: an axiomatic reasoning

$$p_1 x_1^1 + p_2 x_1^2 + p_3 x_1^3 + p_4 x_1^4 + p_5 x_1^5$$

$$p_1 x_2^1 + p_2 x_2^2 + p_3 x_2^3 + p_4 x_2^4 + p_5 x_2^5$$

$$p_1 x_3^1 + p_2 x_3^2 + p_3 x_3^3 + p_4 x_3^4 + p_5 x_3^5$$

$$p_1 x_4^1 + p_2 x_4^2 + p_3 x_4^3 + p_4 x_4^4 + p_5 x_4^5$$

$$p_1 x_5^1 + p_2 x_5^2 + p_3 x_5^3 + p_4 x_5^4 + p_5 x_5^5$$

$$x_1^1 , x_2^1 , x_3^1 , ..., x_T^1$$

$$x_1^2 , x_2^2 , x_3^2 , ..., x_T^2$$

$$x_1^3 , x_2^3 , x_3^3 , ..., x_T^3$$

$$x_1^4 , x_2^4 , x_3^4 , ..., x_T^4$$

$$x_1^5 , x_2^5 , x_3^5 , ..., x_T^5$$

$$t=1, t=2, t=3, ..., t=T$$
Reconsider: an axiomatic reasoning

∃ $u_i$ s.th. $u_1(x_1^1) - u_2(x_2^1) - u_3(x_3^1)$ \ldots $u_T(x_T^1) \rightarrow \sum_t$

$p_1$

$p_2$

$p_3$

$p_4$

$p_5$

$t=1$ $t=2$ $t=3$ \ldots $t=T$
Reconsider: an axiomatic reasoning

However: In general $u_i$
Reconsider: an axiomatic reasoning

However: In general $u_i \neq u_i$!
Reconsider: an axiomatic reasoning

However: In general $u_i \neq u_i$!

Thus: Intertemporally additive expected utility model only represents very particular preferences where $u_i = u_i$!
Question: What preferences are missing in the standard model?
Question: What preferences are missing in the standard model?

Answer: Those with a non-trivial intertemporal risk attitude!
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For a definition assume $T = 2$ and define:

- $X$: (connected compact metric) space of goods
- $\Delta(\cdot)$: Set of Borel probability measures on space $\cdot$ (Prohorov metric)
Question: What preferences are missing in the standard model?

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For a definition assume $T = 2$ and define:

- $X$: (connected compact metric) space of goods
- $\Delta(\cdot)$: Set of Borel probability measures on space ‘.’ (Prohorov metric)
- $P_2 = \Delta(X)$: Probability measures $p_2$ on $X$
- $P_1 = \Delta(X \times P_2)$: Probability measures $p_1$ on $X \times P_2$
- $\succeq_2$: 2nd period preference relation on $P_2$ with elements $p_2$
- $\succeq_1$: 1st period preference relation on $P_1$ with elements $p_1$
**Question**: What preferences are missing in the standard model?

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For a definition assume $T = 2$ and define:

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- $\succeq_2$: 2nd period preference relation on $P_2$ with elements $p_2$
- $\succeq_1$: 1st period preference relation on $P_1$ with elements $p_1$
- $C^0(\cdot)$: Set of all continuous functions from space ‘·’ to $\mathbb{R}$
**Question:** What preferences are missing in the standard model?

**Answer:** Those with a non-trivial *intertemporal risk attitude*!

**Definition:** A decision maker is called *intertemporal risk averse* iff:

For all $\overline{x}_1, x_1, \overline{x}_2, x_2 \in X$ such that $\overline{x}_2 \succ_2 x_2$ and

$$ \overline{x}_1 \sim_1 x_1 \sim_1 \overline{x}_2 $$

it holds that
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\[
\begin{array}{c}
\bar{x}_1 & \sim_1 & x_1 & \sim_1 & \bar{x}_2 \\
\end{array}
\]

it holds that

\[
\begin{array}{c}
\bar{x}_1 & \succ_1 & x_1 \\
\frac{1}{2} & \begin{array}{c}
\bar{x}_1 \\
\bar{x}_2
\end{array} & \frac{1}{2} & \begin{array}{c}
x_1 \\
x_2
\end{array}
\end{array}
\]
Question: What preferences are missing in the standard model?

Answer: Those with a non-trivial *intertemporal risk attitude*!

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For all $x_1, x_1, x_2, x_2 \in X$ such that $x_2 \succeq_2 x_2$ and

```
\[ \overline{x}_1 \sim_1 - x_2 \quad \overline{x}_1 \sim_1 - x_2 \]
```

it holds that

```
\[ \overline{x}_1 \prec_1 - x_2 \quad \frac{1}{2} \quad \overline{x}_1 \sim_1 - x_2 \]
```
**Question:** What preferences are missing in the standard model?

**Answer:** Those with a non-trivial *intertemporal risk attitude*!

**Definition:** A decision maker is called *intertemporal risk neutral* iff:

For all $\overline{x}_1, x_1, \overline{x}_2, x_2 \in X$ such that $\overline{x}_2 \succeq_2 x_2$ and

$$
\overline{x}_1 \sim_1 x_1 \sim_1 \overline{x}_2
$$

it holds that

$$
\overline{x}_1 \sim_1 x_1 \sim_1 \overline{x}_2
$$
Uncertainty Aggregation Rule

For $f : \mathbb{R} \to \mathbb{R}$ continuous and strictly increasing and some compact metric space $Y$ define

1. $\mathcal{M}^f : \Delta(Y) \times C^0(Y) \to \mathbb{R}$
2. $\mathcal{M}^f(p, u) = f^{-1} \left[ \int_Y f \circ u \, dp \right]$
Uncertainty Aggregation Rule

For \( f : \mathbb{R} \to \mathbb{R} \) continuous and strictly increasing and some compact metric space \( Y \) define

\[
\mathcal{M}^f : \Delta(Y) \times C^0(Y) \to \mathbb{R} \\
\mathcal{M}^f(p, u) = f^{-1} \left[ \int_Y f \circ u \, dp \right]
\]

The uncertainty aggregation rule satisfies:

\[
\mathcal{M}^f(y, u) = u(y) \quad \forall y \in Y
\]
Representation I

Uncertainty Aggregation Rule

For \( f : \mathbb{R} \rightarrow \mathbb{R} \) continuous and strictly increasing and some compact metric space \( Y \) define

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\mathcal{M}^f(p, u) = f^{-1}\left[ \int_Y f \circ u \, dp \right]
\]

The uncertainty aggregation rule satisfies:

\[
\mathcal{M}^f(y, u) = u(y) \quad \forall \, y \in Y
\]

It includes rules corresponding to

- expected value \( (f = \text{id}) \)
- geometric mean \( (f = \ln) \)
- power mean \( (f^\alpha = \text{id}^\alpha) \) ‘CRRA type’

Example
Theorem 1:
The set of preference relations \((\succeq_1, \succeq_2)\) on \((P_1, P_2)\) satisfies

i) vNM axioms

ii) additive separability for certain consumption paths

iii) time consistency
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The set of preference relations \((\succeq_1, \succeq_2)\) on \((P_1, P_2)\) satisfies

1. vNM axioms
2. additive separability for certain consumption paths
3. time consistency

if and only if, there exist continuous functions \(u_t : X \rightarrow U_t \subset \mathbb{R}\) and a strictly increasing and continuous function \(f_t : U_t \rightarrow \mathbb{R}\) for \(t \in \{1, 2\}\) such that with defining \(\tilde{u}_2(x_2) = u_2(x_2)\) and
**Theorem 1:**
The set of preference relations \((\succeq_1, \succeq_2)\) on \((P_1, P_2)\) satisfies

1. **vNM axioms**
2. **additive separability for certain consumption paths**
3. **time consistency**

if and only if, there exist continuous functions \(u_t : X \to U_t \subset \mathbb{R}\) and a strictly increasing and continuous function \(f_t : U_t \to \mathbb{R}\) for \(t \in \{1, 2\}\) such that with defining \(\tilde{u}_2(x_2) = u_2(x_2)\) and

\[
\tilde{u}_1(x_1, p_2) = u_1(x_1) + M^{f_2}(p_2, \tilde{u}_2)
\]

it holds that for \(t \in \{1, 2\}\)

\[
p_t \succeq_t p_t' \iff M^{f_t}(p_t, \tilde{u}_t) \geq M^{f_t}(p_t', \tilde{u}_t) \quad \forall p_t, p_t' \in P_t.
\]
Theorem 2:

In the representation of theorem 1, a decision maker is intertemporal risk averse, if and only if, $f_1$ is strictly concave.
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In the representation of theorem 1, a decision maker is intertemporal risk averse, if and only if, $f_1$ is strictly concave.

Define the measure of relative intertemporal risk aversion as:

$$RIRA_t(z) = -\frac{f''_t(z)}{f'_t(z)} z.$$ 

The measures $RIRA_t$

▷ are uniquely defined if a zero welfare level is fixed, i.e. if $x^{zero}$ is chosen such that $u_t(x^{zero}) = 0$.

▷ It is independent of the commodity under observation and its measure scale.
The representation in theorem 1 uses the time additive utility functions $u_t$.

In a one commodity world and with $X \subset \mathbb{R}$ can also choose $u_t = \text{id}$. 
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→ recursion relation: $	ilde{u}_t = N^g_t(x_t, M^{f_{t+1}}(p_{t+1}, \tilde{u}_t))$

Where $N^g_t$ is a nonlinear intertemporal aggregation rule similar to $M^{f_t}$ parameterized by the function $g_t : U_t \rightarrow \mathbb{R}$. (slightly sloppy)
The representation in theorem 1 uses the time additive utility functions \( u_t \).

In a one commodity world and with \( X \subset \mathbb{R} \) can also choose \( u_t = \text{id} \).

\[ \rightarrow \text{recursion relation: } \tilde{u}_t = \mathcal{N}^{g_t}(x_t, \mathcal{M}^{f_t+1}(p_t+1, \tilde{u}_t)) \]

Where \( \mathcal{N}^{g_t} \) is a nonlinear intertemporal aggregation rule similar to \( \mathcal{M}^{f_t} \) parameterized by the function \( g_t : U_t \rightarrow \mathbb{R} \). (slightly sloppy)

For \( g_t(z) = \beta^t z^\rho \) and \( f_t(z) = z^\alpha \) Epstein & Zin’s (1991) generalized isoelastic model.

Can interpret

- \( 1 - \alpha \) as measure of Arrow-Pratt relative risk aversion
- \( (1 - \rho)^{-1} \) as intertemporal elasticity of substitution
Epstein Zin interpretation depends crucially on $u = \text{id}$ in
\[
\tilde{u}_t = \mathcal{N}^{g_t}(x_t, \mathcal{M}^{f+1}_t(p_{t+1}, \tilde{u}_t)).
\]
Multi-commodity usually: Directional derivative $\Rightarrow$ Doesn’t work!
Epstein Zin interpretation depends crucially on \( u = \text{id} \) in
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\]

Multi-commodity usually: Directional derivative \( \Rightarrow \) Doesn’t work!
Instead: Linearize utility in commodity \( i \) and determine \( f^i_t \) and \( g^i_t \).
As in standard model, risk measure depends on
\>
- particular commodity under observation
\>
- measure scale
Representation - Epstein Zin II

Epstein Zin interpretation depends crucially on \( u = \text{id} \) in
\[
\tilde{u}_t = \mathcal{N}^{g_t}(x_t, \mathcal{M}^{f_{t+1}}(p_{t+1}, \tilde{u}_t)).
\]

Multi-commodity usually: Directional derivative \( \Rightarrow \) Doesn’t work!

Instead: Linearize utility in commodity \( i \) and determine \( f_t^i \) and \( g_t^i \).

As in standard model, risk measure depends on
- particular commodity under observation
- measure scale

**Intertemporal risk aversion** in the ‘Epstein Zin form’ is characterized by strict concavity of \( f_t \circ g_t^{-1} \) and is independent of the commodity.

The IRA measure captures the difference between the propensity to smooth certain consumption over time and the propensity to smooth consumption between different risk states.
Example: Power mean (‘CRRA type function’):

Let \( f_\alpha(z) = z^\alpha \) and \( p \) simple (finite support):

\[
\mathcal{M}^\alpha(p, u) = \left( \sum_x p(x) u(x)^\alpha \right)^{\frac{1}{\alpha}}, \quad \alpha \in [-\infty, \infty], \text{“RRA on welf = 1 – } \alpha \text{”}
\]
Example: Power mean (‘CRRA type function’):

Let \( f_\alpha(z) = z^\alpha \) and \( p \) simple (finite support):

\[
\mathcal{M}^\alpha(p, u) = \left( \sum_x p(x) u(x)^\alpha \right)^\frac{1}{\alpha}, \quad \alpha \in [-\infty, \infty], \quad \text{“RRA on welf = 1 - \alpha”}
\]

Consider a given, cardinal welfare function \( u \) and the lottery

\[
\begin{align*}
\bar{u} &= u(\bar{x}) = 100 \text{ with probability } \bar{p} = p(\bar{x}) = .9 \\
\underline{u} &= u(\underline{x}) = 10 \text{ with probability } \underline{p} = p(\underline{x}) = .1
\end{align*}
\]
Example: Power mean (‘CRRA type function’):

Let \( f_\alpha(z) = z^\alpha \) and \( p \) simple (finite support):

\[ M^\alpha(p, u) = \left( \sum_x p(x)u(x)^\alpha \right)^{\frac{1}{\alpha}}, \alpha \in [-\infty, \infty], \text{“RRA on welf = 1} - \alpha” \]

Consider a given, cardinal welfare function \( u \) and the lottery

\[ \bar{u} = u(\bar{x}) = 100 \text{ with probability } \bar{p} = p(\bar{x}) = .9 \]
\[ \underline{u} = u(\underline{x}) = 10 \text{ with probability } \underline{p} = p(\underline{x}) = .1 \]

and find

\[ M^\infty(p, u) \equiv \lim_{\alpha \to \infty} M^\alpha(p, u) = \max_x u(x) = 100 \]
\[ M^1(p, u) = \mathbb{E}_p u = 91 \]
\[ M^0(p, u) \equiv \lim_{\alpha \to 0} M^\alpha(p, u) = \prod_x u(x)^{p(x)} = 73.2 \]
\[ M^{-10}(p, u) = \left( \sum_x p(x)u(x)^{-10} \right)^{-\frac{1}{10}} = 12.6 \]
\[ M^{-\infty}(p, u) \equiv \lim_{\alpha \to -\infty} M^\alpha(p, u) = \min_x u(x) = 10 \]
In the certainty additive representation of theorem 1 \((g = \text{id})\), define measure of absolute intertemporal risk aversion as:

\[
\text{AIRA}(z) = -\frac{(f \circ g^{-1})''(z)}{(f \circ g^{-1})'(z)}.
\]

▷ \text{AIRA} is uniquely defined if unit of welfare is fixed, i.e. if \(x^1\) and \(x^2\) (non-indifferent) are chosen such that \(u(x^1) - u(x^2) = 1\).

▷ It is independent of the commodity under observation and its measure scale.
Good and measure scale dependence of risk measures

Space of goods

Coordinate space $\mathbb{R}$
(space of numbers)

Robinson’s preferences $\succeq$

The (economist’s) utility function $u$ on $\mathbb{R}$

curvature describes risk attitude

$\Phi$: Coordinate system
Good and measure scale dependence of risk measures

Space of goods

Coordinate space $\mathbb{R}^n$

Robinson’s preferences $\succ$

The (economist’s) utility function $u$ on $\mathbb{R}$

describes risk attitude

differs for litchis

$\Phi$: Coordinate system

litchi quantity

coconut quantity

$\Phi_1$

$\Phi_2$
Good and measure scale dependence of risk measures

Space of goods

Coordinate space $\mathbb{R}^n$

Robinson's preferences $\succeq$

The (economist's) utility function $u$ on $\mathbb{R}$

describes risk attitude

$\Phi$: Coordinate system

arbitrary if coordinate $\Phi_3$ arbitrary
Good and measure scale dependence of risk measures

Space of goods

Coordinate space $\mathbb{R}^n$

Coconut quality

Litchi quantity

Is there a risk attitude measure that is independent of coordinate system $\Phi$ and the particular good?

I.e. that describes attitude with respect to risk rather than with respect to coconuts, litchis or money?