Intertemporal Risk Attitude

Christian Traeger
Department of Agricultural & Resource Economics, UC Berkeley,
Department of Economics, UC Berkeley

- Related Literature & Overview
- Aggregating over Time & Uncertainty
- Intertemporal Risk Aversion
- Preference Representations
- Conclusions
Motivation: How to evaluate climate change?
Stern Review, prominent responses in JEL September 2007:

- Nordhaus: Criticizes the low rate of pure time preference
- Weitzman: Uncertainty more important than time preference
Motivation: How to evaluate climate change?

Stern Review, prominent responses in JEL September 2007:

▷ Nordhaus: Criticizes the low rate of pure time preference
▷ Weitzman: Uncertainty more important than time preference

All argue in standard framework, however introducing intertemporal risk aversion, we will find:

▷ Standard framework involves implicit assumption of risk neutrality
▷ Intertemporal risk aversion + standard rationality constraints
  ⇒ zero rate of pure time preference
Related Evaluation Frameworks / Decision Models

Certainty:

▷ ‘Standard’: Certainty additive model \( \left( \sum_t u(x_t) \right) \)

▷ Koopmans (1960, 1972):
  Recursive evaluation \( (u_t = V(u(x), u_{t+1})) \)
Related Evaluation Frameworks / Decision Models I

Certainty:

- ‘Standard’: Certainty additive model \( \left( \sum_t u(x_t) \right) \)
- Koopmans (1960, 1972):
  Recursive evaluation \( (u_t = V(u(x), u_{t+1})) \)

Uncertainty:

- ‘Standard’: Certainty additive expected utility model
  \( (E_p \sum_t u_t(x_t)) \)
  Recursive expected utility evaluation \( (u_t = V(u(x), E_p u_{t+1})) \)
Certainty:

▷ ‘Standard’: Certainty additive model \( \left( \sum_{t} u(x_t) \right) \)

▷ Koopmans (1960, 1972):

Recursive evaluation \( (u_t = V(u(x), u_{t+1})) \)

Uncertainty:

▷ ‘Standard’: Certainty additive expected utility model

\( \left( E_p \sum_{t} u_t(x_t) \right) \)

▷ Kreps & Porteus (1978): Koopmans + von Neumann-Morgenstern

Recursive expected utility evaluation \( (u_t = V(u(x), E_p u_{t+1})) \)

Today:

▷ Certainty additivity + von Neumann-Morgenstern

▷ Still implies recursivity (not standard model!)

▷ Simpler representation than Kreps and Porteus (1978)
Risk Aversion:

- Kihlstrom & Mirman (1974): Multi-commodity risk aversion (Risk measure good and measure scale dependent)
Overview II

Risk Aversion:

- Kihlstrom & Mirman (1974): Multi-commodity risk aversion (Risk measure good and measure scale dependent)
- Weil (1990): Standard model ties relative risk aversion (RRA) to the inverse of the intertemporal elasticity of substitution (IES)
Risk Aversion:

- Kihlstrom & Mirman (1974): Multi-commodity risk aversion (Risk measure good and measure scale dependent)
- Weil (1990): Standard model ties relative risk aversion (RRA) to the inverse of the intertemporal elasticity of substitution (IES)
- Epstein & Zin (1989, 1991): Use a one commodity version of Kreps & Porteus to disentangle RRA from IES
Risk Aversion:

- Kihlstrom & Mirman (1974): Multi-commodity risk aversion (Risk measure good and measure scale dependent)
- Weil (1990): Standard model ties relative risk aversion (RRA) to the inverse of the intertemporal elasticity of substitution (IES)
- Epstein & Zin (1989, 1991): Use a one commodity version of Kreps & Porteus to disentangle RRA from IES

Today:

- Take Epstein & Zin back to multi-commodity world (& Find good and measure scale dependence of RRA)
Overview II

Risk Aversion:

- Kihlstrom & Mirman (1974): Multi-commodity risk aversion (Risk measure good and measure scale dependent)

- Weil (1990): Standard model ties relative risk aversion (RRA) to the inverse of the intertemporal elasticity of substitution (IES)

- Epstein & Zin (1989, 1991): Use a one commodity version of Kreps & Porteus to disentangle RRA from IES

Today:

- Take Epstein & Zin back to multi-commodity world (& Find good and measure scale dependence of RRA)

- Introduce measure of Intertemporal Risk Aversion (IRA)

- Find that IRA is good and measure scale independent, i.e. Describes attitude with respect to risk rather than a commodity
Overview II

Risk Aversion:

▷ Kihlstrom & Mirman (1974): Multi-commodity risk aversion (Risk measure good and measure scale dependent)

▷ Weil (1990): Standard model ties relative risk aversion (RRA) to the inverse of the intertemporal elasticity of substitution (IES)

▷ Epstein & Zin (1989, 1991): Use a one commodity version of Kreps & Porteus to disentangle RRA from IES

Today:

▷ Take Epstein & Zin back to multi-commodity world (& Find good and measure scale dependence of RRA)

▷ Introduce measure of Intertemporal Risk Aversion (IRA)

▷ Find that IRA is good and measure scale independent, i.e. Describes attitude with respect to risk rather than a commodity

▷ Find that standard model assumes intertemporal risk neutrality
Overview II

Risk Aversion:
- Kihlstrom & Mirman (1974): Multi-commodity risk aversion (Risk measure good and measure scale dependent)
- Weil (1990): Standard model ties relative risk aversion (RRA) to the inverse of the intertemporal elasticity of substitution (IES)
- Epstein & Zin (1989,1991): Use a one commodity version of Kreps & Porteus to disentangle RRA from IES

Today / Lecture:
- Take Epstein & Zin back to multi-commodity world (& Find good and measure scale dependence of RRA)
- Introduce measure of Intertemporal Risk Aversion (IRA)
- Find that IRA is good and measure scale independent, i.e. Describes attitude with respect to risk rather than a commodity
- Find that standard model assumes intertemporal risk neutrality
Atemporal Uncertainty

How to evaluate?

\[ u(x_1), u(x_2), u(x_3), u(x_4), u(x_5) \]

\[ p_1, p_2, p_3, p_4, p_5 \]

\[ f \]
Atemporal Uncertainty

How to evaluate?
Atemporal Uncertainty

How to evaluate?

von Neumann & Morgenstern (1944):

'vNM axioms' (weak order, continuity, independence)

exists $u$ such that

$$U \equiv E_p u = \sum_i p_i u(x^i) = \int_X u \, dp$$

represents preferences
‘Intertemporal Certainty’ How to evaluate?
‘**Intertemporal Certainty’**  How to evaluate?

**Additive Separability**

\[ u_1(x_1) u_2(x_2) u_3(x_3) u_T(x_T) \]

Koopmans (1960), ..., (Wakker 1988): Additive Separability (also: Certainty Additivity, Coordinate Independence, ...)

\[ U \equiv \sum_t u_t(x_t) \]

represents preferences
‘Intertemporal Certainty’ How to evaluate?

Koopmans (1960), ..., (Wakker 1988):

**Additive Separability** (also: Certainty Additivity, Coordinate Independence, ...)

\[
\begin{align*}
&u_1(x_1) \quad u_2(x_2) \quad u_3(x_3) \quad \ldots \quad u_T(x_T) \quad \rightarrow \sum_t \\
&\text{exist } u_1, \ldots, u_T \text{ such that } U \equiv \sum_t u_t(x_t) \text{ represents preferences}
\end{align*}
\]
'Real World': intertemporal uncertainty
‘Real World’: intertemporal uncertainty

\[ x_1^1 \ldots x_3^3 \ldots x_T^1 \]

\[ x_1^2 \ldots x_3^2 \ldots x_T^2 \]

\[ x_1^3 \ldots x_3^3 \ldots x_T^3 \]

\[ x_1^4 \ldots x_3^4 \ldots x_T^4 \]

\[ x_1^5 \ldots x_3^5 \ldots x_T^5 \]

\[ t=1 \quad t=2 \quad t=3 \quad \ldots \quad t=T \]
‘Real World’: intertemporal uncertainty

\[ x_1^1 \quad x_2^1 \quad x_3^1 \quad \ldots \quad x_T^1 \]

\[ x_1^2 \quad x_2^2 \quad x_3^2 \quad \ldots \quad x_T^2 \]

\[ x_1^3 \quad x_2^3 \quad x_3^3 \quad \ldots \quad x_T^3 \]

\[ x_1^4 \quad x_2^4 \quad x_3^4 \quad \ldots \quad x_T^4 \]

\[ x_1^5 \quad x_2^5 \quad x_3^5 \quad \ldots \quad x_T^5 \]

\[ t=1 \quad t=2 \quad t=3 \quad \ldots \quad t=T \]
Time & Uncertainty Aggregation

\[ u_1(x_1^t) \quad u_2(x_2^t) \quad u_3(x_3^t) \quad \ldots \quad u_T(x_T^t) \]

\[ p_1 \quad u_1(x_1^2) \quad u_2(x_2^2) \quad u_3(x_3^2) \quad \ldots \quad u_T(x_T^2) \]

\[ p_2 \quad u_1(x_1^3) \quad u_2(x_2^3) \quad u_3(x_3^3) \quad \ldots \quad u_T(x_T^3) \]

\[ p_3 \quad u_1(x_1^4) \quad u_2(x_2^4) \quad u_3(x_3^4) \quad \ldots \quad u_T(x_T^4) \]

\[ p_4 \quad u_1(x_1^5) \quad u_2(x_2^5) \quad u_3(x_3^5) \quad \ldots \quad u_T(x_T^5) \]

\[ t=1 \quad t=2 \quad t=3 \quad \ldots \quad t=T \]
Time & Uncertainty Aggregation

Standard Model: intertemporally additive expected utility

\[ U = E_p \sum_t u_t(x_t) \]
Reconsider: an axiomatic reasoning
Reconsider: an axiomatic reasoning

\[\begin{align*}
&\text{vNM axioms} \\
&\begin{array}{cccc}
x_1^1 & x_2^1 & x_3^1 & \cdots & x_T^1 \\
x_1^2 & x_2^2 & x_3^2 & \cdots & x_T^2 \\
x_1^3 & x_2^3 & x_3^3 & \cdots & x_T^3 \\
x_1^4 & x_2^4 & x_3^4 & \cdots & x_T^4 \\
x_1^5 & x_2^5 & x_3^5 & \cdots & x_T^5 \\
\end{array} \\
&\begin{array}{cccc}
t=1 & t=2 & t=3 & \cdots & t=T \\
\end{array}
\end{align*}\]
Reconsider: an axiomatic reasoning

\[ \exists u \text{ s.th.} \]

\[ u(x_1) \quad u(x_2) \quad u(x_3) \quad \ldots \quad u(T) \]

\[ \sum p \]

\[ E_p \quad E_p \quad E_p \quad \ldots \quad E_p \]

\[ t=1 \quad t=2 \quad t=3 \quad \ldots \quad t=T \]
Reconsider: an axiomatic reasoning

\[ p_1 \quad p_2 \quad p_3 \quad p_4 \quad p_5 \]

\[ x_1^1 \quad x_2^1 \quad x_3^1 \quad \ldots \quad x_T^1 \]

\[ x_1^2 \quad x_2^2 \quad x_3^2 \quad \ldots \quad x_T^2 \]

\[ x_1^3 \quad x_2^3 \quad x_3^3 \quad \ldots \quad x_T^3 \]

\[ x_1^4 \quad x_2^4 \quad x_3^4 \quad \ldots \quad x_T^4 \]

\[ x_1^5 \quad x_2^5 \quad x_3^5 \quad \ldots \quad x_T^5 \]

\[ t=1 \quad t=2 \quad t=3 \quad \ldots \quad t=T \]
Reconsider: an axiomatic reasoning

\[ \exists \, u_i \text{ s.th.} \quad u_1(x_1^1) \quad \cdots \quad u_3(x_3^1) \quad \cdots \quad u_T(x_T^1) \to \sum_t \]

\[ \quad u_1(x_1^2) \quad \cdots \quad u_3(x_3^2) \quad \cdots \quad u_T(x_T^2) \to \sum_t \]

\[ \quad u_1(x_1^3) \quad \cdots \quad u_3(x_3^3) \quad \cdots \quad u_T(x_T^3) \to \sum_t \]

\[ \quad u_1(x_1^4) \quad \cdots \quad u_3(x_3^4) \quad \cdots \quad u_T(x_T^4) \to \sum_t \]

\[ \quad u_1(x_1^5) \quad \cdots \quad u_3(x_3^5) \quad \cdots \quad u_T(x_T^5) \to \sum_t \]

\[ t=1 \quad t=2 \quad t=3 \quad \cdots \quad t=T \]
Reconsider: an axiomatic reasoning

However: In general $u_i$
Reconsider: an axiomatic reasoning

However: In general $u_i \neq u_i$!
Reconsider: an axiomatic reasoning

However: In general $u_i \neq u_i$!

Thus: Intertemporally additive expected utility model only represents very particular preferences where $u_i = u_i$!
Question: What preferences are missing in the standard model?
Question: What preferences are missing in the standard model?

Answer: Those with a non-trivial intertemporal risk attitude!
Question: What preferences are missing in the standard model?

Answer: Those with a non-trivial intertemporal risk attitude!

For a definition assume $T = 2$ and define:

- $X$: (connected compact metric) space of goods
- $\Delta(\cdot)$: Set of Borel probability measures on space ‘$\cdot$’ (Prohorov metric)
**Question**: What preferences are missing in the standard model?

**Answer**: Those with a non-trivial *intertemporal risk attitude*!

For a definition assume $T = 2$ and define:

- $X$: (connected compact metric) space of goods
- $\Delta(\cdot)$: Set of Borel *probability* measures on space $\cdot$ (Prohorov metric)
- $P_2 = \Delta(X)$: Probability measures $p_2$ on $X$
- $P_1 = \Delta(X \times P_2)$: Probability measures $p_1$ on $X \times P_2$
- $\succeq_2$: 2nd period preference relation on $P_2$ with elements $p_2$
- $\succeq_1$: 1st period preference relation on $P_1$ with elements $p_1$
**Question:** What preferences are missing in the standard model?

**Answer:** Those with a non-trivial *intertemporal risk attitude*!

For a definition assume $T = 2$ and define:

- $X$: (connected compact metric) space of goods
- $\Delta(\cdot)$: Set of Borel *probability* measures on space ‘·’ (*Prohorov metric*)
- $P_2 = \Delta(X)$: Probability measures $p_2$ on $X$
- $P_1 = \Delta(X \times P_2)$: Probability measures $p_1$ on $X \times P_2$
- $\succeq_2$: $2^{nd}$ period preference relation on $P_2$ with elements $p_2$
- $\succeq_1$: $1^{st}$ period preference relation on $P_1$ with elements $p_1$
- $C^0(\cdot)$: Set of all continuous functions from space ‘·’ to $\mathbb{R}$
**Question**: What preferences are missing in the standard model?

**Answer**: Those with a non-trivial *intertemporal risk attitude*!

**Definition**: A decision maker is called *intertemporal risk averse* iff:

For all $\bar{x}_1, x_1, \bar{x}_2, x_2 \in X$ such that $\bar{x}_2 \succeq_2 x_2$ and

$$
\begin{array}{ccc}
\bar{x}_1 & \sim_1 & \bar{x}_2 \\
\end{array}
\begin{array}{ccc}
x_1 & \sim_1 & \bar{x}_2 \\
\end{array}
$$

it holds that
**Question:** What preferences are missing in the standard model?

**Answer:** Those with a non-trivial *intertemporal risk attitude*!

**Definition:** A decision maker is called *intertemporal risk averse* iff:

For all \( x_1, x_1, x_2, x_2 \in X \) such that \( x_2 \succeq_2 x_2 \) and

\[
\overrightarrow{x_1} \sim_1 \overrightarrow{x_2} \quad \overrightarrow{x_1} \approx_1 \overrightarrow{x_2}
\]

it holds that

\[
\overrightarrow{x_1} \succeq_1 \overrightarrow{x_2}
\]
Question: What preferences are missing in the standard model?

Answer: Those with a non-trivial intertemporal risk attitude!

Definition: A decision maker is called intertemporal risk seeking iff:

For all $\overline{x}_1, \overline{x}_1, \overline{x}_2, \overline{x}_2 \in X$ such that $\overline{x}_2 \succeq_2 \overline{x}_2$ and

\[\overline{x}_1 \sim_1 \overline{x}_1 \sim_2 \overline{x}_2 \]

it holds that

\[\overline{x}_1 \prec_1 \overline{x}_1 \prec_2 \overline{x}_2 \]
**Question**: What preferences are missing in the standard model?

**Answer**: Those with a non-trivial *intertemporal risk attitude*!

**Definition**: A decision maker is called *intertemporal risk neutral* iff:

For all $\bar{x}_1, x_1, \bar{x}_2, x_2 \in X$ such that $\bar{x}_2 \succeq_2 x_2$ and

\[
\bar{x}_1 \sim_1 x_1 \sim_1 \bar{x}_2
\]

it holds that

\[
\bar{x}_1 \sim_1 \frac{1}{2} \bar{x}_1 \frac{1}{2} \bar{x}_2
\]

\[
\bar{x}_1 \sim_1 \frac{1}{2} x_1 \frac{1}{2} x_2
\]

\[
\bar{x}_1 \sim_1 \frac{1}{2} x_1 \frac{1}{2} x_2
\]

\[
\bar{x}_1 \sim_1 \frac{1}{2} \bar{x}_1 \frac{1}{2} \bar{x}_2
\]

\[
\bar{x}_1 \sim_1 \frac{1}{2} x_1 \frac{1}{2} x_2
\]

\[
\bar{x}_1 \sim_1 \frac{1}{2} \bar{x}_1 \frac{1}{2} \bar{x}_2
\]

\[
\bar{x}_1 \sim_1 \frac{1}{2} x_1 \frac{1}{2} x_2
\]
Uncertainty Aggregation Rule

For \( f : \mathbb{R} \to \mathbb{R} \) continuous and strictly increasing and some compact metric space \( Y \) define

\[
\mathcal{M}^f : \Delta(Y) \times C^0(Y) \to \mathbb{R}
\]

\[
\mathcal{M}^f(p, u) = f^{-1} \left[ \int_Y f \circ u \, dp \right]
\]
Uncertainty Aggregation Rule

For $f : \mathbb{R} \to \mathbb{R}$ continuous and strictly increasing and some compact metric space $Y$ define

\[ M^f : \Delta(Y) \times C^0(Y) \to \mathbb{R} \]

\[ M^f (p, u) = f^{-1} \left[ \int_Y f \circ u \, dp \right] \]

The uncertainty aggregation rule satisfies:

\[ M^f (y, u) = u(y) \quad \forall y \in Y \]
Uncertainty Aggregation Rule

For $f : \mathbb{R} \to \mathbb{R}$ continuous and strictly increasing and some compact metric space $Y$ define

\[ M_f : \Delta(Y) \times C^0(Y) \to \mathbb{R} \]
\[ M_f (p, u) = f^{-1} \left[ \int_Y f \circ u \, dp \right] \]

The uncertainty aggregation rule satisfies:

\[ M_f (y, u) = u(y) \quad \forall y \in Y \]

It includes rules corresponding to

\[ \text{expected value (} f = \text{id} \text{)} \]
\[ \text{geometric mean (} f = \ln \text{)} \]
\[ \text{power mean (} f_\alpha = \text{id}^\alpha \text{) ‘CRRA type’} \]

Example
Theorem 1:
The set of preference relations \((\succeq_1, \succeq_2)\) on \((P_1, P_2)\) satisfies

\(i\)  vNM axioms
\(ii\) additive separability for certain consumption paths
\(iii\) time consistency
Theorem 1:

The set of preference relations \((\succeq_1, \succeq_2)\) on \((P_1, P_2)\) satisfies

\(i\) \textbf{vNM axioms}

\(ii\) additive separability for certain consumption paths

\(iii\) time consistency

if and only if, there exist continuous functions \(u_t : X \to U_t \subset \mathbb{R}\) and a strictly increasing and continuous function \(f_t : U_t \to \mathbb{R}\) for \(t \in \{1, 2\}\) such that with defining \(\tilde{u}_2(x_2) = u_2(x_2)\) and
Theorem 1:
The set of preference relations $(\succeq_1, \succeq_2)$ on $(P_1, P_2)$ satisfies

(i) vNM axioms

(ii) additive separability for certain consumption paths

(iii) time consistency

if and only if, there exist continuous functions $u_t : X \to U_t \subset \mathbb{R}$ and a strictly increasing and continuous function $f_t : U_t \to \mathbb{R}$ for $t \in \{1, 2\}$ such that with defining $\tilde{u}_2(x_2) = u_2(x_2)$ and

$$\tilde{u}_1(x_1, p_2) = u_1(x_1) + M^{f_2}(p_2, \tilde{u}_2)$$

it holds that for $t \in \{1, 2\}$

$$p_t \succeq_t p'_t \iff M^{f_t}(p_t, \tilde{u}_t) \geq M^{f_t}(p'_t, \tilde{u}_t) \quad \forall p_t, p'_t \in P_t.$$
Theorem 2:
In the representation of theorem 1, a decision maker is intertemporal risk averse, if and only if, $f_1$ is strictly concave.
Theorem 2:
In the representation of theorem 1, a decision maker is intertemporal risk averse, if and only if, $f_1$ is strictly concave.

Define the measure of relative intertemporal risk aversion as:

$$\text{RIRA}_t(z) = -\frac{f''_t(z)}{f'_t(z)} z.$$ 

The measures $\text{RIRA}_t$

- are uniquely defined if a zero welfare level is fixed, i.e. if $x^{zero}$ is chosen such that $u_t(x^{zero}) = 0$.
- It is independent of the commodity under observation and its measure scale.
The representation in theorem 1 uses the time additive utility functions $u_t$.
In a one commodity world and with $X \subset \mathbb{R}$ can also choose $u_t = \text{id}$. 
The representation in theorem 1 uses the time additive utility functions $u_t$.

In a one commodity world and with $X \subset \mathbb{R}$ can also choose $u_t = \text{id}$.

\[ \to \text{recursion relation: } \tilde{u}_t = \mathcal{N}_t^{g_t}(x_t, \mathcal{M}_t^{f_{t+1}}(p_{t+1}, \tilde{u}_t)) \]

Where $\mathcal{N}_t^{g_t}$ is a nonlinear intertemporal aggregation rule similar to $\mathcal{M}_t^{f_t}$ parameterized by the function $g_t : U_t \to \mathbb{R}$. (slightly sloppy)
The representation in theorem 1 uses the time additive utility functions $u_t$.

In a one commodity world and with $X \subset \mathbb{R}$ can also choose $u_t = \text{id}$.

→ recursion relation: \( \tilde{u}_t = N^g_t \left( x_t, M^{f_{t+1}}(p_{t+1}, \tilde{u}_t) \right) \)

Where $N^g_t$ is a nonlinear intertemporal aggregation rule similar to $M^{f_t}$ parameterized by the function $g_t : U_t \rightarrow \mathbb{R}$. (slightly sloppy)

For $g_t(z) = \beta^t z^\rho$ and $f_t(z) = z^\alpha$ Epstein & Zin’s (1991) generalized isoelastic model.

Can interpret

▷ $1 - \alpha$ as measure of Arrow-Pratt relative risk aversion
▷ $(1 - \rho)^{-1}$ as intertemporal elasticity of substitution
Epstein Zin interpretation depends crucially on $u = \text{id}$ in
$$\tilde{u}_t = \mathcal{N}^{g_t}(x_t, \mathcal{M}^{f_{t+1}}(p_{t+1}, \tilde{u}_t)).$$

Multi-commodity usually: Directional derivative $\Rightarrow$ Doesn’t work!
Epstein Zin interpretation depends crucially on $u = \text{id}$ in
\[
\tilde{u}_t = \mathcal{N}^{g_t} \left( x_t, \mathcal{M}^{f_{t+1}}(p_{t+1}, \tilde{u}_t) \right).
\]
Multi-commodity usually: Directional derivative $\Rightarrow$ Doesn’t work!
Instead: Linearize utility in commodity $i$ and determine $f_t^i$ and $g_t^i$.
As in standard model, risk measure depends on

$\triangleright$ particular commodity under observation

$\triangleright$ measure scale
Epstein Zin interpretation depends crucially on $u = \text{id}$ in
\[
\tilde{u}_t = \mathcal{N}^{g_t}\left(x_t, \mathcal{M}^{f_{t+1}}(p_{t+1}, \tilde{u}_t)\right).
\]
Multi-commodity usually: Directional derivative $\Rightarrow$ Doesn’t work!
Instead: Linearize utility in commodity $i$ and determine $f_t^i$ and $g_t^i$.
As in standard model, risk measure depends on
  ▶ particular commodity under observation
  ▶ measure scale

**Intertemporal risk aversion** in the ‘Epstein Zin form’ is characterized by strict concavity of $f_t \circ g_t^{-1}$ and is independent of the commodity.

The IRA measure captures the difference between the propensity to smooth certain consumption over time and the propensity to smooth consumption between different risk states.
Evaluate future needs similar to current needs
Evaluate future needs similar to current needs

**Definition:** A decision maker’s evaluation is called *risk stationary* iff:

\[
\begin{align*}
\frac{1}{2} & \quad \overline{x} \quad x \\
\frac{1}{2} & \quad x \quad x
\end{align*}
\begin{align*}
\preceq_{1} & \quad x^{*} \quad x \\
\iff
\end{align*}
\]
Evaluate future needs similar to current needs

*Definition:* A decision maker’s evaluation is called *risk stationary* iff:

\[ \frac{1}{2} x \quad \frac{1}{2} x \quad \frac{1}{2} x \quad \frac{1}{2} x \]

\[ \begin{array}{c}
\frac{1}{2} x \\
\leq_1 \quad x^* \quad x \\
\frac{1}{2} x \\
\end{array} \quad \iff \quad \begin{array}{c}
\frac{1}{2} x \\
\geq_2 \quad x^* \\
\frac{1}{2} x \\
\end{array} \]
Evaluate future needs similar to current needs

**Definition:** A decision maker’s evaluation is called **risk stationary** iff:

\[
\begin{align*}
\frac{1}{2} & \quad \overline{x} & \quad x \\
\frac{1}{2} & \quad x & \quad \underbrace{x}_1 \quad x^* & \quad x \quad \Leftrightarrow \quad \frac{1}{2} & \quad \overline{x} \\
\frac{1}{2} & \quad x & \quad \underbrace{x}_2 \quad x^*
\end{align*}
\]

▷ Finite time horizon!
▷ For more periods take \( \overline{x}, \underline{x} \) and \( x^* \) as consumption paths
▷ Allows to define pure time preference (‘utility discount factor’)

setup
A sequence of binary relations \((\succeq_t)_{t \in \{1, \ldots, T\}}\) on \((P_t)_{t \in \{1, \ldots, T\}}\) satisfies

\(i\) vNM axioms
\(i\!i\) additive separability
\(i\!i\!i\) time consistency
\(i\!v\) indifference to the timing of risk resolution
\(v\) risk stationarity
\(vi\) intertemporal risk aversion
A sequence of binary relations \((\succeq_t)_{t \in \{1,\ldots,T\}}\) on \((P_t)_{t \in \{1,\ldots,T\}}\) satisfies

\begin{itemize}
  \item[i)] vNM axioms
  \item[ii)] additive separability
  \item[iii)] time consistency
  \item[iv)] indifference to the timing of risk resolution
  \item[v)] risk stationarity
  \item[vi)] intertemporal risk aversion
\end{itemize}

if and only if, there exists a continuous function \(u : X \rightarrow \mathbb{IR}\), a discount factor \(\beta = 1\), and \(\xi < 0\) such that the function

\[
u_t(x_t) = \sum_{\tau=t}^{T} \beta^{\tau-1} u(x^t_{\tau})
\]

represent choice over certain consumption paths \(x^t = (x_t, x_{t+1}, \ldots, x_T)\)
Stationarity II

A sequence of binary relations \((\succeq_t)_{t \in \{1, \ldots, T\}}\) on \((P_t)_{t \in \{1, \ldots, T\}}\) satisfies

i) vNM axioms

ii) additive separability

iii) time consistency

iv) indifference to the timing of risk resolution

v) risk stationarity

vi) intertemporal risk aversion

if and only if, there exists a continuous function \(u : X \to \mathbb{R}\), a discount factor \(\beta = 1\), and \(\xi < 0\) such that the function

\[
u_t(x^t) = \sum_{\tau=t}^{T} \beta^{\tau-1} u(x_{\tau}^t)\]

represent choice over certain consumption paths \(x^t=(x_t, x_{t+1}, \ldots, x_T)\) and

\[
\mathcal{M}_{\Delta P_t}^{\xi} u_t = \frac{1}{\xi} \ln \left[ \int dP_t^X \exp[\xi u_t(x^t)] \right]
\]

represents choice over lotteries in period \(t \in \{1, \ldots, T\}\).
A sequence of binary relations \((\succeq_t)_{t \in \{1, \ldots, T\}}\) on \((P_t)_{t \in \{1, \ldots, T\}}\) satisfies

\[
\begin{align*}
  &i) \quad \text{vNM axioms} \\
  &ii) \quad \text{additive separability} \\
  &iii) \quad \text{time consistency} \\
  &iv) \quad \text{indifference to the timing of risk resolution} \\
  &v) \quad \text{risk stationarity} \\
  &vi) \quad \text{intertemporal risk aversion}
\end{align*}
\]

if and only if, there exists a continuous function \(u : X \rightarrow \mathbb{R}\), a discount factor \(\beta = 1\), and \(\xi < 0\) such that the function

\[
u_t(x^t) = \sum_{\tau=t}^{T} \beta^{\tau-1} u(x^\tau_t)\]

represent choice over certain consumption paths \(x^t = (x_t, x_{t+1}, \ldots, x_T)\) and

\[
\mathcal{M}_{p_t}^{\xi} u_t = \frac{1}{\xi} \ln \left[ \int dp_t^\chi \exp[\xi \ u_t(x^t)] \right]
\]

Standard model: \( \mathcal{M}_{p_t}^{0} u_t \equiv \lim_{\xi \to 0} \mathcal{M}_{p_t}^{\xi} u_t = E_{p_t}^\chi u_t = \int dp_t^\Delta u_t(x^t) \) (with \(\beta = 1\))
A sequence of binary relations \((\succeq_t)_{t\in\{1,\ldots,T\}}\) on \((P_t)_{t\in\{1,\ldots,T\}}\) satisfies

\begin{itemize}
  \item[i)] vNM axioms
  \item[ii)] additive separability
  \item[iii)] time consistency
  \item[iv)] indifference to the timing of risk resolution
  \item[v)] risk stationarity
  \item[vi)] intertemporal risk aversion
\end{itemize}

if and only if, there exists a continuous function \(u : X \rightarrow \mathbb{R}\), a discount factor \(\beta = 1\), and \(\xi < 0\) such that the function

\[
u_t(x^t) = \sum_{\tau=t}^{T} \beta^{\tau-1} u(x^\tau_{\tau})
\]

represent choice over certain consumption paths \(x^t = (x_t, x_{t+1}, \ldots, x_T)\) and

\[
M_{p_t}^\xi u_t = \frac{1}{\xi} \ln \left[ \int dp_t x \exp[\xi u_t(x^t)] \right]
\]

represents choice over lotteries in period \(t \in \{1, \ldots, T\}\).
A sequence of binary relations \((\succeq_t)_{t\in\{1,\ldots,T\}}\) on \((P_t)_{t\in\{1,\ldots,T\}}\) satisfies

\begin{itemize}
  \item[i)] vNM axioms
  \item[ii)] additive separability
  \item[iii)] time consistency
  \item[iv)] indifference to the timing of risk resolution
  \item[v)] certainty stationarity
  \item[vi)] intertemporal risk aversion
\end{itemize}

if and only if, there exists a continuous function \(u : X \to \mathbb{R}\), a discount factor \(\beta \in \mathbb{R}_{++}\), and \(\xi < 0\) such that the function

\[ u_t(x^t) = \sum_{\tau=t}^{T} \beta^{\tau-1} u(x^\tau) \]

represent choice over certain consumption paths \(x^t=(x_t,x_{t+1},\ldots,x_T)\) and

\[ \mathcal{M}_{p_t}^{\xi} u_t = \frac{1}{\xi} \ln \left[ \int dp_t^{X} \exp[\xi u_t(x^t)] \right] \]

represents choice over lotteries in period \(t \in \{1, \ldots, T\}\).
Note: an intertemporal risk averse decision maker discounts for reasons of uncertainty
Note: an intertemporal risk averse decision maker discounts for reasons of uncertainty

Consider the following setup, where $X \subset \mathbb{R}$:

- Let $\beta = 1$ (no pure time preference) and $\xi < 0$ (IRA).
- Let uncertainty increase over time (more weight on the tails).
- Let, however, $E u(x_t) = \bar{u}$ for all $t \in \{1, \ldots, T\}$.

Then: dm puts effectively less weight on expected welfare, the further it is in the future (the more uncertain it is)
Note: an intertemporal risk averse decision maker discounts for reasons of uncertainty

Consider the following setup, where $X \subset \mathbb{R}$:

- Let $\beta = 1$ (no pure time preference) and $\xi < 0$ (IRA).
- Let uncertainty increase over time (more weight on the tails).
- Let, however, $E_u(x_t) = \bar{u}$ for all $t \in \{1, ..., T\}$.

Then: dm puts effectively less weight on expected welfare, the further it is in the future (the more uncertain it is)

Somewhat resembles discounting, however:

- future gains less weight the less *can* be known
- preference for a *less uncertain* future over a *more uncertain* future, even if in every period *expected welfare* is the *same*
Conclusions:

- The intertemporally additive expected utility standard model implicitly assumes a form of risk neutrality.
Conclusions:

- The intertemporally additive expected utility standard model implicitly assumes a form of risk neutrality.
- Intertemporal risk aversion captures difference between propensity to substitute consumption over time and between risk states.
Conclusions:

- The intertemporally additive expected utility standard model implicitly assumes a form of risk neutrality.
- Intertemporal risk aversion captures the difference between the propensity to substitute consumption over time and between risk states.
- The measure of intertemporal risk aversion is independent of the commodity under observation and its measure scale.
Conclusions:

- The intertemporally additive expected utility standard model implicitly assumes a form of risk neutrality.
- Intertemporal risk aversion captures the difference between the propensity to substitute consumption over time and between risk states.
- The measure of intertemporal risk aversion is independent of the commodity under observation and its measure scale.
- Have given a general representation theorem for intertemporally risk averse (and seeking) decision makers.
Conclusions:

- The intertemporally additive expected utility standard model implicitly assumes a form of risk neutrality.
- Intertemporal risk aversion captures difference between propensity to substitute consumption over time and between risk states.
- The measure of intertemporal risk aversion is independent of the commodity under observation and its measure scale.
- Have given a general representation theorem for intertemporally risk averse (and seeking) decision makers.
- Some standard assumptions, intertemporal risk aversion and risk stationarity imply zero rate of pure time preference.
Conclusions:

- The intertemporally additive expected utility standard model implicitly assumes a form of risk neutrality.

- **Intertemporal risk aversion** captures difference between propensity to substitute consumption over time and between risk states.

- The measure of intertemporal risk aversion is independent of the commodity under observation and its measure scale.

- Have given a general representation theorem for intertemporally risk averse (and seeking) decision makers.

- Some standard assumptions, intertemporal risk aversion and risk stationarity imply zero rate of pure time preference.

- IRA DM discounts for reasons of increasing uncertainty, but very differently from standard model discounter.
Lecture Series “Intertemporal Risk Attitude”

consists of 4 + 4 lectures

1 A Toy Model
2 Intertemporal Risk Aversion
3 Epstein Zin and Many Commodities
4 The General Model
5 The Isoelastic Model
6 A Preference for the Timing of Risk Resolution
7 Stationarity in a Finite Time Horizon
8 Timing Indifference, Stationarity and Discounting

Meets: Wednesday Sept 12th, 4:10pm~5:00pm, 103 Mulford
Website: http://are.berkeley.edu/~chris/lecture.html
Further remarks:

- Standard model allows $\beta < 1$ under risk stationary evaluation because DM is essentially risk neutral (risk stat no more implications than certainty stat)
Further remarks:

- Standard model allows $\beta < 1$ under risk stationary evaluation because DM is essentially risk neutral (risk stat no more implications than certainty stat).

- Have given representation using certainty additive utility ($u$), alternatively can use $u$ and nonlinear time aggregation.

- For certainty additive choice of utility can interpret IRA = risk aversion with respect to utility gains and losses.
Further remarks:

- Standard model allows $\beta < 1$ under risk stationary evaluation because DM is essentially risk neutral (risk stat no more implications than certainty stat)
- Have given representation using certainty additive utility ($u$), alternatively can use $u$ and nonlinear time aggregation
- For certainty additive choice of utility can interpret $IRA = risk aversion with respect to utility gains and losses$
- Concept of IRA closely relates to disentanglement of atemporal risk aversion and intertemporal substitutability
- Generalized isoelastic (one-commodity) model relaxes timing indifference and uses recursive uncertainty description
Further Remarks

Further remarks:

- Standard model allows $\beta < 1$ under risk stationary evaluation because DM is essentially risk neutral (risk stat no more implications than certainty stat)
- Have given representation using certainty additive utility ($u$), alternatively can use $u$ and nonlinear time aggregation
- For certainty additive choice of utility can interpret IRA = risk aversion with respect to utility gains and losses
- Concept of IRA closely relates to disentanglement of atemporal risk aversion and intertemporal substitutability
- Generalized isoelastic (one-commodity) model relaxes timing indifference and uses recursive uncertainty description
- IRA is a multicommodity risk concept
Axioms:

A1 (weak order) \( \succeq_t \) is transitive and complete

A2 (independence) \( \forall p, q, r \in P_t : \)

\[
p \sim_t q \implies \lambda p + (1 - \lambda) r \sim_t \lambda q + (1 - \lambda) r \quad \forall \lambda \in [0, 1]
\]

A3 (continuity) \( \forall p \in P_t : \)

The sets \( \{ q \in P_t : q \succeq_t p \} \) and \( \{ q \in P_t : p \succeq_t q \} \) are closed in \( P_t \)
Axioms:

\( A1 \) (weak order) \( \succeq_t \) is transitive and complete

\( A2 \) (independence) \( \forall p, q, r \in P_t : \)

\[ p \sim_t q \implies \lambda p + (1 - \lambda) r \sim_t \lambda q + (1 - \lambda) r \quad \forall \lambda \in [0, 1] \]

\( A3 \) (continuity) \( \forall p \in P_t : \)

The sets \( \{ q \in P_t : q \succeq_t p \} \) and \( \{ q \in P_t : p \succeq_t q \} \) are closed in \( P_t \)

\( A4 \) (additive separability for \( T \geq 3, T = 2 \) see paper)

For all \( x, x' \in X^1, x, x' \in X \) and \( i \in \{1, \ldots, T\} \) it holds that

\[ (x_{-i}, x) \succeq_1 (x'_{-i}, x) \iff (x_{-i}, x') \succeq_1 (x'_{-i}, x') \]
Axioms:

**A1** (weak order) \( \succeq_t \) is transitive and complete

**A2** (independence) \( \forall p, q, r \in P_t : \)
\[
p \sim_t q \quad \Rightarrow \quad \lambda p + (1 - \lambda) r \sim_t \lambda q + (1 - \lambda) r \quad \forall \lambda \in [0, 1]
\]

**A3** (continuity) \( \forall p \in P_t : \)
The sets \( \{ q \in P_t : q \succeq_t p \} \) and \( \{ q \in P_t : p \succeq_t q \} \) are closed in \( P_t \)

**A4** (additive separability for \( T \geq 3, T = 2 \) see paper)
For all \( x, x' \in X^1, x, x' \in X \) and \( i \in \{1, ..., T\} \) it holds that
\[
(x_{-i}, x) \succeq_1 (x'_{-i}, x) \iff (x_{-i}, x') \succeq_1 (x'_{-i}, x')
\]

**A5** (time consistency) For all certain consumption \( x \in X \) and \( t \in \{1, ..., T - 1\} : \)
\[
(x, p) \succeq_t (x, p') \iff p \succeq_{t+1} p' \quad \forall p, p' \in P_{t+1}.
\]
Definition: A decision maker is called **timing indifferent** iff:

For all $x, \bar{x}, \underline{x} \in X$

\[ \frac{1}{2} x \sim_1 \frac{1}{2} \underline{x} \]

\[ \frac{1}{2} \bar{x} \sim_1 \frac{1}{2} \bar{x} \]

\[ \frac{1}{2} x \sim_1 \frac{1}{2} \underline{x} \]

\[ \frac{1}{2} \bar{x} \sim_1 \frac{1}{2} \underline{x} \]

▷ Uncertainty resolves ('coin toss takes place') in first versus second period
Definition: A decision maker’s evaluation is called certainty stationary iff:

For all \( \bar{x}, x, \underline{x} \in X \)

\[
\begin{align*}
\bar{x} & \succeq_1 x & \underline{x} & \preceq_2 x \\
\bar{x} & \succeq_2 x & \underline{x} & \preceq_1 x
\end{align*}
\]

▷ ’Risk stationarity with degenerate lotteries´ (Finite time horizon!)
▷ For more periods take \( \bar{x} \) and \( \underline{x} \) as consumption paths
▷ Allows to define pure time preference (‘utility discount factor’)
Example: Power mean (‘CRRA type function’):

Let \( f_\alpha(z) = z^\alpha \) and \( p \) simple (finite support):

\[
\mathcal{M}_\alpha(p, u) = \left( \sum_x p(x)u(x)^\alpha \right)^{\frac{1}{\alpha}}, \quad \alpha \in [-\infty, \infty], \quad \text{"RRA on welf} = 1 - \alpha\text{"}
\]
Example: Power mean (‘CRRA type function’):

Let \( f_\alpha(z) = z^\alpha \) and \( p \) simple (finite support):

\[
\mathcal{M}^\alpha(p, u) = \left( \sum_x p(x) u(x)^\alpha \right)^{1/\alpha}, \quad \alpha \in [-\infty, \infty], \quad "RRA \text{ on } welf = 1 - \alpha"
\]

Consider a given, cardinal welfare function \( u \) and the lottery

\[
\begin{align*}
\mathbb{u} &= u(\bar{x}) = 100 \text{ with probability } \bar{p} = p(\bar{x}) = .9 \\
\mathbb{u} &= u(\underline{x}) = 10 \text{ with probability } \underline{p} = p(\underline{x}) = .1
\end{align*}
\]
Example: Power mean (‘CRRA type function’):

Let \( f_\alpha(z) = z^\alpha \) and \( p \) simple (finite support):

\[
M_\alpha(p, u) = \left( \sum_x p(x)u(x)^\alpha \right)^{\frac{1}{\alpha}}, \quad \alpha \in [-\infty, \infty], \quad " RRA on welf = 1 - \alpha "
\]

Consider a given, cardinal welfare function \( u \) and the lottery

\[
\begin{align*}
\bar{u} &= u(x) = 100 \text{ with probability } \bar{p} = p(x) = .9 \\
u &= u(x) = 10 \text{ with probability } p = p(x) = .1 \\
\end{align*}
\]

and find

\[
\begin{align*}
M_\infty(p, u) &\equiv \lim_{\alpha \to \infty} M_\alpha(p, u) = \max_x u(x) = 100 \\
M_1(p, u) &= = \mathbb{E}_p u = 91 \\
M_0(p, u) &\equiv \lim_{\alpha \to 0} M_\alpha(p, u) = \prod_x u(x)^{p(x)} = 73.2 \\
M^{-10}(p, u) &= \left( \sum_x p(x)u(x)^{-10} \right)^{-\frac{1}{10}} = 12.6 \\
M^{-\infty}(p, u) &\equiv \lim_{\alpha \to -\infty} M_\alpha(p, u) = \min_x u(x) = 10
\end{align*}
\]
In the certainty additive representation of theorem 1 \((g = \text{id})\), define measure of absolute intertemporal risk aversion as:

\[
\text{AIRA}(z) = -\frac{(f \circ g^{-1})''(z)}{(f \circ g^{-1})'(z)}.
\]

\(\textbf{AIRA}\) is uniquely defined if unit of welfare is fixed, i.e. if \(x^1\) and \(x^2\) (non-indifferent) are chosen such that \(u(x^1) - u(x^2) = 1\).

\(\textbf{AIRA}\) is independent of the commodity under observation and its measure scale.
Good and measure scale dependence of risk measures

Space of goods

Coordinate space $\mathbb{R}$
(space of numbers)

Robinson’s preferences $\succeq$

The (economist’s) utility function $u$ on $\mathbb{R}$

curvature describes risk attitude

$\Phi$: Coordinate system
Good and measure scale dependence of risk measures

Space of goods

Coordinate space $\mathbb{R}^n$

Robinson’s preferences $\succeq$

$\Phi$: Coordinate system

$\Phi_1$, $\Phi_2$

litchi quantity

coconut quantity

The (economist’s) utility function $u$ on $\mathbb{R}$

differs for litchis

curvature describes risk attitude
Good and measure scale dependence of risk measures

Space of goods

Coordinate space $\mathbb{R}^n$

coconut quality

Robinson’s preferences $\succeq$

The (economist’s) utility function $u$ on $\mathbb{R}$

curvature describes risk attitude

$\Phi$: Coordinate system

arbitrary if coordinate $\Phi_3$ arbitrary
Good and measure scale dependence of risk measures

Space of goods

Coordinate space $\mathbb{R}^n$

coconut quality

litchi quantity

coconut quantity

Is there a risk attitude measure that is independent of coordinate system $\Phi$ and the particular good?

I.e. that describes attitude with respect to risk rather than with respect to coconuts, litchis or money?