ARE Math Review Lecture 3: Determinants

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1 Defining the Determinant

The **determinant** is a real-valued function[1] of a square matrix. Various mathematically equivalent definitions are possible[2]; I give two below.

1.1 Recursive Definition

The determinant[3] of an $n \times n$ square matrix A is defined in Simon and Blume by the following recursive formula:

$$\det A = a_{11}C_{11} + a_{12}C_{12} + \ldots + a_{1n}C_{1n}. \tag{1}$$

The scalars C_{1j} are called **cofactors**, and require further definition. Let A_{ij} be the $(n-1) \times (n-1)$ matrix obtained by deleting row i and column j from A. Then the (i, j)th **minor** of A is given by

$$M_{ij} = \det A_{ij}, \tag{2}$$

and the cofactors (some of which appear in the determinant formula) are given by

$$C_{ij} = (-1)^{i+j} M_{ij}. \tag{3}$$

[1]Recall that a function is a rule which assigns a unique value to a given argument to the function. In the case at hand, the argument is a square matrix and the assigned value is a real number.

[2]For instance, the Simon and Blume definition is stated in terms of the first row of A, but this can be generalized to admit expansion about the entries in other rows.

[3]Alternate notation is to put A inside of vertical bars: $\det A \equiv |A|$.

This formula looks rather hairy, but is not too bad if you take it a piece at a time:

1. The minors are easy to understand.

2. If we can figure out how to compute the determinants of the minors, the cofactors are given by these determinants multiplied by either 1 or -1, depending on whether $i + j$ is even or odd.

3. With the cofactors in hand, the formula for the determinant is straightforward.

The biggest problem is that we appear to have a circular definition: The determinant formula depends on some other determinants. The important thing to note is that the determinants on the RHS are computed on smaller matrices (of size $(n-1) \times (n-1)$). After enough recursive applications of the formula, we are left with the task of computing the determinant of 1×1 matrices: If $A = [c]$, then $\det A = c$. So it is possible to avoid the hassle of writing down an explicit formula for the determinant if we are willing to apply this recursive definition a sufficient number of times to get to the bottom of things.

1.2 Explicit Formula

On the other hand, the explicit formula for the determinant is not too difficult to grasp, once we acquire the requisite background understanding of permutation functions. A permutation is a function which reorders a sequence of counting numbers on the range 1 to n. They are customarily denoted generically with the lower-case sigma, σ. We can summarize the action of a particular permutation function by displaying the reordering scheme.

For illustration, consider the permutation of the positive integers {1, 2, 3} described by

$$\sigma = (3, 1, 2). \tag{4}$$

The result of applying this permutation function to each number in the domain may be listed as follows:

$$\begin{aligned} \sigma(1) &= 3 \\ \sigma(2) &= 1 \\ \sigma(3) &= 2. \end{aligned}$$

Generally for the first n positive integers, there are $n!$ different permutation functions, one for each possible distinct ordering.

For any particular permutation function, the number of position interchanges of the numbers 1 through n needed to achieve the specified reordering is either always even or always odd[4]. For example, for $\sigma = (3, 1, 2)$, we could go through the steps $(1, 2, 3) \rightarrow (3, 2, 1) \rightarrow (3, 1, 2)$, so two position interchanges are required, and the parity is even. Based on this, we can define a sign function, denoted $\operatorname{sgn}(\sigma)$, which returns 1 if the parity for σ is even and -1 if the parity for σ is odd.

Now we have all the necessary machinery to write down the explicit formula for the determinant of an arbitrary $n \times n$ matrix A:

$$\det(A) = \sum_{\sigma} \operatorname{sgn}(\sigma) a_{1\sigma(1)} a_{2\sigma(2)} \dots a_{n\sigma(n)}. \tag{5}$$

The notation employs shorthand: The σ under the summation sign generically represents all possible permutations of the first n integers. We have one term in the sum for each of these $n!$ different permutations (e.g., a 4×4 matrix has $4! = 24$ terms in the explicit formula).

Each term includes a + or - sign depending on the parity of σ. The n factors in each term include one column entry from each row of A, with the particular column entries specified by the permutation function under consideration. Hence all rows and columns are present and accounted for among the factors in each term, but the assignment of column subscripts to row subscripts varies between terms over all possible permutation functions σ on the set $\{1, 2, \dots, \text{n}\}$.

2 Computing the Determinant

2.1 Two-by-two Case

For

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

we have

$$\det A = ad - cd. \tag{6}$$

[4]The eveness or oddness of the required number of position interchanges can be succinctly described as the **parity** of σ.

Note that we add a term with factors chosen by moving down to the right and subtract a term formed by moving down to the left through the matrix. Further, the factors in each term are chosen from mutually distinct rows and columns.

2.2 Three-by-three Case

For

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix},$$

we have

$$\begin{aligned} \det A &= a_{11}a_{22}a_{33} + a_{21}a_{32}a_{13} + a_{12}a_{23}a_{31} \\ &- a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} - a_{23}a_{32}a_{11}. \end{aligned} \tag{7}$$

Similar comments apply as for the 2×2 case. Each term involves three factors, chosen from mutually distinct row / column positions; for example, the last term includes a factor in row 1 from column 1, from row 2 in column 3, and from row 3 in column 2. We see that each row and each column only appears once for the factors in each term. All possible assignments (permutations) of columns to row positions are included over the six terms. Terms whose factors are located by "down to the right" paths through the matrix A enter with positive signs, while those corresponding to "down to the left" paths through A enter with negative signs. See Figure 9.1 in Simon and Blume on p. 193 for a nice visual mnemonic description of the algorithm.

2.3 Larger than Three-by-Three

Applying the definition to an $n \times n$ square matrix recursively recasts the calculation in terms of determinants of $(n-1)\times(n-1)$ square matrices. Repeat this recursion enough times and you reach a calculation which only depends on 3×3 (or 2×2) square matrices. So don't worry about determinants of big square matrices; with patience and effort their calculation can be recast in terms of the simpler cases mentioned above.

3 Algebraic Properties of the Determinant

Some algebraic properties of the determinant are listed here without proof because they are assumed knowledge, and are likely to appear with no introduction in discussions of econometrics.

Let A and B be square matrices of the same dimensions. Then

(a) $\det(A^T) = \det(A)$;

(b) $\det(AB) = \det(A)\det(B)$.

Exercise: We might feel inclined to include the property $\det(A + B) = \det(A) + \det(B)$ on the list. Provide examples to show cases where this property works and where it does not. (Thus it is not a general algebraic property of determinants).

4 Uses of the Determinant

4.1 Matrix Inversion

The determinant appears in an explicit formula for the inverse of a matrix. First we need a preliminary definition. Let C_{ij} refer to the (i, j)th cofactor of A, as before. Then the **adjoint** of A is given by $\operatorname{adj} A = [C_{ji}] = [C_{ij}]^T$ – the adjoint equals the transpose of the matrix of cofactors.

With this definition in hand, the inverse formula is

$$A^{-1} = \frac{1}{\det A} \cdot \operatorname{adj} A. \tag{8}$$

4.2 Testing for Nonsingularity

The determinant gives rise to a simple test for whether a square matrix is nonsingular. In particular, a square matrix A is nonsingular if and only if it has a nonzero determinant. To see this, note that the formula for a matrix index is undefined if the determinant is zero; hence a matrix with a zero determinant is not invertible.

Conversely, a square matrix is nonsingular if and only if it is invertible. But then the determinant (the denominator in the inverse formula) cannot be zero.

4.3 Kramer's Rule

Know this cold and you have one of the potential obstacles to passing the prelim under control. (Jeff Perloff is fond of questions where the equilibrium solution can be obtained by first applying Kramer's rule. Then derivatives are employed to analyze the comparative statics of the equilibrium.).

The idea here is that the solution to a system of linear equations may be written down explicitly using determinants.

Given the linear system $A\mathbf{x} = \mathbf{b}$, Kramer's rule says the ith component of the solution is

$$x_j = \frac{\det(B_j)}{\det(A)}, \tag{9}$$

where B_j is the matrix formed by replacing column j of A with the RHS vector b.

Exercise: Use Kramer's rule to write out an explicit formula for the general solution to a system of three equations in three unknowns.

Exercise (Bushenomics 101): A variant of the IS-LM model which incorporates taxes is introduced in problem 9.18 of Simon and Blume. Apply the Linear Implicit Function Theorem to this version of the IS-LM model in order to explore the predicted effect of (1) increased government spending and (2) tax cuts the on interest rates and output. Do the results support the Bush administration's assertion that larger deficits do not result in higher interest rates? Are there any plausible restrictions on the model in which interest rates do not increase? (For background, read section 9.3).