Location-Price Games when Consumers have Heterogeneous Tastes and Incomes

Gordon C. Rausser, Leo K. Simon and Ross Wildes

August 30, 2003

PRELIMINARY DRAFT: DO NOT CITE WITHOUT PERMISSION OF THE AUTHORS

JEL CODES: C72 D43, L13, L15, R39

Contact author: Leo Simon. 207 Giannini Hall, University of California at Berkeley, Berkeley, CA 94720.
Email: simon@are.berkeley.edu. Phone: (510) 642 8230. Fax: 510 643 8911.

ABSTRACT

1. INTRODUCTION

This paper combines computational and analytical techniques to study a model of product differentation and spatial competition. We analyze the pure-strategy, subgame perfect equilibria of a two-stage game in which producers first choose location then price, and consumers' characteristics are heterogeneous with respect to both tastes and incomes. In contrast to many specifications with a one-dimensional space of consumer characteristics, one cannot obtain explicit solutions for equilibrium prices in the second-stage subgames. Consequently, one cannot explicitly represent firms' profit functions in the induced single-stage location game, in which the second-stage subgames are replaced by equilibrium payoff vectors. Computational methods are required, therefore, in order to solve the game. Having computed a numerical solution, and in the process obtained numerical values for the derivatives required for comparative statics analysis, we then apply standard analytical tools to study the nature of interfirm competition in our setting.

In a location-then-price model such as ours, a core issue for comparative statics analysis is the inherent tension between location- and price-competition. Holding prices constant, location competition between two firms will be more intense, the greater is the rate at which consumers will shift from one product to the other in response to shifts in location. As a general rule, however, the more responsive is the consumer sector to location shifts, the more responsive will it be to price shifts. Consequently, as a general rule, parameter changes which intensify the degree of location competition, holding prices constant, will also intensify the degree of price competition. But as price-competition intensifies, equilibrium prices will decline, reducing the incentives for location competition! This, then, is the tension: any comparative statics effect that we study necessarily involves a balance between these two countervailing components (and others as well). One of our primary goals in this paper is to explore the nature of this tension, and the factors which determine how it is resolved.

The starting point for our work is the seminal paper by Caplin and Nalebuff (1991). This paper identified conditions on consumer preferences and the distribution of consumer characteristics across the population which guaranteed existence and, under certain conditions uniqueness, of a pure-strategy equilibrium of the Bertrand game in which suppliers of differentiated products compete by price. Armed with this technical tool, a number of papers have obtained explicit representations for the pure-strategy equilibria of various location-then-price games in which the distribution of consumer characteristics is non-uniform.[1] To our knowledge, all of these contributions maintain the assumptions that: (a) consumers are distinguished by a one-dimensional taste parameter; implicitly, they all have the same income level; in fact, typically, income

---

[1] The papers most closely related to ours are Tabuchi and Thisse (1995) and **?**

is not explicitly modeled; (b) each consumer is required to purchase one unit from one of two competing producers; Our paper relaxes both of these assumptions: income heterogeneity is included in the space of consumer characteristics and consumers have the option of purchasing neither good, either because they prefer not to or because they cannot afford to. The benefit of these extensions is that they considerably enrich the context in which our two firms compete; the cost is that in this enriched context, an explicit representation of the solution to our problem cannot be obtained.

To illustrate how spatial competition models can be enriched by relaxing assumption (a), consider the standard Hotelling (1929) duopoly model on the unit interval, with consumers distributed nonatomically over the interval and quadratic transportation costs (d'Aspremont, Gabszewicz and Thisse, 1979). Suppose that consumer *i* prefers good #1 to #2, but that #1's price is higher. The consumer will purchase good #1 if the price differential does not exceed the differential between good #1's and #2's Euclidian distance from the consumers' location/ideal point (check the lit).[2] The distance differential reflects the degree to which consumer *i* prefers good #1 to #2; all consumers are different in this regard. The *price* differential is a proxy for the incremental utility which consumer *i* would obtain from spending this differential on the some third, numeraire good; the model we are discussing treats all consumers as *the same* in this regard. Expressed rather imprecisely, setups of this kind model consumers as having different preferences with respect to the differentiated commodity but *identical* preferences with respect to the numeraire commodity. Clearly this is only half the story, since under diminishing marginal utility, agents with identical preferences but different incomes will view a given price differential differently: the higher the income the less utility will be derived from the additional purchasing power represented by this differential. When heterogeneous incomes are added to spatial competition models, this second half of the story can be incorporated, increasing the explanatory power of the models. To illustrate, it would be difficult within an homogeneous income model to explain the ubiquituous fact that in markets for a commodities with a wide quality spectrum—cars, wine, restaurants, etc.—the markup over costs is much higher at the top end of the spectrum than at the bottom. Using a heterogeneous income model exhibiting diminishing marginal utility for the numeraire good, the property just described follows immediately from the preceding discussion: the lower is the incremental utility derived by the "marginal consumer" from a given price differential, the less intense will be the level of price competition.[3]

---

[2] In symbols, the cost to a consumer with ideal point $\bar{x_i}$ of purchasing good j is $p_j + (x_j - \bar{x_i})^2$. Good #1 is purchased if $p_1 - p_2 \leq \big((x_2 - \bar{x_i})^2 - (x_1 - \bar{x_i})^2\big)$.

[3] In this paper, we confine ourselves to consumer sector specifications satisfying the conditions set out in Caplin and Nalebuff (1991). One of these (assumption A1') is that the marginal utility of income is independent of the level of income. Thus, we must postpone our investigation of the implications of the above remarks until the followup to this paper (See extensions section)

We now discuss the implications of relaxing assumption (b) above. One of these is related to the preceding discussion. When consumers are obliged to purchase one of the two differentiated products, the only tradeoff which matters is the one between these two products. Once we introduce the option of purchasing neither, the consumer has a second tradeoff to consider, between purchasing one of the differentiated products versus buying more of the numeraire good. In the presence of this additional option, the level of both price and location competition intensifies. A producer can now gain market share at the expense of *two* competitors by either lowering price or increasing location: the supplier of the other differentiated product and the supplier of the numeraire good. That is, by relaxing assumption (b) we are, in effect, adding a third firm into the marketplace. This addition significantly complicates the nature of spatial competition. In particular, the familiar monotone relationship between the distance between firm locations and equilibrium prices no longer obtains: in the solution to our model firm #2 is located to the left of #1; in a neighborhood of this solution, when #2 moves closer to #1, the equilibrium price of good #1 declines, as one would expect, but #2's price actually increases. The reason is that while the shift intensifies the degree of #2's price competition with firm #1, it diminishes the degree of competition with the numeraire good, and the latter effect dominates.

A second effect of relaxing assumption (b) relates to the possibility that consumers in our model may be income-constrained. Some may prefer the more expensive product but be unable to afford it, and so purchase the cheaper one. Others may be unable to afford even the cheaper one, and so purchase neither. (Note that this possibility only becomes interesting to model once assumption (a) has been relaxed: under assumption (a), if a product is unaffordable to one consumer it is unaffordable to all of them!) The fraction of consumers who are income-constrained is an important factor in determining the relationship between location- and price-competition, discussed above (page 1). Other things equal, the larger is this fraction, the less intense will be the level of location-competition: location shifts that increase the desirability of a product will have less impact on market shares, the smaller is the fraction of consumers whose product choices are made on the basis of desirability. The effect of income constraints on the intensity of price-competition is more subtle. For cross-price competition, the previous observation also applies: a reduction in lower quality good's price will have no impact on those consumers who would be induced to shift from the higher-quality good if they could afford it in the first place, but who, because they are income-constrained, are already buying the lower-quality good. On the other hand, depending on circumstances, a firm's incentive to reduce its *own* price may be increased or decreased by the presence of income-constrained consumers, since a price reduction will induce some consumers to shift allegience because they prefer to, and others because they can now afford to. Because of the issues just discussed, the comparative statics properties of our model, in particular, the tension between price- and quality-competition, will be quite sensitive to the fraction of consumers who are income-constrained (see in particular §6.2 below).

The paper is organized as follows. Section 3 is a brief discussion of computational methods and issues. In particular, we explain why and how we solve a “smoothed” version of our original problem. In section 4 we plot firms’ profits and reaction curves, as a numerical check of our computational solution. The reaction curve graphs provide strong evidence that the equilibria we compute are unique up to firm labels. Section 5 is a detailed analysis of the first order conditions in both the price subgames and the location game. In section 6, we study the comparative statics effects of varying the five key variables which parameterize the base case for our numerical simulations. We conclude and discuss extensions in section 7.

2. THE MODEL

There are two identical firms, indexed by *i*, each producing an homogeneous good. The *location* of good *i* is represented by a point $x_i \in \mathbb{R}_+$, $x_i \geq 1$.[4] A *location vector* is a pair $\mathbf{x} = (x_1, x_2)$. We will restrict attention to equilibria in which good #1 is located to the right of good #2, with the interpretation that good #1 is the higher quality product. Since our firms are identical, any equilibrium that we identify with $x_1 > x_2$ must have a mirror-image counterpart in which $x_2 > x_1$.

In order to produce x_i, firm *i* incurs a fixed cost of $C(x_i) = c{x_i}^{\gamma}$, with $c > 0$ and $\gamma \geq 1$. Marginal costs are assumed to be zero.[5] Once a location vector $\mathbf{x}$ has been determined, firms engage in Bertrand price competition, choosing a price vector $\mathbf{p}(\mathbf{x}) = (p_1(\mathbf{x}), p_2(\mathbf{x})) \in \mathbb{R}^2_+$. For a location vector $\mathbf{x}^\star$ to be a pure-strategy subgame perfect equilibrium (PSSPE), the following requirements must be satisfied: (a) for every location vector $\mathbf{x}$, the vector $\mathbf{p}(\mathbf{x})$ must be a Bertrand-Nash equilibrium; (b) given that firms choose Bertrand Nash prices in every subgame, the vector $\mathbf{x}^\star$ must be a Nash equilibrium for the single-stage location game induced by replacing second-stage subgames with the payoffs generated by Bertrand equilibrium prices.

Caplin and Nalebuff (1991) identify a pair of sufficient conditions for existence of a pure-strategy equilibrium in every price subgame. Our consumer sector is specified so that these conditions are satisfied. Because consumers' incomes in our model are heterogeneous, the basic versions of Caplin-Nalebuff's conditions—A1 and A2—need to be strengthened to A1' and A2'. There is a continuum of consumers, with mass normalized to unity. A consumer in our model has three options: she can purchase one unit of either good, or purchase neither good. As Caplin and Nalebuff (1991, page 45) observe, the latter option can be interpreted as the purchase of a third differentiated product, located at the origin, at at a price of zero. (Given the availability of this option, our differentiated product market is not strictly a duopoly but a three firm oligopoly, in which the third firm has only one admissible strategy.) Each consumer is characterized by a *taste parameter* $\alpha \in [\alpha_\ell, \alpha_u] \subset \mathbb{R}_+$ and an *income level* $y \in [y_\ell, y_h] \subset \mathbb{R}_+$. Given a price vector $\mathbf{p}$, a consumer with characteristics (α, y) derives utility $V(0, \mathbf{p}; \alpha, y) = y$ if she purchases neither good and $V(x_i, \mathbf{p}; \alpha, y)$ from the purchase of good *i*, where:

$$V(x_i, \mathbf{p}; \alpha, y) \quad = \quad (\alpha + y - p_i)x_i \tag{1}$$

---

[4] Under the utility specification (1) below, if x_i were less than unity, there would be an open set of consumers who would strictly prefer *not* to purchase good *i*, even if its price were zero. By requiring $x_i \geq 1$, we ensure that good *i* is at least weakly desirable to all consumers.

[5] In this respect, our model differs from models of vertical differentiation such as **?** and Cremer and Thisse (1991), which follow Mussa and Rosen (1986): in these papers, fixed costs are zero while marginal costs are constant with respect to output, and increasing with respect to quality. As will become apparent, the distinction has very little significance.

The term $(y-p_i)$ is interpreted as the quantity of an undifferentiated numeraire commodity that a consumer with income y can purchase after spending p_i on good i. With a little manipulation, we obtain:

Observation O1. *V satisfies Assumption A1' in Caplin and Nalebuff (1991, pages 29 and 44).* [6]

To economize on notation, it will be convenient for the next definition to define $x_0 = 0$, and to say that a consumer "purchases good 0" (at a price of zero) if she purchases neither of the two differentiated goods. Given a location vector **x** and prices **p**, a consumer with characteristics (α, y) will purchase good i if for each of the other two alternatives, either the alternative is unaffordable or it yields less utility than i. In symbols, the set of consumers who purchase good i is $I_i(\mathbf{x},\mathbf{p})$, defined by

$$I_i(\mathbf{x},\mathbf{p}) \qquad = \qquad \cap_{j\neq i}\left\{(\alpha,y) : V(x_j,\mathbf{p};\alpha,y) > V(x_i,\mathbf{p};\alpha,y) \text{ implies } p_j > y\right\}.^7 \tag{3}$$

The budgetary requirement that $(\alpha,y) \in I_i(\mathbf{x},\mathbf{p})$ implies $y \geq p_i$, while logically necessary, has not been emphasized in the spatial competition literature to date. Presumably the reason is that this literature has focused on the case in which all consumers have the same income level and in this context the constraint binds either all consumers or none. As we shall see, however, when incomes are heterogeneous it plays a significant role in the model, affecting in particular the interaction between price- and location-competition.[8,9]

The distribution of consumer characteristics is a truncation of the bivariate normal distribution with mean $(E[\alpha], E[y]) \in [\alpha_\ell, \alpha_u] \times [y_\ell, y_h]$ and variance-covariance matrix $\begin{bmatrix} \sigma & 0 \\ 0 & \sigma \end{bmatrix}$, where $\sigma > 0$. Let $g(\alpha, y)$ denote the

---

[6] To establish this, we need to reformulate our utility function in the form of Caplin and Nalebuff (1991)'s display (3.1), modified to satisfy A1':

$$U(\chi,\mathbf{p};\beta,y) \qquad = \qquad \sum_{k=1}^{n}\beta_k t_k(\chi) \quad + \quad (y-p_i)t_{n+1}(\chi) \tag{2}$$

where χ is an n-vector of product attributes, β is an n-vector of consumer characteristics while and t satisfies the conditions set out in Caplin and Nalebuff (1991, page 29). In order to do this, we need to reformulate our one-dimensional product space as a two-dimensional space. Accordingly, we will set $n = 2$ and for $i = 1,2$, let $\chi^i = (x_i, x_i)$. Also, let $\chi^3 = (0,1)$ denote the "third," zero-priced good, mentioned above. Also let $\beta = (\alpha, 0)$. Finally, let $t : \mathbb{R}^2 \to [0,1]^3$ be defined by $t(\chi^i) = \begin{bmatrix}\chi^i_1 & 0 & \chi^i_2\end{bmatrix}$. Now note that for $i = 1,2$, $U(\chi^i,\mathbf{p};\beta,y) = \alpha x_i + (y-p_i)x_i$ while $U(\chi^3,\mathbf{p};\beta,y) = 0 + (y-p_i)1 = (y-p_i)$. Clearly, U is equivalent to V, establishing that Observation **O1** is valid.

[7] When any of these weak inequalities hold with equality, the consumer's optimal decision is not uniquely defined. We can, however, ignore this problem because the set of consumer characteristics for which it arises has measure zero.

[8] One might argue that the budget restriction is unimportant in practical applications, on the grounds that most people's income exceed the price of any single product that they might wish to purchase. When interpreted less literally, however, budgetary constraints arise in a wide variety of contexts. In particular, when purchasing major products such as houses and cars, many consumers face a binding credit constraint. More generally, the idea that consumers create "mental accounts" or "budget categories" has gained widespread acceptance in the behavioral economics and marketing literatures. See, for example, Thaler (1985), Thaler (1999). Under the mental accounting hypothesis, consumers allocate their income among a number of categories of goods and services, and "balance the budget" within each of these categories. To the extent that this hypothesis is valid, we would expect that the budget constraint modeled in this paper would be a significant determinant of consumer behavior.

[9] Caplin and Nalebuff do not explicitly include a budget constraint in their model of the consumer sector. Nor can it be inferred from their specification of consumer preferences (display 3.1 on page 29). This is a potential source of concern: their existence argument is an extremely delicate balancing act and the addition of a further restriction could, in principle, destroy the balance. Fortunately for us, our budget requirement enters their model unannounced, as part of their specification of the "reservation price property" (R_i on page 35).

p.d.f. obtained by truncating this distribution to the rectangle $[\alpha_\ell, \alpha_u] \times [y_\ell, y_h]$. From Caplin and Nalebuff (1991, pages 30, 44) and their comment about truncations on page 31, we have:

Observation O2. *g is $\frac{-1}{3}$-concave, with convex, compact support and positive volume and hence satisfies Caplin and Nalebuff (1991)'s Assumption A2'.*

The *demand* for firm i's product at locations $\mathbf{x}$ and prices $\mathbf{p}$, is now obtained by integrating its customer base with respect to the p.d.f. g. Letting $\mathrm{D}_i(\mathbf{x},\mathbf{p}) = \int_{I_i(\mathbf{x},\mathbf{p})} g(\alpha, y) d(\alpha, y)$ denote this demand, the *profit function* for firm i can now be written as:

$$\pi_i(\mathbf{x},\mathbf{p}) \quad = \quad p_i \mathrm{D}_i(\mathbf{x},\mathbf{p}) \quad - \quad c {x_i}^{\gamma} \tag{4}$$

A Nash equilibrium for the Bertrand price game defined by $\mathbf{x}$ is a pair $\mathbf{p}^\star = (p_1^\star, p_2^\star)$ such that for each i, $p_i^\star$ maximizes $\pi_i(\mathbf{x}, (\cdot, p_{\neg i}^\star))$. Caplin and Nalebuff (1991, Proposition 10), combined with observations **O1** and **O2**, yield:

Observation O3. *For every location pair* $\mathbf{x}$*, there is a pure-strategy Nash equilibrium for the Bertrand price game defined by* $\mathbf{x}$.

Because our model is not, technically, a duopoly model (see page 5), the uniqueness results in Caplin and Nalebuff (1991, Section 6) are not applicable. Nonetheless, our simulations provide a great deal of evidence that the price equilibria in our model are indeed unique, provided that each firm has a positive market share. We shall henceforth assume uniqueness and write $\mathbf{p}^\star(\cdot)$ to denote the mapping from location vectors to equilibrium prices. A PSSPE for the two-stage location-price game is a pair $\mathbf{x}^\star = (x_1^\star, x_2^\star)$ such that for each i, $x_i^\star$ maximizes $\pi_i((\cdot, x_{\neg i}^\star), \mathbf{p}^\star(\cdot, x_{\neg i}^\star))$. While uniqueness of equilibria in the price games does imply existence of a *mixed*-strategy subgame perfect equilibrium, we are not aware of any analytical results that guarantee existence of a PSSPE in the present context.

In the basecase for our numerical simulations, the parameters of the model are set to the following values:

Parameter name	Parameter symbol	Parameter value
Cost coefficient	c	0.05
Cost exponent	γ	1.00
Lower bound on taste distribution	α_ℓ	0.00
Upper bound on taste distribution	α_u	1.00
Mean of taste distribution	$E[\alpha]$	0.50
Lower bound on income distribution	y_ℓ	0.00
Upper bound on income distribution	y_h	9.00
Mean of income distribution	$E[y]$	4.50
Variance of joint distribution	σ	15.00

3. COMPUTATION AND SMOOTHING

Our computational algorithm for solving our two-stage game makes nested calls to matlab's fmincon routine. The fmincon program is designed for solving optimization problems subject to constraints. We ignore the optimization component of the program and "minimize" a constant function subject to the constraints that the first order conditions for our firms are satisfied with equality. (Our algorithm cannot identify corner solutions.) In the outer loop of the algorithm, the first order conditions identify a Nash equilibrium in prices holding locations constant. In the inner loop, we solve for a Nash equilibrium in locations, assuming Nash equilibrium prices for each location pair. Since the neither the price nor the location game has a closed-form solution, the algorithm computes numerical gradients of the first order conditions. To evaluate these gradients in the inner, location loop, we need information on how equilibrium prices change with locations. To obtain this, we compute, via the implicit function theorem, the first- and second-order derivatives of the function mapping locations to equilibrium prices. We then use a second-order Taylor expansion to obtain the required estimates of price changes.

Because our objective function can only be computed by means of numerical integration methods, technical difficulties arise in the computation of derivatives of these functions. To compute integrals, we use the Gauss-Legendre method described in **?**. This involves taking a weighted average of function values, evaluated on a finite grid of points. While the method can be used to compute integral *levels* to a high degree of accuracy, it is, without further modification, highly unreliable as a technique for numerically computing first and, especially, second derivatives of an integral, particularly when, as in our case, the limits of integration are only piecewise differentiable. To illustrate the problem, let $x(t) = \min(0, 1 - t)$ and consider the problem of twice numerically differentiating with respect to t the numerical estimate of $\int_0^{x(t)} f(x')dx'$, in a neighborhood of $t = 1$.

The key points to make about the algorithm are:

- we are using matlab's fmincon routine, but this is an optimization package. But we need to optimize two things simultaneously, i.e., each party's payoff given the other's action. Since fmincon is only set up to optimize a single objective function, we put in a dummy function to minimize (f = 1), and solve the following minimization problem: minimize f subject to the constraints that both players' first order conditions are zero.
- the way the solution algorithm works for this class of problem is, very roughly, as follows. Our task is so choose $x^* \in \mathbb{R}^2$ such that $v'(x) = 0$, where $v' = (v_1, v_2)$, and v_i is the derivative of player

i's payoff function w.r.t. i's choice variable, z_i. We choose a starting value of x, i.e., x_0. Now, given a value x_t, the algorithm numerically computes $Jv'(x_t)$ and $v'(x_t)$, where the i'th row of Jv' is the derivative of v_i w.r.t. x_t, and chooses x_{t+1} so that $dx = x_{t+1} - x_t$ solves $Jv'(x_t)dx = -v'(x_t)$. Thus, if all but the first term in the Taylor expansion of v' about x_t were zero, then $v'(x_{t+1})$ would be zero, and the algorithm would haved converged to the solution in one iteration. Of course, these higher order terms will not be zero in general, and further iterations will be required. The algorithm continues until the norm of the adjustment term dx is is sufficiently small that it is inferior to some pre-specified tolerance level. It will be clear from this description that the performance of the algorithm will depend critically on the accuracy with which the numerical derivatives are computed. In particular, it will be readily apparent that unless these derivatives are computed to a high degree of accuracy, the algorithm will be unable to satisfy anything but an extremely slack convergence criterion.

- because we are modelling a two-stage game, we need to use a nested, two-stage solution algorithm. Specifically, in our problem, i's payoff function u_i is a function of locations x and prices p. The requirement of subgame perfection is that for each location pair x and each i, $\frac{\partial u_i(x,p)}{\partial p_i} = 0$. Thus, in the process of obtaining a numerical solution for the *location game*, in which players choose locations with the understanding that prices will adjust to equilibrium levels given these locations, the solution algorithm must, at every iteration, identify a solution to the *price subgame* given these locations. In short, a solution algorithm for the subprice game must be embedded in the solution algorithm for the location game.
- a major technical difficulty arises in the construction of this algorithm. It relates to the numerical computation of the first derivatives (the u_i's) and the matrix of second derivatives u'. The problem is that solution prices vary with x, but not in a closed-form way, so that in the process of computing derivatives of the u_i's with respect to x's, it is, in principle, necessary to compute solutions to multiple price subgames. Indeed, in our algorithm, in order to compute a full set of derivatives of u_i, given $\bar{x}$, it is necessary to evaluate u_i at eleven different nearby values of x. In a "brute-force" version of our algorithm, therefore, we would need to identify equilibrium prices for twenty-two different subgames, every time we computed a derivative of u'! This approach would be prohibitively slow. Our alternative approach is to obtain approximate solutions to nearby price subgames, using the implicit function theorem, and feed these solutions into our derivative computation routine. It turns out, however, that *linear* approximations are insufficiently accurate for our purposes. In order to obtain satisfactory standards of accuracy, we needed to compute second-order Taylor expansions about our initial price equilibria.

- <u>second order conditions and corner solutions</u>: our algorithm is not very general. It is not equipped to handle corner solutions. Nor is it equipped to steer away from extreme points at which second order conditions are violated. While it is relatively easy to choose parameter values to ensure that all solutions must be interior, the latter problem is harder to control for. In order to impose the additional constraints on our problem that for each i, $u_{ii} \leq 0$, we would have needed to compute a full set of *third*-order derivatives. Since this would have slowed the algorithm considerably and would have been prohibitively tedious to code, we contented ourselves with a trap in the code to alert us to solutions which fail the second order conditions.

4. COMPUTATION CHECK

In order to confirm that we have indeed computed a valid solution to our two-stage game, we performed two numerical checks. The results of these checks are graphed in Fig. **??** and Fig. **??**. The left panel of Fig. **??** plots each firm's payoff as a function of its own price, holding constant both firms' locations and the other firm's price at their numerical solution levels for the two stage game. The right panel plots each firm's payoff as a function of its own location, holding constant the other firms' location at its numerical solution level for the two stage game, and solving for equilibrium prices at each location vector. Note that in each stage of the game, each player's payoff is indeed maximized at the value identified by our numerical simulation. Together these graphs provide numerical confirmation that the solution to our simulation constitutes an equilibrium for the two-stage game. In addition to confirming the existence of an equilibrium, the graphs in Fig. **??** provide useful additional information. Because the distribution of consumers is ρ-concave, Caplin and Nalebuff (1991, Theorem 1) guarantees that the functions graphed in the left panel will be quasi-concave. In fact, they are concave. As far as we are aware, there are no analytical results pertaining to the functions graphed in the right panel. In fact, however, these turn out to be concave also. Indeed, these graphs suggest the surprising possibility that analytical conditions might be identified which would guarantee concavity of payoffs with respect to own locations in the two-stage game. These conditions would, in turn, imply existence of pure-strategy equilibria in the two-stage game.

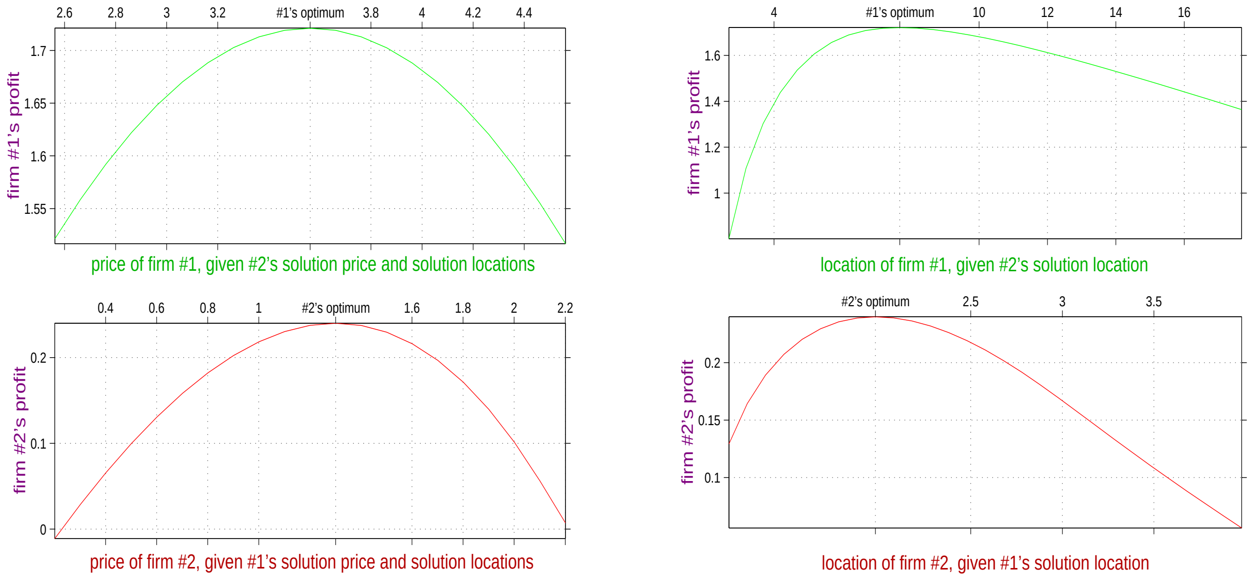

FIGURE 1. Payoff functions in the equilibrium-path price subgame and the location game

The two panels of Fig. **??** plot players’ reaction curves in price and location space. In the left panel, locations are fixed at their solution values in our basecase simulation. In the right panel, locations vary and we solve for equilibrium prices for each location pair. Note that the curves in the right panel are contained in the region where firm #1’s location exceeds #2’s. Because firms’ payoffs are identical in the basecase of our simulation, there is, necessarily, a mirror image of these curves (with firms’ identities reversed) in the region where #2’s location exceeds #1’s. The numerical technique we used to compute these curves was different from the fixed-point algorithm we used to obtain our original solution. The “$\star$’s” in the left and right panels denote, respectively, our computed fixed-point in price space, given the solution locations, and the fixed-point in location space itself. The fact that the “$\star$’s” coincide with the intersection of the reaction curves in each panel is a further confirmation that our fixed point algorithm indeed delivered a subgame-perfect equilibrium. Once again, the figure conveys useful additional information. In each panel, the two curves have a unique intersection within the range plotted. Indeed, in each case, the shapes of the two curves within this range provides strong numerical evidence that the equilibrium we have identified is the unique pure-strategy sub-game perfect equilibrium in which #1’s location exceeds #2’s.[10]

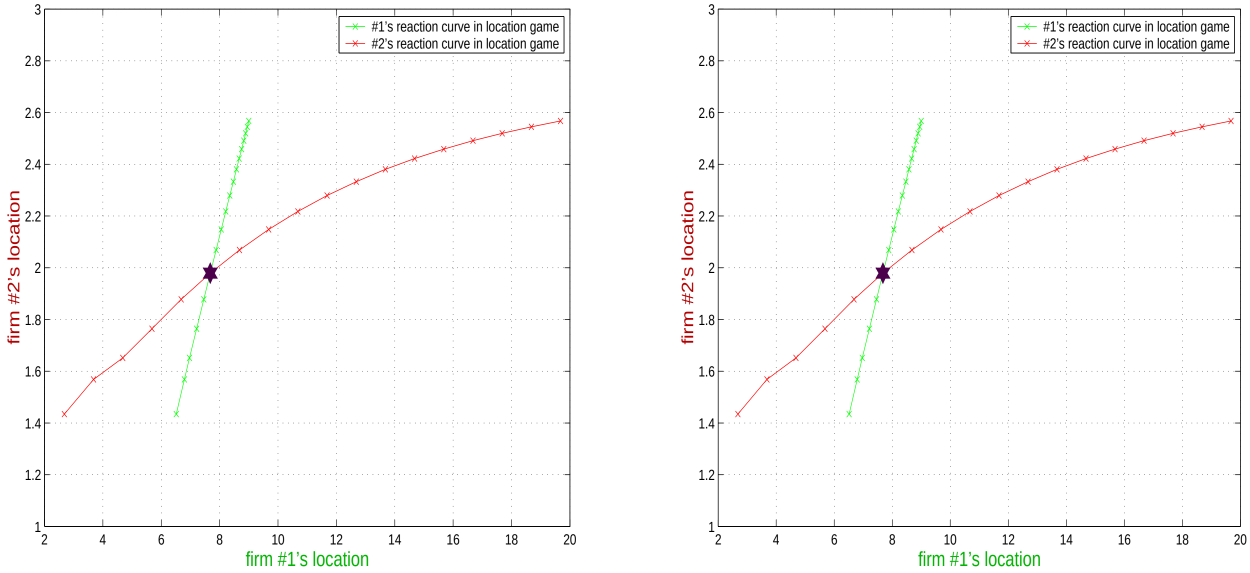

FIGURE 2. Reaction curves in the equilibrium-path price subgame and the location game

---

[10] We have, of course, displayed the price reaction curves for the on-the-equilibrium-path subgame. Comparable graphs for the other subgames exhibit the same characteristics and are available on request.

5. EQUILIBRIUM

Somehow up front in this write-up we need to explain our philosophy. We can’t get a closed-form solution. We can’t expect to be able to sign all the terms in the comparative statics. But we can sign the important terms and decompose the game into its component parts. More important, the numerical results provide us with a guide about what to focus on, i.e., what turns out to be important and what isn’t. For example, we find below that when we analyze the Jacobian of what we will call the Nash location function, the effects that are decisive in determining our comparative statics results are the partials of this function with respect to locations, holding prices constant. When prices adjust to changes in locations, nothing much changes in a qualitative sense. In short, the numerics direct us to what’s important analytically. So in what follows, we will sign what we can sign, point out what the ambiguities are and focus on the terms that turn out to really matter in our context. We also need to say something about the fact that all of the analytics we provide below apply to the unsmoothed model but the numbers we use are from the smoothed model

5.1. **Market Regions.** Fig. **??** illustrates the allocation of customers to the two firms. (Note that #’1 market region is much larger than depicted in the figures, since the maximum level of income, $y_h = 9$.) The left panel of the figure depicts the allocation for the equilibrium of our computational model. It reflects the fact that consumer boundaries are smoothed for computational purposes. The right panel represents the corresponding regions for our analytical (i.e., unsmoothed) model, in which the boundaries of firms’ market regions are piecewise linear, and illustrates the various terms terms defined below. Let $\mathrm{D}_i(x,p)$ denote the measure of the set of consumers which purchase i’s product (henceforth we shall refer to $\mathrm{D}_i(x,p)$ as i’s

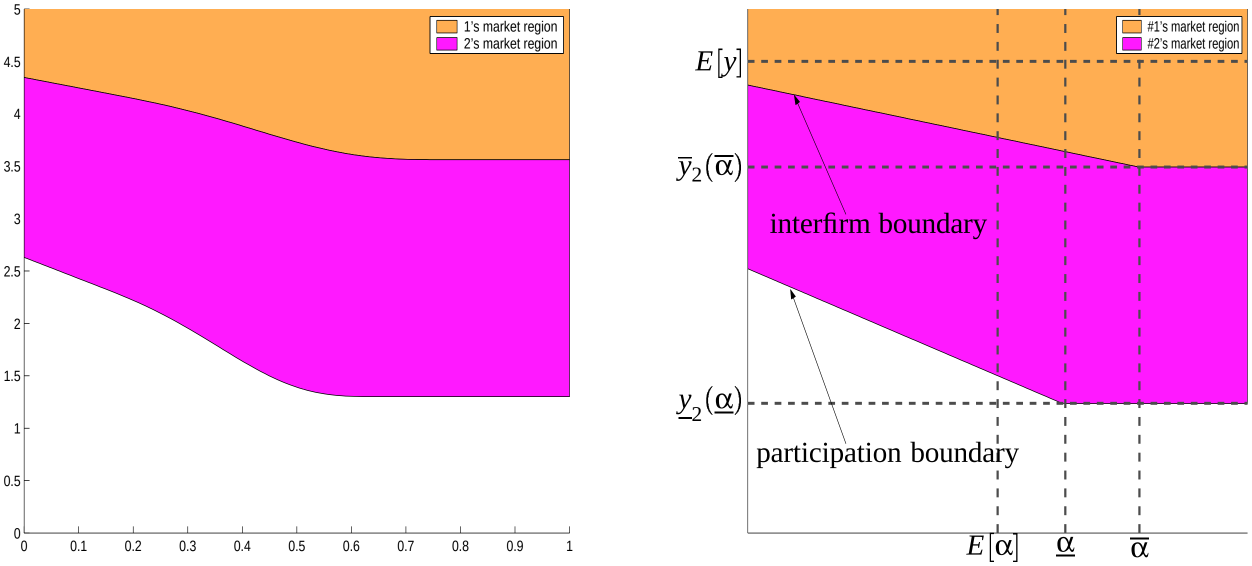

FIGURE 3. Market Regions In Equilibrium

market region). Player *i*'s *revenue* at the location-price quadruple (x,p) is the product of his price p_i and $\mathrm{D}_i(x,p)$, defined below. Observe from Fig. **??** that consumers with higher income levels purchase firm #1's good, middle income levels purchase #2's, and the poorest consumers purchase neither good. Note also that the income levels at which consumers switch from one firm to another decline as their taste parameters increase. For each *i* and α, let $\underline{y}_i(\alpha;x,p)$ and $\bar{y}_i(\alpha;x,p)$ denote respectively the lowest and highest income level at which a consumer with taste parameter α weakly prefers to purchase good *i* rather than either of his two alternatives, i.e., purchasing good $\neg i$ or nothing at all. (The symbol $\neg i$ denotes "the player who is not *i*", i.e., $\neg 1 = 2$ and $\neg 2 = 1$.) In Fig. **??**, $\bar{y}_2(\cdot|x,p)$ separates the two shaded areas while $\underline{y}_2(\cdot|x,p)$ separates the lower shaded area from the unshaded area. We shall henceforth refer to the line $\bar{y}_2(\cdot|x,p)$ as the *interfirm boundary*, i.e., the boundary between the two firms' market regions, and to $\underline{y}_2(\cdot|x,p)$ as the *participation boundary*, i.e., the boundary separating firm #2's clientele from the consumers who do not participate in the market at all. To reduce notation, we will henceforth suppress the dependence of $\underline{y}_i$ and $\bar{y}_i$ on (x,p)

The functions $\bar{y}_2(\cdot)$ and $\underline{y}_2(\cdot)$ are defined as follows:

$$\bar{y}_i(\alpha) = \begin{cases} y_h & \text{if } i=1 \\ \max\left(p_1, \frac{x_1\Delta p}{\Delta x} + p_2 - \alpha\right) & \text{if } i=2 \end{cases} \qquad (\bar{5})$$

$$\underline{y}_i(\alpha) = \begin{cases} \bar{y}_2(\alpha) & \text{if } i=1 \\ \max\left(p_2, \frac{p_2-\alpha}{1-1/x_2}\right) & \text{if } i=2 \end{cases} \qquad (\underline{5})$$

where $\Delta x = x_1 - x_2$ and $\Delta p = p_1 - p_2$. Note that by assumption Δx is positive, and, by implication, Δp will be positive also. Firm *i*'s market region can now be written as $\mathrm{D}_i(x,p) = \int_{\alpha_\ell}^{\alpha_u}\int_{\underline{y}_i(\alpha)}^{\bar{y}_i(\alpha)} g(y,\alpha)dyd\alpha$, where *g* is the bivariate normal distribution with parameters specified above, truncated to the rectangle $[\alpha_\ell \;\; \alpha_u] \times [y_\ell \;\; y_h]$. In the analysis that follows, it will be significant that the interfirm boundary, $\bar{y}_2(\cdot)$, lies every below the average level of income.

While the local properties analyzed in this and later subsections appear to be quite robust numerically, they depend in principle on certain global characteristics of the solution to our model. In particular, we will proceed below on the assumption that in equilibrium, the market regions for the two firms exhibit the characteristics illustrated in Fig. **??**. One of these characteristics has important distibutional implications: at every level of tastes, the income of the consumer who is indifferent between goods one and two is below the average income. There is no reason to expect that all of the qualitative characteristics of Fig. **??** hold for all parameter configurations. Our treatment below can readily be modified to take acount of marginal modifications to this picture. For example, if the boundary of either firm's market region has no flat spots, the expressions below simplify in obvious ways. Conceivably, however, the left intercepts for either of the lower boundary lines might not belong to the interior of the support of the income distribution. In this case, our analysis would need to be modified significantly.

It is useful to rewrite the expression for $\mathrm{D}_i(x,p)$ as $\overline{\mathrm{D}}_i(x,p) - \underline{\mathrm{D}}_i(x,p)$, where $\overline{\mathrm{D}}_i(x,p) = \int_{\alpha_\ell}^{\alpha_u} \int_{y_\ell}^{\bar{y}_i(\alpha)} g(y,\alpha)dyd\alpha$. and $\underline{\mathrm{D}}_i(x,p) = \int_{\alpha_\ell}^{\alpha_u} \int_{y_\ell}^{\underline{y}_i(\alpha)} g(y,\alpha)dyd\alpha$. Let $\overline{\alpha}(x,p) = \frac{x_1 \Delta p}{\Delta x} - \Delta p$ denote the largest value of α for which $\bar{y}_2(\alpha) = \underline{y}_1(\alpha)$ depends on values other than p_1 and let $\underline{\alpha}(x,p) = \frac{p_2}{x_2}$ denote the largest value of α for which $\underline{y}_2(\alpha)$ depends on values other than p_2 (see Fig. **??**). Consumers with taste parameters greater than $\overline{\alpha}$ and incomes in a neighborhood of $\bar{y}_2(\alpha)$ would strictly prefer to purchase good #1 than #2 but have insufficient funds to do so; similarly, consumers with taste parameters greater than $\underline{\alpha}$ and incomes in a neighborhood of $\underline{y}_2(\alpha)$ would strictly prefer to purchase good #2 than nothing, but cannot for the same reason. Note that at the given prices, firms are competing *with each other* only for consumers with taste parameters to the left of $\overline{\alpha}$, so that interfirm competition will be more intense, the further to the right is the location of $\overline{\alpha}$.

Observe for future reference that

$$\frac{\partial \bar{y}_2(\alpha)}{\partial p_j} = \begin{cases} \frac{x_1}{\Delta x} & \text{if } j = 1 \ \& \ \alpha \leq \overline{\alpha} \\ \frac{-x_2}{\Delta x} & \text{if } j = 2 \ \& \ \alpha \leq \overline{\alpha} \\ 0 & \text{otherwise} \end{cases} \qquad \frac{\partial \overline{\alpha}(x,p)}{\partial p_j} = \begin{cases} \frac{x_2}{\Delta x} & \text{if } j = 1 \ \& \ \alpha \leq \overline{\alpha} \\ \frac{-x_2}{\Delta x} & \text{if } j = 2 \ \& \ \alpha \leq \overline{\alpha} \\ 0 & \text{otherwise} \end{cases} \qquad (\overline{6})$$

$$\frac{\partial \underline{y}_2(\alpha)}{\partial p_j} = \begin{cases} \frac{x_2}{(x_2 - 1)} & \text{if } j = 2 \ \& \ \alpha \leq \underline{\alpha} \\ 0 & \text{otherwise} \end{cases} \qquad \frac{\partial \underline{\alpha}(x,p)}{\partial p_j} = \begin{cases} \frac{1}{x_2} & \text{if } j = 2 \ \& \ \alpha \leq \underline{\alpha} \\ 0 & \text{otherwise} \end{cases} \qquad (\underline{6})$$

while

$$\frac{\partial \bar{y}_2(\alpha)}{\partial x_j} = \begin{cases} \frac{-x_2 \Delta p}{(\Delta x)^2} & \text{if } j = 1 \ \& \ \alpha \leq \overline{\alpha} \\ \frac{x_1 \Delta p}{(\Delta x)^2} & \text{if } j = 2 \ \& \ \alpha \leq \overline{\alpha} \\ 0 & \text{otherwise} \end{cases} \qquad \frac{\partial \overline{\alpha}(x,p)}{\partial x_j} = \begin{cases} \frac{-x_2 \Delta p}{(\Delta x)^2} & \text{if } j = 1 \ \& \ \alpha \leq \overline{\alpha} \\ \frac{x_1 \Delta p}{(\Delta x)^2} & \text{if } j = 2 \ \& \ \alpha \leq \overline{\alpha} \\ 0 & \text{otherwise} \end{cases} \qquad (\overline{7})$$

$$\frac{\partial \underline{y}_2(\alpha)}{\partial x_j} = \begin{cases} \frac{-(p_2 - \alpha)}{(x_2 - 1)^2} & \text{if } j = 2 \ \& \ \alpha \leq \underline{\alpha} \\ 0 & \text{otherwise} \end{cases} \qquad \frac{\partial \underline{\alpha}(x,p)}{\partial x_j} = \begin{cases} \frac{-p_2}{x_2^2} & \text{if } j = 2 \ \& \ \alpha \leq \underline{\alpha} \\ 0 & \text{otherwise} \end{cases} \qquad (\underline{7})$$

All of the signs in (6) and (7) are obvious (since Δx and Δp are positive) except for $\frac{\partial \underline{y}_2(\alpha)}{\partial x_2}$, which is negative for $\alpha \leq \underline{\alpha}$. To see this recall that $\underline{\alpha} = \frac{p_2}{x_2}$ and x_2 is restricted to be greater than unity.

We now further decompose $\overline{\mathrm{D}}_2(x,p)$ and $\underline{\mathrm{D}}_i(x,p)$ as follows:

$$\overline{\mathrm{D}}_2(x,p) = \int_{\alpha_\ell}^{\overline{\alpha}}\int_{y_\ell}^{\overline{y}_2(\alpha)} g(y,\alpha)dyd\alpha + \int_{\overline{\alpha}}^{\alpha_u}\int_{y_\ell}^{p_1} g(y,\alpha)dyd\alpha = \underline{\mathrm{D}}_1(x,p) \qquad (\overline{8})$$

$$\underline{\mathrm{D}}_2(x,p) = \int_{\alpha_\ell}^{\underline{\alpha}}\int_{y_\ell}^{\underline{y}_2(\alpha)} g(y,\alpha)dyd\alpha + \int_{\underline{\alpha}}^{\alpha_u}\int_{y_\ell}^{p_2} g(y,\alpha)dyd\alpha \qquad (\underline{8})$$

Let $\xi(\theta)_{[a\;b]}$ denote the function that takes the value θ on the interval $[a\;\;b]$ and unity elsewhere. Applying Leibnitz's rule, we have

$$\frac{\partial\overline{\mathrm{D}}_i(x,p)}{\partial p_i} = \begin{cases} 0 & \text{if } i=1 \\ -\frac{x_2}{\Delta x}\int_{\alpha_\ell}^{\overline{\alpha}} g(\overline{y}_2(\alpha),\alpha)d\alpha & \text{if } i=2 \end{cases} \qquad (\overline{9})$$

$$\frac{\partial\underline{\mathrm{D}}_i(x,p)}{\partial p_i} = \begin{cases} \int_{\alpha_\ell}^{\alpha_u}\xi\left(\frac{x_1}{\Delta x}\right)_{[\alpha_\ell\;\overline{\alpha}]} g(\overline{y}_2(\alpha),\alpha)d\alpha & \text{if } i=1 \\ \int_{\alpha_\ell}^{\alpha_u}\xi\left(\frac{x_2}{x_2-1}\right)_{[\alpha_\ell\;\underline{\alpha}]} g(\underline{y}_2(\alpha),\alpha)d\alpha & \text{if } i=2 \end{cases} \qquad (\underline{9})$$

Similarly

$$\frac{\partial\overline{\mathrm{D}}_i(x,p)}{\partial x_i} = \begin{cases} 0 & \text{if } i=1 \\ \frac{x_1\Delta p}{(\Delta x)^2}\int_{\alpha_\ell}^{\overline{\alpha}} g(\overline{y}_2(\alpha),\alpha)d\alpha & \text{if } i=2 \end{cases} \qquad (\overline{10})$$

$$\frac{\partial\underline{\mathrm{D}}_i(x,p)}{\partial x_i} = \begin{cases} \frac{-x_2\Delta p}{(\Delta x)^2}\int_{\alpha_\ell}^{\overline{\alpha}} g(\overline{y}_2(\alpha),\alpha)d\alpha & \text{if } i=1 \\ -\int_{\alpha_\ell}^{\underline{\alpha}}\frac{p_2-\alpha}{(x_2-1)^2} g(\underline{y}_2(\alpha),\alpha)d\alpha & \text{if } i=2 \end{cases} \qquad (\underline{10})$$

Using (9), the derivative of firm *i*'s market region with respect to his own price is:

$$\frac{\partial\mathrm{D}_1(x,p)}{\partial p_1} = -\int_{\alpha_\ell}^{\alpha_u}\xi\left(\frac{x_1}{\Delta x}\right)_{[\alpha_\ell\;\overline{\alpha}]} g(\overline{y}_2(\alpha),\alpha)d\alpha < 0 \qquad (11\text{-}1\text{-}1)$$

$$\frac{\partial\mathrm{D}_2(x,p)}{\partial p_2} = -\left(\frac{x_2}{\Delta x}\int_{\alpha_\ell}^{\overline{\alpha}} g(\overline{y}_2(\alpha),\alpha)d\alpha + \int_{\alpha_\ell}^{\alpha_u}\xi\left(\frac{x_2}{x_2-1}\right)_{[\alpha_\ell\;\underline{\alpha}]} g(\underline{y}_2(\alpha),\alpha)d\alpha\right) < 0 \qquad (11\text{-}2\text{-}2)$$

As p_1 increases, firm #1 loses market share as the interfirm boundary shifts north (see Fig. **??**): consumers with α's to the left of $\overline{\alpha}$ switch to good #2 by preference, and those with α's to the right of $\overline{\alpha}$ switch to good #2 because they become income constrained. As p_2 increases, firm #2 loses market share in two directions,as the interfirm boundary shifts south while the participation boundary shifts north: #2's upper-end customers shift to good #1, while the lower-end customers shift to purchasing nothing.

Similarly, using (10), the derivative of firm *i*'s market region with respect to his location is:

$$\frac{\partial \mathrm{D}_1(x,p)}{\partial x_1} = \frac{x_2 \Delta p}{(\Delta x)^2} \int_{\alpha_\ell}^{\overline{\alpha}} g(\bar{y}_2(\alpha),\alpha) d\alpha > 0 \tag{12-1}$$

$$\frac{\partial \mathrm{D}_2(x,p)}{\partial x_2} = \frac{x_1}{x_2} \frac{\partial \mathrm{D}_1(x,p)}{\partial x_1} + (x_2 - 1)^{-2} \int_{\alpha_\ell}^{\underline{\alpha}} (p_2 - \alpha) g(\underline{y}_2(\alpha),\alpha) d\alpha > 0 \tag{12-2}$$

To see that (12-2) is positive observe that $\alpha < \underline{\alpha}$ implies $\alpha < \frac{p_2}{x_2} < p_2$ since x_2 is restricted to be greater than unity. Note that expression (12-2) involves two effects while (12-2) involves only one. In both instances, the first term relates to interfirm competition, i.e., the location of the interfirm boundary. The additional term for firm #2 captures the effect of location shifts on the participation boundary.

We are also going to need the derivative of *i*'s market share with respect to ¬*i*'s price. Differentiating (8) and using (12-1) it is straightforward to check that

$$\frac{\partial \mathrm{D}_1(x,p)}{\partial p_2} = \frac{\Delta x}{\Delta p} \frac{\partial \mathrm{D}_1(x,p)}{\partial x_1} > 0 \tag{13-1-2}$$

$$\frac{\partial \mathrm{D}_2(x,p)}{\partial p_1} = \frac{x_1 \Delta x}{x_2 \Delta p} \frac{\partial \mathrm{D}_1(x,p)}{\partial x_1} + \int_{\overline{\alpha}}^{\alpha_u} g(\bar{y}_2(\alpha),\alpha) d\alpha > 0 \tag{13-2-1}$$

Positivity of both terms is to be expected: holding both firm's location constant, an increase in firm ¬*i*'s price transfers market share to firm *i*. Since *i*'s price is held constant, *i*'s profit must increase.

5.2. **The second stage (price) subgames.** The expression $\frac{\partial \pi_i(\mathbf{x},\mathbf{p})}{\partial p_i} = \mathrm{D}_i(x,p) + p_i \frac{\partial \mathrm{D}_i(x,p)}{\partial p_i}$ is the change in firm *i*'s profits resulting from a change in *i*'s price in the subgame defined by the location vector *x*. Henceforth we shall refer to $f^p(x,p) = \begin{pmatrix} \frac{\partial \pi_1(x,p)}{\partial p_1} & \frac{\partial \pi_2(x,p)}{\partial p_2} \end{pmatrix}$ as the *Nash Price function*, and denote its Jacobian by $\mathrm{J} f^p(x,p) = \begin{bmatrix} \frac{\partial^2 \pi_1(\mathbf{x},\mathbf{p})}{\partial p_1^2} & \frac{\partial^2 \pi_1(\mathbf{x},\mathbf{p})}{\partial p_2 \partial p_1} \\ \frac{\partial^2 \pi_2(\mathbf{x},\mathbf{p})}{\partial p_1 \partial p_2} & \frac{\partial^2 \pi_2(\mathbf{x},\mathbf{p})}{\partial p_2^2} \end{bmatrix}$. Let $P(x)$ denote the set of Nash equilibrium price vectors given the location vector *x*, i.e., the set of *p*'s such that $f^p(x,p) = 0$. Given assumptions A1 and A2, Caplin and Nalebuff (1991, Theorem 1) guarantees that $P(x)$ is non-empty, for every *x*. Our simulation results provide strong evidence that $P(\cdot)$ is in fact singleton-valued, i.e., that each subgame has a *unique* pure-strategy equilibrium. We will henceforth assume this to be the case and let $p(\cdot)$ denote the function mapping location vectors to equilibrium price vectors.

Because $f^p(\cdot, p(\cdot))$ is identically zero across price subgames, we can apply the implicit function theorem to approximate the change in equilibrium prices resulting from a change in *x*, as follows:

$$\begin{bmatrix} \frac{dp_1(\mathbf{x})}{dx_1} & \frac{dp_1(\mathbf{x})}{dx_2} \\ \frac{dp_2(\mathbf{x})}{dx_1} & \frac{dp_2(\mathbf{x})}{dx_2} \end{bmatrix} = -\left(\mathrm{J} f^p(x, p(x))\right)^{-1} \begin{bmatrix} \frac{\partial^2 \pi_1(\mathbf{x},\mathbf{p}(\mathbf{x}))}{\partial p_1 \partial x_1} & \frac{\partial^2 \pi_1(\mathbf{x},\mathbf{p}(\mathbf{x}))}{\partial p_1 \partial x_2} \\ \frac{\partial^2 \pi_2(\mathbf{x},\mathbf{p}(\mathbf{x}))}{\partial p_2 \partial x_1} & \frac{\partial^2 \pi_2(\mathbf{x},\mathbf{p}(\mathbf{x}))}{\partial p_2 \partial x_2} \end{bmatrix} \tag{14}$$

Our task in this subsection is to derive explicit expressions for the derivatives on the right-hand side of (14). The elements of $\mathrm{J}f^{p}(x,p(x))$ are:

$$\begin{aligned}
\frac{\partial^2\pi_1(x,p(x))}{\partial p_1^2} &= 2\frac{\partial \mathrm{D}_1(x,p(x))}{\partial p_1} \quad - && \text{(15-1-1)}\\
&\qquad p_1\left[\left(\frac{x_2}{\Delta x}\right)^2 g(\overline{y}_2(\overline{\alpha}),\overline{\alpha}) \quad + \quad \frac{x_1}{\Delta x}\int_{\alpha_\ell}^{\alpha_u}\xi\left(\frac{x_1}{\Delta x}\right)_{[\alpha_\ell\ \overline{\alpha}]}\frac{dg(\overline{y}_2(\alpha),\alpha)}{d\overline{y}_2}d\alpha\right] \quad < \quad 0 &&\\
\frac{\partial^2\pi_1(x,p(x))}{\partial p_1\,\partial p_2} &= \frac{x_1}{\Delta x}\int_{\alpha_\ell}^{\overline{\alpha}} g(\overline{y}_2(\alpha),\alpha)d\alpha \quad + && \text{(15-1-2)}\\
&\qquad \frac{p_1 x_2}{(\Delta x)^2}\left[x_2 g(\overline{y}_2(\overline{\alpha}),\overline{\alpha}) \quad + \quad x_1\int_{\alpha_\ell}^{\overline{\alpha}}\frac{dg(\overline{y}_2(\alpha),\alpha)}{d\overline{y}_2}d\alpha\right] \quad > \quad 0 &&\\
\frac{\partial^2\pi_2(x,p(x))}{\partial p_2\,\partial p_1} &= -\left(\frac{\partial \mathrm{D}_1(x,p(x))}{\partial p_1} \quad + \quad p_2\left(\frac{x_2}{\Delta x}\right)^2\left[g(\overline{y}_2(\overline{\alpha}),\overline{\alpha}) \quad + \quad \frac{x_1}{x_2}\int_{\alpha_\ell}^{\overline{\alpha}}\frac{dg(\overline{y}_2(\alpha),\alpha)}{d\overline{y}_2}d\alpha\right]\right) && \text{(15-2-1)}\\
\frac{\partial^2\pi_2(x,p(x))}{\partial p_2^2} &= 2\frac{\partial \mathrm{D}_2(x,p(x))}{\partial p_2} \quad + \quad p_2\left(\frac{x_2}{\Delta x}\right)^2\left[g(\overline{y}_2(\overline{\alpha}),\overline{\alpha}) \quad + \quad \int_{\alpha_\ell}^{\overline{\alpha}}\frac{dg(\overline{y}_2(\alpha),\alpha)}{d\overline{y}_2}d\alpha\right] \quad - && \text{(15-2-2)}\\
&\qquad \frac{p_2}{x_2-1}\left(g(\underline{y}_2(\underline{\alpha}),\underline{\alpha})/x_2 \quad + \quad x_2\int_{\alpha_\ell}^{\alpha_u}\xi\left(\frac{x_2}{x_2-1}\right)_{[\alpha_\ell\ \underline{\alpha}]}\frac{dg(\underline{y}_2(\alpha),\alpha)}{d\underline{y}_2}d\alpha\right) \quad < \quad 0 &&
\end{aligned}$$

The signs of (15-1-1) and (15-1-2) follow from the facts that $\frac{\partial \mathrm{D}_1(x,p(x))}{\partial p_1} < 0$ and $\frac{dg(\overline{y}_2(\cdot),\cdot)}{dp_1}$ is everywhere positive (since $\overline{y}_2(\cdot) < Ey$). While (15-2-2) cannot be signed from first principles, negativity is guaranteed by Caplin and Nalebuff (1991, Theorem 1). Expression (15-2-1) would be unambiguously positive were it not for the term within square brackets.[11] In our simulations, however, the sum of these terms is very small relative to $\left|\frac{\partial \mathrm{D}_1(x,p(x))}{\partial p_1}\right|$, so that (15-2-1) turns out to be positive. Note also that since $\left|\frac{\partial^2\pi_1(x,p(x))}{\partial p_1^2}\right|$ clearly dominates $\left|\frac{\partial^2\pi_1(x,p(x))}{\partial p_1\,\partial p_2}\right|$, and the square bracketed term in (15-2-1) dominates the corresponding term in (15-2-2), the following condition is sufficient, but by no means necessary, for $\det(\mathrm{J}f^{p}(x,p(x)))$ to be positive:

$$\left|\frac{\partial \mathrm{D}_1(x,p(x))}{\partial p_1}\right| \quad < \quad 2\left|\frac{\partial \mathrm{D}_2(x,p(x))}{\partial p_2}\right| \tag{16}$$

(At the solution, $x^\star$, to the basecase of our numerical simulations, $\left|\frac{\partial \mathrm{D}_1(x^\star,p(x^\star))}{\partial p_1}\right| \quad < \quad \left|\frac{\partial \mathrm{D}_2(x^\star,p(x^\star))}{\partial p_2}\right|$, so that (16) is easily satisfied.)

---

[11] These terms reflect the facts, respectively, that as $\substack{p_1 \text{ increases} \\ p_2 \text{ decreases}}$ the region of interfirm competition shifts to the $\substack{\text{right} \\ \text{left}}$ and consumers become $\substack{\text{more} \\ \text{less}}$ dense in a neighborhood of the line dividing the firms' market regions. Both facts $\substack{\text{increase} \\ \text{diminish}}$ the intensity of interfirm competition, so that as $\substack{p_1 \\ p_2}$ increases, the marginal impact of p_1 on #1's market share $\substack{\text{increases} \\ \text{decreases}}$ while the marginal impact of p_2 on #2's market share $\substack{\text{decreases} \\ \text{increases}}$.

We now specify the elements of the second matrix on the right hand side of (14).

$$\begin{aligned}\frac{\partial^2\pi_1(x,p(x))}{\partial p_1\,\partial x_1} &= \frac{x_2(p_1+\Delta p)}{(\Delta x)^2}\int_{\alpha_\ell}^{\overline{\alpha}} g(\overline{y}_2(\alpha),\alpha)d\alpha \quad + \\ &\qquad \frac{p_1x_2^2\Delta p}{(\Delta x)^3}\left[g(\overline{y}_2(\overline{\alpha}),\overline{\alpha}) \quad + \quad \frac{\Delta p}{\Delta x}\int_{\alpha_\ell}^{\overline{\alpha}}\frac{dg(\overline{y}_2(\alpha),\alpha)}{d\overline{y}_2}d\alpha\right] \quad > \quad 0\end{aligned} \tag{17-1-1}$$

$$\begin{aligned}\frac{\partial^2\pi_1(x,p(x))}{\partial p_1\,\partial x_2} &= -\frac{x_1}{x_2}\left\{\frac{x_2(p_1+\Delta p)}{(\Delta x)^2}\int_{\alpha_\ell}^{\overline{\alpha}} g(\overline{y}_2(\alpha),\alpha)d\alpha \quad + \right. \\ &\qquad \left.\frac{p_1x_2}{\Delta x}\left[\frac{x_2\Delta p}{(\Delta x)^2}g(\overline{y}_2(\overline{\alpha}),\overline{\alpha}) \quad + \quad \int_{\alpha_\ell}^{\alpha_u}\frac{dg(\underline{y}_2(\alpha),\alpha)}{d\underline{y}_2}d\alpha\right]\right\} \quad < \quad 0\end{aligned} \tag{17-1-2}$$

$$\begin{aligned}\frac{\partial^2\pi_2(x,p(x))}{\partial p_2\,\partial x_1} &= \frac{x_2(p_2-\Delta p)}{(\Delta x)^2}\int_{\alpha_\ell}^{\overline{\alpha}} g(\overline{y}_2(\alpha),\alpha)d\alpha \quad + \\ &\qquad \frac{p_2x_2^2\Delta p}{(\Delta x)^4}\left[g(\overline{y}_2(\overline{\alpha}),\overline{\alpha}) \quad + \quad \int_{\alpha_\ell}^{\alpha_u}\frac{dg(\overline{y}_2(\alpha),\alpha)}{d\overline{y}_2}d\alpha\right]\end{aligned} \tag{17-2-1}$$

$$\begin{aligned}\frac{\partial^2\pi_2(x,p(x))}{\partial p_2\,\partial x_2} &= \frac{\partial \mathrm{D}_2(x,p(x))}{\partial x_2} \quad - \quad \frac{p_2x_1}{(\Delta x)^2}\left(\int_{\alpha_\ell}^{\overline{\alpha}} g(\overline{y}_2(\alpha),\alpha)d\alpha \quad + \right. \\ &\qquad \left.\frac{x_2\Delta p}{\Delta x}\left[g(\overline{y}_2(\overline{\alpha}),\overline{\alpha}) \quad + \quad \frac{\Delta p}{\Delta x}\int_{\alpha_\ell}^{\overline{\alpha}}\frac{dg(\overline{y}_2(\alpha),\alpha)}{d\overline{y}_2}d\alpha\right]\right) \quad + \quad \frac{1}{(x_2-1)^2}\times \\ &\qquad \left\{\frac{p_2-\underline{\alpha}}{x_2}g(\underline{y}_2(\underline{\alpha}),\underline{\alpha}) \;+\; \int_{\alpha_\ell}^{\underline{\alpha}}\left(g(\underline{y}_2(\alpha),\alpha) \;+\; \frac{\alpha-p_2}{(x_2-1)^2}\frac{dg(\underline{y}_2(\alpha),\alpha)}{d\underline{y}_2}\right)d\alpha\right\}\end{aligned} \tag{17-2-2}$$

The issues involved in signing (17) are related to those involved in signing (15). Expressions (17-1-1) and (17-1-2) are, respectively, unambiguously positive and negative: an increase in $\genfrac{}{}{0pt}{}{x_1}{x_2}$ has the dual effect of $\genfrac{}{}{0pt}{}{\text{increasing}}{\text{decreasing}}$ firm #1's initial market share, and $\genfrac{}{}{0pt}{}{\text{decreasing}}{\text{increasing}}$ the intensity of competition thus $\genfrac{}{}{0pt}{}{\text{increasing}}{\text{decreasing}}$ the benefits to #1 of raising price, while $\genfrac{}{}{0pt}{}{\text{decreasing}}{\text{increasing}}$ the cost. On the other hand, expressions (17-2-1) and (17-2-2) cannot be signed unambiguously: an increase in $\genfrac{}{}{0pt}{}{x_2}{x_1}$ has the dual effect of $\genfrac{}{}{0pt}{}{\text{increasing}}{\text{decreasing}}$ firm #2's initial market share, while $\genfrac{}{}{0pt}{}{\textit{increasing}}{\textit{decreasing}}$ the intensity of competition thus, once again, $\genfrac{}{}{0pt}{}{\text{increasing}}{\text{decreasing}}$ the benefits to #2 of raising price, but in this case $\genfrac{}{}{0pt}{}{\textit{increasing}}{\textit{decreasing}}$ the cost. In expression (17-2-2), an additional factor—the positive term in the curly brackets—contributes to sign ambiguity. This term has nothing to do with inter-firm competition. Rather it relates to consumers at the lower end of the income scale who are deciding between purchasing the inferior good #2 and purchasing no good at all. As the quality of good #2 increases, there is a decline in the rate at which consumers switch from the former option to the latter in response to a unit increase in p_2, i.e., there is a reduction in one element of the cost to firm #2 of raising its price. We shall refer to this factor below as the *participation boundary effect.*

In the numerical solution to our base case, it turns out that (17-2-1) is negative and (17-2-2) is positive. The magnitude of $\frac{\partial^2 \pi_2(x,p(x))}{\partial p_2\, \partial x_1}$ is very small, and its sign can change with small perturbations. Its sign, however, is not important; what matters is that $\frac{\partial^2 \pi_1(x,p(x))}{\partial p_1\, \partial x_2}$, the sum of two positive effects is larger than $\frac{\partial^2 \pi_2(x,p(x))}{\partial p_2\, \partial x_1}$, the sum of a positive and a negative effect. This will have important ramifications when we come to sign the effects of changing locations on prices. Indeed, returning to (14), we have

$$\begin{bmatrix} \frac{dp_1(\mathbf{x})}{dx_1} & \frac{dp_1(\mathbf{x})}{dx_2} \\ \frac{dp_2(\mathbf{x})}{dx_1} & \frac{dp_2(\mathbf{x})}{dx_2} \end{bmatrix} = -\left(\Delta_{J_{f^p}}\right)^{-1} \begin{bmatrix} \frac{\partial^2 \pi_2(\mathbf{x},\mathbf{p}(\mathbf{x}))}{\partial p_2^2} & -\frac{\partial^2 \pi_1(\mathbf{x},\mathbf{p}(\mathbf{x}))}{\partial p_1 \partial p_2} \\ -\frac{\partial^2 \pi_2(\mathbf{x},\mathbf{p}(\mathbf{x}))}{\partial p_1 \partial p_2} & \frac{\partial^2 \pi_1(\mathbf{x},\mathbf{p}(\mathbf{x}))}{\partial p_1^2} \end{bmatrix} \begin{bmatrix} \frac{\partial^2 \pi_1(\mathbf{x},\mathbf{p}(\mathbf{x}))}{\partial p_1 \partial x_1} & \frac{\partial^2 \pi_1(\mathbf{x},\mathbf{p}(\mathbf{x}))}{\partial p_1 \partial x_2} \\ \frac{\partial^2 \pi_2(\mathbf{x},\mathbf{p}(\mathbf{x}))}{\partial p_2 \partial x_1} & \frac{\partial^2 \pi_2(\mathbf{x},\mathbf{p}(\mathbf{x}))}{\partial p_2 \partial x_2} \end{bmatrix} \tag{14$'$}$$

which, in the solution to our basecase simulation,

$$= \begin{bmatrix} 2.5832 & 0.4424 \\ 0.7452 & 2.1046 \end{bmatrix} \begin{bmatrix} 0.0542 & -0.2104 \\ 0.0020 & 0.2070 \end{bmatrix} = \begin{bmatrix} 0.1410 & -0.4521 \\ 0.0446 & 0.2788 \end{bmatrix} \tag{14-N}$$

In (14-N), note first the off-diagonal terms: in the expression for $\frac{dp_1(\mathbf{x})}{dx_2}$ the negative term dominates the positive term, while the reverse is true for $\frac{dp_2(\mathbf{x})}{dx_1}$. Hence we obtain

$$\frac{dp_1(\mathbf{x})}{dx_2} \quad < \quad 0 \quad < \quad \frac{dp_2(\mathbf{x})}{dx_1}. \tag{18}$$

As we have just explained, the source of this asymmetry is that an increase in $\begin{array}{c} x_1 \text{ diminishes} \\ x_2 \text{ intensifies} \end{array}$ the degree of inter-firm competition. This factor also explains why $\frac{dp_1(\mathbf{x})}{dx_1}$ is positive. On the other hand, the sign of $\frac{dp_2(\mathbf{x})}{dx_2}$ is a surprise. The inter-firm competition argument we have been stressing would imply that this term should be negative, since increased competition should lower p_2 as well as p_1. The reason that it does not can be attributed to the participation boundary effect. To demonstrate this, we have computed how expression (14-N) would change if the participation boundary were independent of both p_2 and x_2, that is, if all terms involving either $\underline{y}_2(\cdot)$ or $\underline{\alpha}$ in expressions (11-2-2), (15-2-2) and (17-2-2) were set to zero. In this case, the numbers recorded in (14-N) would change to the following ones:

$$\begin{bmatrix} 2.5474 & 0.6228 \\ 0.9294 & 2.7012 \end{bmatrix} \begin{bmatrix} 0.0625 & -0.2407 \\ 0.0023 & -0.0089 \end{bmatrix} = \begin{bmatrix} 0.1606 & -0.6187 \\ 0.0642 & -0.2477 \end{bmatrix} \tag{14-N$'$}$$

That is, in the absence of the participation boundary effect, an increase in $\begin{array}{c} x_1 \\ x_2 \end{array}$ would result in an $\begin{array}{c} \text{increase} \\ \text{decrease} \end{array}$ in *both* prices, a result which would be consistent with our intuition about firm locations and inter-firm competition.

5.3. **The first stage (location) game.** A necessary condition for a location vector $x^\star > 0$ to be a subgame perfect equilibrium in the two stage game is that for each i

$$\frac{d\pi_i(x^\star, p(x^\star))}{dx_i} = \frac{\partial\pi_i(x^\star, p(x^\star))}{\partial x_i} + \frac{\partial\pi_i(x^\star, p(x^\star))}{\partial p_i}\frac{dp_i(x^\star)}{dx_i} + \frac{\partial\pi_i(x^\star, p(x^\star))}{\partial p_{\neg i}}\frac{dp_{\neg i}(x^\star)}{dx_i} = 0 \tag{19}$$

Because $p(\cdot)$ is a mapping from locations to equilibrium price vectors, $\frac{\partial\pi_i(\cdot,p(\cdot))}{\partial p_i}$ is identically zero. Consequently, the equilibrium condition (19) reduces to

$$\begin{aligned} 0 &= \frac{\partial\pi_i(x^\star, p(x^\star))}{\partial x_i} && + \frac{\partial\pi_i(x^\star, p(x^\star))}{\partial p_{\neg i}}\frac{dp_{\neg i}(x^\star)}{dx_i} \\ &= \left(p_i(x^\star)\frac{\partial \mathrm{D}_i}{\partial x_i} - \mathrm{MC}(x_i^\star)\right) && + \left(p_i(x^\star)\frac{\partial \mathrm{D}_i(x^\star, p(x^\star))}{\partial p_{\neg i}}\frac{dp_{\neg i}(x^\star)}{dx_i}\right) \end{aligned} \tag{20}$$

where $\mathrm{MC}(x_i)$ denotes the marginal cost of increasing x_i. Expression (20) has an immediate implication for the relationship between equilibrium prices and equilibrium locations: if a parameter change were to increase equilibrium prices at the equilibrium location vector, $x^\star$, to $p' \gg p(x^\star)$ then *if all terms in* (20) *other than prices remained constant*, the $\frac{d\pi_i(x,\cdot)}{dx_i}$'s would be positive when evaluated at p'. Now as the right panel of Fig. **??** illustrates, the $\pi_i(\cdot, p(\cdot))$'s are concave in own locations. Provided that cross-location effects are dominated by own-location effects, it follows that equilibrium locations and equilibrium prices at the original equilibrium locations will, all else being equal, move in the same direction. In the analysis below, we will refer to this relationship as the *price level effect*. Intuitively, it makes good sense: price increases enhance the benefits of increasing location, without affecting the costs.

We now return to the decomposition (20) of (19). Following Tirole (1993, p. 281), we will henceforth refer to $\frac{\partial\pi_i(x^\star,p(x^\star))}{\partial x_i}$ as the *demand effect* and to $\frac{\partial\pi_i(x^\star,p(x^\star))}{\partial p_{\neg i}}\frac{dp_{\neg i}}{dx_i}$ as the *strategic effect*. Clearly, the demand effect cannot be signed. On the other hand, since the $\frac{\partial \mathrm{D}_i(x^\star,p(x^\star))}{\partial p_{\neg i}}$'s are both positive (expression (13)), the signs of the strategic effect are determined by the signs of the $\frac{dp_{\neg i}(x^\star)}{dx_i}$'s. Indeed, in any subgame perfect equilibrium such that (18) is satisfied, the strategic effects for the two players will have opposite signs:

$$\text{\#2's strategic effect} = \frac{\partial\pi_2}{\partial p_1}\frac{dp_1}{dx_2} < 0 < \frac{\partial\pi_1}{\partial p_2}\frac{dp_2}{dx_1} = \text{\#1's strategic effect} \tag{21}$$

This asymmetry can be traced to the usual source: as $\begin{array}{c} x_1 \\ x_2 \end{array}$ increases, competition $\begin{array}{c} \text{diminishes} \\ \text{intensifies} \end{array}$, resulting in $\begin{array}{c} \text{an increase in } p_2 \\ \text{a decrease in } p_1 \end{array}$, which has a $\begin{array}{c} \text{positive} \\ \text{negative} \end{array}$ impact on $\begin{array}{c} \text{\#1's} \\ \text{\#2's} \end{array}$ profits. Expression (21), together with the first order condition (20), imply that

$$p_1(x^\star)\frac{\partial \mathrm{D}_1(x^\star, p(x^\star))}{\partial x_1} \qquad < \qquad \mathrm{MC}(x_1^\star) \tag{22-1}$$

$$p_2(x^\star)\frac{\partial \mathrm{D}_2(x^\star, p(x^\star))}{\partial x_2} \qquad > \qquad \mathrm{MC}(x_2^\star) \tag{22-2}$$

Assuming D_i is concave in x_i (as it is in our case), firm #1 is induced to keep increasing his location beyond the point at which, *holding prices constant* his marginal increase in market share from a further increase in location just offsets the marginal cost of the increase.

When costs are linear in location, as they are in the base case of our model, (22) implies that $p_1(x^\star)\frac{\partial \mathrm{D}_1(x^\star,p(x^\star))}{\partial x_1} < p_2(x^\star)\frac{\partial \mathrm{D}_2(x^\star,p(x^\star))}{\partial x_2}$. This property is especially striking because the price of the higher quality good $p_1(x^\star)$ significantly exceeds $p_2(x^\star)$. Hence $\frac{\partial \mathrm{D}_1(x^\star,p(x^\star))}{\partial x_1} \ll \frac{\partial \mathrm{D}_2(x^\star,p(x^\star))}{\partial x_2}$. (Indeed, in our base case, the ratio $\frac{\partial \mathrm{D}_1(x^\star,p(x^\star))}{\partial x_1} \Big/ \frac{\partial \mathrm{D}_2(x^\star,p(x^\star))}{\partial x_2}$ is approximately 1/6.) This asymmetry can be traced back to the same source as the one that explained (18): the demand effect motivates both firms to increase location; the strategic effect, however, motivates firm #1 to increase, while motivating #2 to decrease location.

To analyze the comparative statics properties of our model, we need first to analyze the Jacobian, $\mathrm{J}f^x(x)$, of the *Nash location function* $f(x) = \begin{pmatrix} \frac{d\pi_1(x,p(x))}{dx_1} & \frac{d\pi_2(x,p(x))}{dx_2} \end{pmatrix}$, that is, $\mathrm{J}f^x(x) = \begin{bmatrix} \frac{d^2\pi_1(x,p(x))}{dx_1{}^2} & \frac{d^2\pi_1(x,p(x))}{dx_1\,dx_2} \\ \frac{d^2\pi_2(x,p(x))}{dx_2\,dx_1} & \frac{d^2\pi_2(x,p(x))}{dx_2{}^2} \end{bmatrix}$, when evaluated at the solution $x^\star$ to the basecase of our numerical simulations. As we shall see below in display (29-N), $\mathrm{J}f^x(x^\star, p(x^\star))$ has the sign pattern $\begin{bmatrix} - & + \\ + & - \end{bmatrix}$. This symmetric pattern is at first sight surprising, since as we have observed, rightward shifts by the two firms have opposite effects on the intensity of competition between them. Indeed, one would expect that $\mathrm{J}f^x(x^\star, p(x^\star))$ to exhibit the symmetric pattern $\begin{bmatrix} - & + \\ - & + \end{bmatrix}$. One of our objectives in this subsection is to explain why it doesn't.

To analyze $\mathrm{J}f^{x}(x^{\star}, p(x^{\star}))$, we decompose it into five component parts: For $i, j = 1, 2$,

$$\begin{aligned}\frac{d^2\pi_i(x^{\star}, p(x^{\star}))}{dx_i\,dx_j} &= \frac{d}{dx_j}\left(\frac{\partial\pi_i}{\partial x_i}\right) + \frac{d}{dx_j}\left(\frac{\partial\pi_i}{\partial p_{\neg i}}\frac{dp_{\neg i}}{dx_i}\right) \qquad (23)\\ &= \left(\frac{dp_i}{dx_j}\frac{\partial \mathrm{D}_i}{\partial x_i} + p_i\frac{\partial^2 \mathrm{D}_i}{\partial x_i\,\partial x_j} - p_i\frac{d\mathrm{MC}(x_i^{\star})}{dx_i}\right)\\ &\quad + \left(\frac{dp_{\neg i}}{dx_i}\left[\frac{dp_i}{dx_j}\frac{\partial \mathrm{D}_i}{\partial p_{\neg i}} + p_i\frac{\partial^2 \mathrm{D}_i}{\partial p_{\neg i}\,\partial x_j}\right] + \frac{d^2 p_{\neg i}}{dx_i\,dx_j}\frac{\partial\pi_i}{\partial p_{\neg i}}\right)\end{aligned}$$

which, by (20)

$$= \underbrace{\frac{dp_i}{dx_j}\frac{\mathrm{MC}(x_i^{\star})}{p_i}}_{\substack{\text{price}\\ \text{level}\\ \text{effect}}} + \underbrace{p_i\frac{\partial^2 \mathrm{D}_i}{\partial x_i\,\partial x_j}}_{\substack{\text{location}\\ \text{competition}\\ \text{effect}}} + \underbrace{p_i\frac{dp_{\neg i}}{dx_i}\frac{\partial^2 \mathrm{D}_i}{\partial p_{\neg i}\,\partial x_j}}_{\substack{\text{crossprice}\\ \text{competition}\\ \text{effect}}} + \underbrace{\frac{d^2 p_{\neg i}}{dx_i\,dx_j}\frac{\partial\pi_i}{\partial p_{\neg i}}}_{\substack{\text{crossprice}\\ \text{sensitivity}\\ \text{effect}}} - \underbrace{p_i\frac{d\mathrm{MC}(x_i^{\star})}{dx_i}}_{\substack{\text{marginal}\\ \text{cost}\\ \text{effect}}} \qquad (23')$$

The five terms on the right-hand side of (23) together determine the effect of an increase in x_j on the net gain to i from increasing x_i. (i) The *price level effect* is the impact of a shift in x_j on i's first order condition (20), holding constant all terms in (20) except the level of p_i. This effect was discussed on page 21, immediately following display (20): if p_i $\begin{array}{c}\text{increases}\\ \text{decreases}\end{array}$ with x_j, then $\frac{d\pi_i(\cdot, p(\cdot))}{dx_i}$ will $\begin{array}{c}\text{increase}\\ \text{decrease}\end{array}$ also. (ii) The *location competition effect* measures the impact of an increase in x_j on the rate at which, holding prices constant, i gains revenue as x_i is increased. Using the terminology of page 21, the effect measures the impact of x_j on i's *demand* incentive to compete-by-location at constant prices. In terms of Fig. **??**, the effect reflects the impact of x_j on the rate at which the interfirm boundary shifts with x_i.[12] Similarly, (iii) the *crossprice competition effect* is the impact of x_j on the rate at which i's revenue changes as the equilibrium value of $p_{\neg i}$ shifts in response to an increase in x_i, holding constant the sensitivity of $p_{\neg i}(\cdot)$ to x_i. This is the first of two impacts of x_j on i's *strategic* incentive to compete-by-location. Once again, in terms of Fig. **??**, the effect reflects the impact of x_j on the rate at which the interfirm boundary shifts with $p_{\neg i}$. (iv) The *crossprice sensitivity effect* measures the impact of x_j on the sensitivity of $p_j(\cdot)$ to x_i; This is the second impact of x_j on i's strategic incentive to compete-by-location. (v) The *marginal cost effect* is self-explanatory.

Of these five terms, the crossprice sensitivity effect is by far the most complex to analyze, since it involves differentiating the already complex expression (14). In none of our comparative statics experiments is this effect decisive, and we will not attempt to derive an explicit expression for it. We will, however, derive

---

[12] When $i = j = 2$, the participation boundary is involved as well.

explicit expressions for the location competition effect and the crossprice competition effect. (The remaining two terms require no further derivations.)

5.3.2. *The location competition effect.* To analyze this effect we compute the partial derivatives of the $\frac{dD_i}{dx_i}$'s w.r.t. the x_j's. Differentiating (12-1), we obtain

$$\begin{aligned}
\frac{\partial^2 D_1(x,p)}{\partial x_1^2} &= -\frac{1}{\Delta x}\Bigg(x_2 \Delta p \int_{\alpha_\ell}^{\overline{\alpha}} \frac{dg(\overline{y}_2(\alpha),\alpha)}{d\overline{y}_2} d\alpha && \text{(24-1-1)}\\
&\qquad + 2\frac{\partial D_1(x,p)}{\partial x_1} + \frac{x_2^2(\Delta p)^2}{(\Delta x)^3} g(\overline{y}_2(\overline{\alpha}),\overline{\alpha})\Bigg) < 0 \\
\frac{\partial^2 D_1(x,p)}{\partial x_1\,\partial x_2} &= \frac{1}{\Delta x}\Bigg(x_1 \Delta p \int_{\alpha_\ell}^{\overline{\alpha}} \frac{dg(\overline{y}_2(\alpha),\alpha)}{d\overline{y}_2} d\alpha && \text{(24-1-2)}\\
&\qquad + \frac{(x_1+x_2)}{x_2}\frac{\partial D_1(x,p)}{\partial x_1} + \frac{x_1 x_2(\Delta p)^2}{(\Delta x)^3} g(\overline{y}_2(\overline{\alpha}),\overline{\alpha})\Bigg) > 0 \\
\frac{\partial^2 D_2(x,p)}{\partial x_2\,\partial x_1} &= -\frac{\partial^2 D_1(x,p)}{\partial x_1\,\partial x_2} < 0 && \text{(24-2-1)}
\end{aligned}$$

All three of these signs can be explained in terms of our usual intuition: as $\begin{matrix} x_1 \\ x_2 \end{matrix}$ increases, competition $\begin{matrix}\text{diminishes}\\ \text{intensifies}\end{matrix}$, thus $\begin{matrix}\text{reducing}\\ \text{increasing}\end{matrix}$ the rate at which the interfirm boundary shifts in response to an increase in x_i. Finally, we differentiate (12-2) with respect to x_2 to obtain

$$\begin{aligned}
\frac{\partial^2 D_2(x,p)}{\partial x_2^2} &= \frac{x_1}{x_2}\frac{\partial^2 D_1(x,p)}{\partial x_1\,\partial x_2} - \frac{x_1}{x_2^2}\frac{\partial D_1(x,p)}{\partial x_1} - \frac{p_2(p_2-\underline{\alpha})}{x_2(x_2-1)^2} g(\underline{y}_2(\underline{\alpha}),\underline{\alpha}) && \text{(24-2-2)}\\
&\qquad - \frac{1}{(x_2-1)^3}\int_{\alpha_\ell}^{\underline{\alpha}} \left\{ \frac{(p_2-\alpha)^2}{x_2-1}\left[\frac{dg(\underline{y}_2(\alpha),\alpha)}{d\underline{y}_2} + 2g(\underline{y}_2(\alpha),\alpha)\right]\right\} d\alpha
\end{aligned}$$

The term in curly brackets is the " participation boundary effect" introduced during the discussion of (17-2-2) on page 20. Recall from ($\underline{7}$) that $(p_2-\alpha)$ is positive on $[0,\underline{\alpha}]$, as is $\frac{dg(\underline{y}_2(\cdot),\cdot)}{d\underline{y}_2}$. It follows that if the positive first term (see (24-1-2)) were excluded, expression (24-2-2) would be unambiguously negative. Indeed, in the numerical results for our basecase, the first term is dominated by the others so that $\frac{\partial^2 D_2(x^\star,p(x^\star))}{\partial x_2^2}$ is indeed negative. That is, when x_2 moves closer to x_1, the positive "increased competition" effect—i.e., the increase in the marginal benefit to #2 from increasing x_2—is dominated by the negative participation boundary effect. Summarizing, the sign pattern for the $\frac{\partial^2 D_i(x,p)}{\partial x_i\,\partial x_j}$'s is

$$\operatorname{sign}\left(\begin{bmatrix} \frac{\partial^2 D_1(x,p)}{\partial x_1^2} & \frac{\partial^2 D_1(x,p)}{\partial x_1\,\partial x_2} \\ \frac{\partial^2 D_2(x,p)}{\partial x_2\,\partial x_1} & \frac{\partial^2 D_2(x,p)}{\partial x_2^2} \end{bmatrix}\right) = \begin{bmatrix} - & + \\ - & (-) \end{bmatrix}. \tag{25}$$

where the parentheses denote a qualitative property of our numerical solution, (x, p) that cannot be established analytically.

5.3.3. *The crossprice competition effect.* Taking partial derivatives of (13) w.r.t. x_j, we obtain

$$\begin{aligned}
\frac{\partial^2 \mathrm{D}_1(x,p)}{\partial p_2\, \partial x_1} &= \frac{-x_2}{(\Delta x)^2}\left(\int_{\alpha_\ell}^{\overline{\alpha}} g(\overline{y}_2(\alpha),\alpha)d\alpha \quad + \quad \frac{x_2 \Delta p}{\Delta x}\left[g(\overline{y}_2(\overline{\alpha}),\overline{\alpha}) + \int_{\alpha_\ell}^{\overline{\alpha}} \frac{dg(\overline{y}_2(\alpha),\alpha)}{d\overline{y}_2} d\alpha\right]\right) && \text{(26-1-1)}\\
\frac{\partial^2 \mathrm{D}_1(x,p)}{\partial p_2\, \partial x_2} &= -\frac{x_1}{x_2}\frac{\partial^2 \mathrm{D}_1(x,p)}{\partial p_2\, \partial x_1} && \text{(26-1-2)}\\
\frac{\partial^2 \mathrm{D}_2(x,p)}{\partial p_1\, \partial x_1} &= \frac{-x_2}{(\Delta x)^2}\left(\int_{\alpha_\ell}^{\overline{\alpha}} g(\overline{y}_2(\alpha),\alpha)d\alpha \quad + \quad \frac{x_2 \Delta p}{\Delta x}\left[g(\overline{y}_2(\overline{\alpha}),\overline{\alpha}) + \frac{x_1}{x_2}\int_{\alpha_\ell}^{\overline{\alpha}} \frac{dg(\overline{y}_2(\alpha),\alpha)}{d\overline{y}_2} d\alpha\right]\right) && \text{(26-2-1)}\\
\frac{\partial^2 \mathrm{D}_2(x,p)}{\partial p_1\, \partial x_2} &= -\frac{x_1}{x_2}\frac{\partial^2 \mathrm{D}_2(x,p)}{\partial p_1\, \partial x_1} && \text{(26-2-2)}
\end{aligned}$$

Using the fact that $\overline{y}_2(\cdot,\cdot)$ lies everywhere below the average level of income, the expressions in (26) can be signed unambiguously as follows:

$$\text{sign}\left(\begin{bmatrix} \frac{\partial^2 \mathrm{D}_1(x,p)}{\partial p_2\, \partial x_1} & \frac{\partial^2 \mathrm{D}_1(x,p)}{\partial p_2\, \partial x_2} \\ \frac{\partial^2 \mathrm{D}_2(x,p)}{\partial p_1\, \partial x_1} & \frac{\partial^2 \mathrm{D}_2(x,p)}{\partial p_1\, \partial x_2} \end{bmatrix}\right) \quad = \quad \begin{bmatrix} - & + \\ - & + \end{bmatrix}. \tag{27}$$

This asymmtric sign pattern is consistent with our intuitive argument about competition, i.e., an increase in $\begin{matrix} x_1 \text{ diminishes} \\ x_2 \text{ intensifies} \end{matrix}$ competition and so $\begin{matrix} \text{weakens} \\ \text{strenghtens} \end{matrix}$ the (unambiguously positive) impact of an increase in firm $\neg i$'s price on firm i's profits. Note, however, that the second part of the expression measuring the impact of dx_j on the strategic effect is $p_i \frac{dp_{\neg i}}{dx_i} \frac{\partial^2 \mathrm{D}_i}{\partial p_{\neg i}\, \partial x_j}$. Since the $\frac{dp_{\neg i}}{dx_i}$'s are also asymmetric, the two asymmetries cancel each other out, yielding a symmetric sign pattern for the second part of the strategic effect:

$$\text{sign}\left(\begin{bmatrix} p_1 \frac{\partial^2 \mathrm{D}_1(x,p)}{\partial p_2\, \partial x_1}\frac{dp_2}{dx_1} & p_1 \frac{\partial^2 \mathrm{D}_1(x,p)}{\partial p_2\, \partial x_2}\frac{dp_2}{dx_1} \\ p_2 \frac{\partial^2 \mathrm{D}_2(x,p)}{\partial p_1\, \partial x_1}\frac{dp_1}{dx_2} & p_2 \frac{\partial^2 \mathrm{D}_2(x,p)}{\partial p_1\, \partial x_2}\frac{dp_1}{dx_2} \end{bmatrix}\right) \quad = \quad \begin{bmatrix} - & + \\ + & - \end{bmatrix}. \tag{28}$$

Intuitively, an increase in $\begin{matrix} x_1 \text{ diminishes} \\ x_2 \text{ increases} \end{matrix}$ the intensity of the strategic effect. But since the strategic effect is $\begin{matrix} \text{positive for firm \#1} \\ \text{negative for firm \#2} \end{matrix}$, (see (21) and the following discussion) a diminution in intensity has a $\begin{matrix} \text{negative} \\ \text{positive} \end{matrix}$ impact on firm $\begin{matrix} \text{\#1's} \\ \text{\#2's} \end{matrix}$ profits, while an increase in intensity has a $\begin{matrix} \text{positive} \\ \text{negative} \end{matrix}$ impact on firm $\begin{matrix} \text{\#1's} \\ \text{\#2's} \end{matrix}$ profits.

We now combine (25), (27) and (28) and consider the sign of the Jacobian matrix of the Nash location function, evaluated at our numerical solution, $x^\star$. Except for the crossprice sensitivity effect in the decomposition

(23), about which we have no analytical information (hence the question marks), we can sign unambiguously all but one of the terms:

$$\operatorname{sign}\left(\begin{bmatrix} \frac{d^2\pi_1(x^\star,p(x^\star))}{dx_1{}^2} & \frac{d^2\pi_1(x^\star,p(x^\star))}{dx_1\,dx_2} \\ \frac{d^2\pi_2(x^\star,p(x^\star))}{dx_2\,dx_1} & \frac{d^2\pi_2(x^\star,p(x^\star))}{dx_2{}^2} \end{bmatrix}\right) = \underbrace{\begin{bmatrix} + & - \\ + & + \end{bmatrix}}_{\substack{\text{price}\\\text{level}\\\text{effect}}} + \underbrace{\begin{bmatrix} - & + \\ - & - \end{bmatrix}}_{\substack{\text{location}\\\text{competition}\\\text{effect}}} + \underbrace{\begin{bmatrix} - & + \\ + & - \end{bmatrix}}_{\substack{\text{crossprice}\\\text{competition}\\\text{effect}}} + \underbrace{\begin{bmatrix} ? & ? \\ ? & ? \end{bmatrix}}_{\substack{\text{crossprice}\\\text{sensitivity}\\\text{effect}}} \tag{29}$$

The fact that the location competition effect and the price level effect matrices have opposite signs reflects a tension that is a fundamental theme of this paper. Any shift in either a location or an exogenous variable which diminishes the degree of competition between firms has two countervailing effects. On the one hand, it decreases each firm's marginal benefit from an increase in its location, holding prices constant. This effect contributes negatively to the left-hand side of (29). On the other hand, it also decreases each firm's marginal benefit from competing by price, and when both firms compete-by-price less aggressively, the resulting effect on the left-hand side of (29) is positive. To illustrate this tension, consider an increase in x_1: as the location gap between firms increases, there is a decline in the rate at which consumers shift from good #i to #$\neg i$ in response to a unit increase in x_i; but at the same time there is a decline in the rate at which consumers shift from #$\neg i$ to #i in response to a unit increase in p_i! In short, the intensity of both location- and price-competition diminishes, with countervailing effects on the Nash location function.

A consequence of the tension just described is that none of the elements of the Jacobian can be signed unambiguously, or at least not by inspection of the signs of its component parts. Notwithstanding this analytical ambiguity, our numerical simulation results indicate that the sign pattern of $\mathrm{J}f^x(\cdot)$ is surprisingly robust. In the basecase solution to our simulations, the numerical values for $\mathrm{J}f^x(x^\star, p(x^\star))$ are as follows:

$$\underbrace{\begin{bmatrix} -.0294 & .0903 \\ .0057 & -.2215 \end{bmatrix}}_{\mathrm{J}f^x(x^\star,p(x^\star))} = \underbrace{\begin{bmatrix} .0020 & -.0064 \\ .0018 & .0110 \end{bmatrix}}_{\substack{\text{price}\\\text{level}\\\text{effect}}} + \underbrace{\begin{bmatrix} -.0270 & .0791 \\ -.0283 & -.1550 \end{bmatrix}}_{\substack{\text{location}\\\text{competition}\\\text{effect}}} + \underbrace{\begin{bmatrix} -.0020 & .0078 \\ .0076 & -.0296 \end{bmatrix}}_{\substack{\text{crossprice}\\\text{competition}\\\text{effect}}} + \underbrace{\begin{bmatrix} -.0023 & .0098 \\ .0246 & -.0480 \end{bmatrix}}_{\substack{\text{crossprice}\\\text{sensitivity}\\\text{effect}}} \tag{29-N}$$

Note that $\mathrm{J}f^x(x^\star, p(x^\star))$ is negative definite. Note also that while the diagonal elements of the price level effect contributes positively to the diagonal elements of the Jacobian, these price level terms are dominated by a factor of more than ten by the negative diagonal elements of the location competition effect. These observations are consistent with the indications from the right panel of Fig. **??** that firms' payoffs are concavity in their own locations in the two-stage game.

Since firm *i*'s first order condition, (20), must hold not just at the equilibrium but at all points along *i*'s location reaction curve, the sign properties of $\mathrm{J}f^x(\cdot)$ determine the slope of this curve. Specifically, along *i*'s location reaction curve, $\bar{x}_i(x_{\neg i})$, we have $\frac{d\bar{x}_i(x_{\neg i})}{dx_{\neg i}} = -\frac{d^2\pi_i(x,p(x))}{dx_i\,dx_{\neg i}} \Big/ \frac{d^2\pi_i(x,p(x))}{dx_i{}^2}$. Hence if the sign pattern (29) held globally—and evidence from our base case simulation is consistent with this supposition—each firm's location reaction curve would be globally upward sloping. Furthermore, plugging the numbers from (29-N) into the expressions for the $\bar{x}_i(\cdot)$, we observe that at $x^\star$, the slope of player #1's reaction curve is significantly greater than unity, while the slope of player #2's is much smaller than unity (cf. the slopes depicted in Fig. **??** above).

6. COMPARATIVE STATICS

Our model has five parameter vectors, which can be classified into three groups: (i) cost parameters, i.e., the coefficients and exponent on the cost function; (ii) the parameters determining first moments of the consumer distribution: the means of the distribution of income and tastes; (iii) the single parameter determining the second moments of the consumer distribution. Let θ denote a list of these parameters with generic element θ_k. In our comparative statics analysis, we shall consider the effects of changing each of these parameters.

Let $x^\star(\theta))$ denote the solution to our model when the parameter set is θ. To avoid confusion between partial derivatives, total derivatives and derivatives that are somewhere in between, it is convenient to rewrite the first order conditions for the location game as $\left(\frac{\partial\Pi_1(x^\star(\theta);\theta)}{\partial x_1} \quad \frac{\partial\Pi_2(x^\star(\theta);\theta)}{\partial x_2}\right) = 0$, where $\Pi_i(x^\star(\theta);\theta) = \pi_i(x^\star(\theta), p(x^\star(\theta);\theta)) = p_i(x^\star(\theta);\theta)\mathrm{D}_i(x^\star(\theta), p(x^\star(\theta);\theta)) - cx_i^\star(\theta)^\gamma$. We shall continue to refer to $\left(\frac{\partial\Pi_1(x^\star(\theta);\theta)}{\partial x_1} \quad \frac{\partial\Pi_2(x^\star(\theta);\theta)}{\partial x_2}\right)$ as the *Nash location function*. The important distinction here is that $\frac{\partial\Pi_i}{\partial x_i}$ involves a change in x_i together with an induced change in the vector p, while $\frac{\partial\pi_i}{\partial x_i}$ involves a change only in x_i. In switching from π's to Π's, we are, in effect, switching from analyzing the subgame-perfect equilibrium of our original two-stage game—in which firms first choose locations, then prices—to analyzing the *Nash* equilibrium of the induced single-stage game in locations, defined by assigning to each location pair x the payoffs that arise when in the original two-stage game, players choose equilibrium prices in the subgame associated with x. The condition for a Nash equilibrium of the induced single-stage game can now be written in terms of the Π's as

$$\frac{\partial\Pi_i(x^\star(\theta);\theta)}{\partial x_i} = \frac{d\pi_i(x^\star(\theta), p(x^\star(\theta));\theta)}{dx_i} = 0, \text{ for } i = 1,2 \tag{20$'$}$$

From the implicit function theorem, a change in an element θ_k of θ induces a change in equilibrium locations according to:

$$\begin{bmatrix} \frac{dx_i^\star(\theta)}{d\theta_k} \\ \frac{dx_i^\star(\theta)}{d\theta_k} \end{bmatrix} = -\begin{bmatrix} \frac{\partial^2\Pi_1(x^\star(\theta);\theta)}{\partial {x_1}^2} & \frac{\partial^2\Pi_1(x^\star(\theta);\theta)}{\partial x_1\,\partial x_2} \\ \frac{\partial^2\Pi_2(x^\star(\theta);\theta)}{\partial x_2\,\partial x_1} & \frac{\partial^2\Pi_2(x^\star(\theta);\theta)}{\partial x_2^2} \end{bmatrix}^{-1} \begin{bmatrix} \frac{\partial^2\Pi_1(x^\star(\theta);\theta)}{\partial x_1\,\partial\theta_k} \\ \frac{\partial^2\Pi_2(x^\star(\theta);\theta)}{\partial x_2\,\partial\theta_k} \end{bmatrix} \tag{30}$$

$$\text{which, inserting the numbers in (29-N)} = \begin{bmatrix} 36.9228 & 15.0511 \\ 0.9442 & 4.8986 \end{bmatrix} \begin{bmatrix} \frac{\partial^2\Pi_1(x^\star(\theta);\theta)}{\partial x_1\,\partial\theta_k} \\ \frac{\partial^2\Pi_2(x^\star(\theta);\theta)}{\partial x_2\,\partial\theta_k} \end{bmatrix}$$

Once (30) has been computed, we can compute the change in equilibrium *prices* induced by a change in θ_k by:

$$\begin{bmatrix} \frac{dp_1(x^\star(\theta);\theta)}{d\theta_k} \\ \frac{dp_2(x^\star(\theta);\theta)}{d\theta_k} \end{bmatrix} = \begin{bmatrix} \frac{\partial p_1(x^\star(\theta);\theta)}{\partial x_1} & \frac{\partial p_1(x^\star(\theta);\theta)}{\partial x_2} \\ \frac{\partial p_2(x^\star(\theta);\theta)}{\partial x_1} & \frac{\partial p_2(x^\star(\theta);\theta)}{\partial x_2} \end{bmatrix} \begin{bmatrix} \frac{dx_i^\star(\theta)}{d\theta_k} \\ \frac{dx_i^\star(\theta)}{d\theta_k} \end{bmatrix} + \begin{bmatrix} \frac{\partial p_1(x^\star(\theta);\theta)}{\partial \theta_k} \\ \frac{\partial p_2(x^\star(\theta);\theta)}{\partial \theta_k} \end{bmatrix} \tag{31}$$

where, mimicking (14)

$$\begin{bmatrix} \frac{\partial p_1(x^\star(\theta);\theta)}{\partial \theta_k} \\ \frac{\partial p_2(x^\star(\theta);\theta)}{\partial \theta_k} \end{bmatrix} = -\left(\mathrm{J}f^p(x^\star(\theta);\theta)\right)^{-1} \begin{bmatrix} \frac{\partial \mathrm{D}_1(x^\star(\theta);\theta)}{\partial \theta_k} & + & p_1 \frac{\partial^2 \mathrm{D}_1(x^\star(\theta);\theta)}{\partial p_1\, \partial \theta_k} \\ \frac{\partial \mathrm{D}_2(x^\star(\theta);\theta)}{\partial \theta_k} & + & p_2 \frac{\partial^2 \mathrm{D}_2(x^\star(\theta);\theta)}{\partial p_2\, \partial \theta_k} \end{bmatrix} \tag{32}$$

Except when necessary, we will henceforth suppress the arguments of the Π's and related terms.

To analyze the comparative statics properties of our game, we need to decompose the expression for $\frac{\partial^2 \Pi_i}{\partial x_i\, \partial \theta_k}$ into its component parts. This decomposition parallels our decomposition (23′) of the $\frac{d^2 \pi_i}{dx_i\, dx_j}$'s sufficiently closely that we can use the same labels for its components:

$$
\begin{aligned}
\frac{d^2\Pi_i}{dx_i\,d\theta_k} &= \frac{d}{d\theta_k}\left(\frac{\partial\Pi_i}{\partial x_i}\right) + \frac{d}{d\theta_k}\left(\frac{\partial\pi_i}{\partial p_{\neg i}}\frac{dp_{\neg i}}{dx_i}\right) \qquad (33)\\
&= \left(\frac{\partial p_i}{\partial\theta_k}\frac{\partial \mathrm{D}_i}{\partial x_i} + p_i\frac{\partial^2 \mathrm{D}_i}{\partial x_i\,\partial\theta_k} + p_i\sum_{j=1}^{2}\frac{\partial^2 \mathrm{D}_i}{\partial x_i\,\partial p_j}\frac{\partial p_j}{\partial\theta_k} - \frac{d\mathrm{MC}}{d\theta_k}\right)\\
&\quad + \left(\frac{dp_{\neg i}}{dx_i}\left[\frac{\partial p_i}{\partial\theta_k}\frac{\partial \mathrm{D}_i}{\partial p_{\neg i}} + p_i\frac{\partial^2 \mathrm{D}_i}{\partial p_{\neg i}\,\partial\theta_k}\right] + \frac{\partial^2 p_{\neg i}}{\partial x_i\,\partial\theta_k}\frac{\partial\pi_i}{\partial p_{\neg i}}\right)
\end{aligned}
$$

which, using the fact (implied by expression (20)) that $\frac{\mathrm{MC}}{p_i} = \frac{\partial \mathrm{D}_i}{\partial x_i} + \frac{\partial \mathrm{D}_i}{\partial p_{\neg i}}\frac{dp_{\neg i}}{dx_i}$

$$
\begin{aligned}
&= \frac{\partial p_i}{\partial\theta_k}\frac{\mathrm{MC}}{p_i} + p_i\left(\frac{\partial^2 \mathrm{D}_i}{\partial x_i\,\partial\theta_k} + \sum_{j=1}^{2}\frac{\partial^2 \mathrm{D}_i}{\partial x_i\,\partial p_j}\frac{\partial p_j}{\partial\theta_k} + \frac{dp_{\neg i}}{dx_i}\frac{\partial^2 \mathrm{D}_i}{\partial p_{\neg i}\,\partial\theta_k}\right)\\
&\quad + \frac{\partial^2 p_{\neg i}}{\partial x_i\,\partial\theta_k}\frac{\partial\pi_i}{\partial p_{\neg i}} - \frac{d\mathrm{MC}}{d\theta_k}
\end{aligned}
$$

which, letting $\Psi_i:\mathbb{R}^2\to\mathbb{R}$ denote the linear function with coefficient vector $\left(\left[\frac{\mathrm{MC}_i}{p_i} + p_i\frac{\partial^2 \mathrm{D}_i}{\partial x_i\,\partial p_i}\right] \quad p_i\frac{\partial^2 \mathrm{D}_i}{\partial x_i\,\partial p_j}\right)$

$$
= \underbrace{\Psi_i\left(\frac{\partial p_1}{\partial\theta_k},\frac{\partial p_2}{\partial\theta_k}\right)}_{\substack{\text{price}\\\text{level}\\\text{effect}}} + \underbrace{p_i\frac{\partial^2 \mathrm{D}_i}{\partial x_i\,\partial\theta_k}}_{\substack{\text{location}\\\text{competition}\\\text{effect}}} + \underbrace{p_i\frac{dp_{\neg i}}{dx_i}\frac{\partial^2 \mathrm{D}_i}{\partial p_{\neg i}\,\partial\theta_k}}_{\substack{\text{crossprice}\\\text{competition}\\\text{effect}}} + \underbrace{\frac{\partial^2 p_{\neg i}}{\partial x_i\,\partial\theta_k}\frac{\partial\pi_i}{\partial p_{\neg i}}}_{\substack{\text{crossprice}\\\text{sensitivity}\\\text{effect}}} - \underbrace{\frac{d\mathrm{MC}}{d\theta_k}}_{\substack{\text{marginal}\\\text{cost}\\\text{effect}}} \qquad (33')
$$

Note that the coefficient matrix of $\boldsymbol{\Psi}$ is common to all of our comparative statics experiments, i.e., is independent of all cross derivatives involving the θ_k's. The $\boldsymbol{\Psi}$ function combines the impacts, from all sources, of changes in equilibrium price levels, holding locations constant, on the first order conditions for the location game. These impacts have three components. The first, $\frac{\mathrm{MC}_i}{p_i}$ is exactly analogous to the first component (the price level effect) in our decomposition(23′) of the $\frac{d^2\pi_i}{dx_i\,dx_j}$'s (see the discussion on page 21, immediately following equation (20)). The other two components are the impacts of p_i and p_j on $\frac{\partial \mathrm{D}_i}{\partial x_i}$. With one exception, the relationship is that $\frac{\partial \mathrm{D}_i}{\partial x_i}$ increases with the gap, Δp, between the two prices: increases in p_1 increase the $\frac{\partial \mathrm{D}_i}{\partial x_i}$'s,while increases in p_2 decrease them. The exception is that an increase in p_2 results in an *increase* in $\frac{\partial \mathrm{D}_2}{\partial x_2}$ because it increases the rate at which the participation boundary moves south in response to an increase in x_2. In the basecase of our numerical simulation, the net effect of these three impacts is that $\boldsymbol{\Psi}\left(\frac{\partial p_1}{\partial\theta_k},\frac{\partial p_2}{\partial\theta_k}\right) = \begin{bmatrix} 0.0531 & -0.0397\\ 0.0594 & 0.1460\end{bmatrix}\begin{bmatrix}\frac{\partial p_1}{\partial\theta_k}\\ \frac{\partial p_2}{\partial\theta_k}\end{bmatrix}$. **Is the less than half reference still applicable?** In all five of our comparative statics experiments, both prices move in the same direction in response to a change in θ_k and $|\frac{\partial p_1}{\partial\theta_k}|$ is no less than half of $|\frac{\partial p_2}{\partial\theta_k}|$. Given these conditions, the Ψ_i's contribute positively or negatively to the $\frac{d^2\Pi_i(x;\theta)}{dx_i\, d\theta_k}$'s, depending on whether prices increase or decrease with θ_k. **What about discussing the other effects?**

6.1. **Impact on equilibrium values of changes in cost parameters.** In this subsection we compare the effects of changing the two parameters, c and γ, of the common cost function cx_i^γ. The effects of these changes are particularly easy to analyze since prices in each subgame are independent of costs. Indeed, when θ_k is either c or γ, all of the terms in (33′) are zero for except the term $-\frac{d\text{MC}}{d\theta_k}$.

6.1.1. *Effect of changing the cost parameter c.* Since $\Pi_i = p_i \text{D}_i - cx_i^\gamma$, we have $\frac{\partial \Pi_i}{\partial c} = -x_i^\gamma$ and $\frac{\partial^2 \Pi_i}{\partial x_i\, \partial c} = -\gamma x_i^{\gamma-1}$. In the base case for our simulations, we assume constant returns to scale, i.e., $\gamma = 1$, so that $\frac{\partial^2 \Pi_i}{\partial c\, \partial x_i} = -1$. Hence from (30)

$$\begin{bmatrix} \frac{dx_1}{dc} \\ \frac{dx_2}{dc} \end{bmatrix} = \begin{bmatrix} 36.9228 & 15.0511 \\ 0.9442 & 4.8986 \end{bmatrix} \begin{bmatrix} -1 \\ -1 \end{bmatrix} = \begin{bmatrix} -51.9739 \\ -5.8428 \end{bmatrix} \tag{34}$$

and, from (31)

$$\begin{bmatrix} \frac{dp_1}{dc} \\ \frac{dp_2}{dc} \end{bmatrix} = \begin{bmatrix} 0.1410 & -0.4521 \\ 0.0446 & 0.2788 \end{bmatrix} \begin{bmatrix} -51.9739 \\ -5.8428 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -4.6850 \\ -3.9473 \end{bmatrix}$$

That is, the effect of increasing marginal cost is to shift equilibrium locations to the left. Since firm #1's location is more sensitive to cost changes than #'2's, the distance between the firms shrinks, increasing the intensity of competition, and lowering prices. Hence as marginal costs rise, consumer welfare declines because lower quality goods are provided, but this decline is mitigated by the lower prices that result from intensified competition.

6.1.2. *Effect of changing the cost exponent* γ. In this case, $\frac{\partial \Pi_i}{\partial \gamma} = -cx_i^\gamma \log(x_i)$ and $\frac{\partial^2 \Pi_i}{\partial \gamma\, \partial x_i} = -cx_i^\gamma(\gamma \log(x_i) + 1)/x_i$. Since in the basecase of our simulations, $\gamma = 1$, this expression reduces to $\frac{\partial^2 \Pi_i}{\partial \gamma\, \partial x_i} = -c(\log(x_i) + 1)$. At our solution value $x^\star = [7.6809 \;\; 1.9792]$ and the common marginal cost paramter $c = 0.0500$, we have $[\frac{\partial^2 \Pi_1}{\partial \gamma\, \partial x_1} \; \frac{\partial^2 \Pi_2}{\partial \gamma\, \partial x_2}] = [-0.1519 \;\; -0.0841]$. Hence from (30)

$$\begin{bmatrix} \frac{dx_1}{d\gamma} \\ \frac{dx_2}{d\gamma} \end{bmatrix} = \begin{bmatrix} 36.9228 & 15.0511 \\ 0.9442 & 4.8986 \end{bmatrix} \begin{bmatrix} -0.1519 \\ -0.0841 \end{bmatrix} = \begin{bmatrix} -6.8762 \\ -0.5556 \end{bmatrix} \tag{35}$$

and, from (31)

$$\begin{bmatrix} \frac{dp_1}{d\gamma} \\ \frac{dp_2}{d\gamma} \end{bmatrix} = \begin{bmatrix} 0.1410 & -0.4521 \\ 0.0446 & 0.2788 \end{bmatrix} \begin{bmatrix} -6.8762 \\ -0.5556 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -0.7181 \\ -0.4616 \end{bmatrix}$$

To facilitate comparison between subsubsections 6.1.1 and 6.1.2, we consider the impacts of changes $dc = 0.0191$ and $d\gamma = 0.1450$. These perturbations are chosen so that the resulting changes in locations will each have unit length. The changes resulting from dc are $dx = [-0.9937 \;\; -0.1117]$ and $dp = [-0.0896 \;\; -$

0.0755]; the corresponding changes resulting from $d\gamma$ are $dx = [-0.9968 \;\; -0.0805]$ and $dp = [-0.1041 \;\; -0.0669]$. That is, relative to a unit increase in c, a comparable increase in γ has a greater effect on the firm producing the higher quality good, and results in a greater reduction the the price gap between the two products. Note also that relative to the change in c, the comparable change in γ also reduces the quality gap between the two products, and so induces a reduction in prices that is greater in norm.

6.2. **Effect of changing the first moments of the income-taste distribution.** Recall that each consumer is characterized by a two-component vector (α, y), representing, respectively, her preference for quality and her income. The distribution of consumer characteristics is a truncated bivariate normal, with mean $\begin{bmatrix} E[\alpha] & E[y] \end{bmatrix}$. and variance $\sigma^2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. In this subsection, we compare the effects of increases in the means of the taste versus the income parameters. The issues that arise in this comparison are much more subtle than the ones considered above. Intuitively, since the two changes we are considering increase either the willingness or the capacity of the average consumer to purchase higher quality goods, we might expect that in both instances, the level of quality provision would increase. In fact, however, we find that the results of the two experiments have opposite signs: when the mean of the taste parameter increases, the location of both goods shifts to the left and prices decline, while the opposite results occur when mean income increases.

The difference between the two results can be traced to a difference between the balance between the conflicting price level effect and the location competition effect. To aid us in explaining this difference, we have, in Fig. 4 below, replicated the right panel of Fig. **??**, with the addition of a neighborhood around the interfirm boundary. The neighborhood represents schematically the set of consumers who are on the verge of shifting from one good to the other. As the legend indicates, the neighborhood is divided into a "sloped" and a "flat" portion; as discussed on page 16 (immediately below equation (11)), a shift in either the *location vector* or firm #2's price will affect only those consumers with characteristics in the sloped portion; consumers in the flat portion are constrained only by the relationship between their incomes and the price of good #1. On the other hand, a shift in firm #1's *price* will affect consumers in the entire neighborhood.

Now compare the effects of increasing either $E[y]$ or $E[\alpha]$. Most consumers in the *sloped* neighborhood lie in the region below $E[y]$ and to the left of $E[\alpha]$. Therefore, an increase in either mean will reduce the total probability mass of consumers in this portion, thus reducing the rate at which consumers can be acquired by increasing location without affecting the marginal cost of this increase. In terms of our decomposition (33′), the location competition effectwill be negative: in either case, firms' incentives to compete-by-location, holding prices constant, will decline. By contrast, consumers in the *flat* neighborhood lie in the region below $E[y]$ but to the right of $E[\alpha]$. Therefore, an increase in $E[y]$ will reduce the total probability mass of consumers in this portion, while an increase in $E[\alpha]$ will raise it. Consequently, an increase in $E[y]$ will

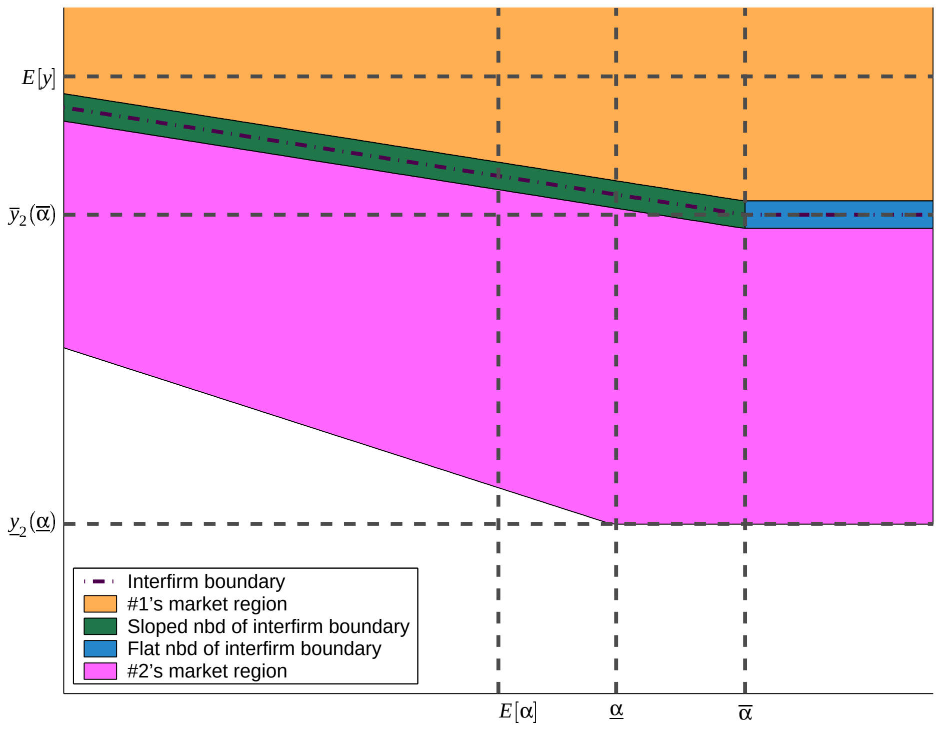

FIGURE 4. A neighborhood of the interfirm boundary

reduce the rate at which consumers are released from their income constraints in response to an decrease in p_1, while an increase in $E[\alpha]$ will raise this rate. It follows that the impact of an increase in $E[y]$ on the flat neighborhood will have a dampening effect on firm #1's incentive to compete by price, while an impact of an increase in $E[\alpha]$ on this portion will have an intensifying effect. This difference will be reflected in the rates at which equilibrium prices change, holding locations constant, in the two instances, and further reflected in the price level effectfor the two cases.

To make these points concrete, we will insert into expressions (32) and (33) the numerical values of the relevant derivatives from the basecase of our simulation, and compare the cases in which θ_k is either $E[\alpha]$ or $E[y]$. Starting with expression (32), we have

$$\begin{bmatrix}\frac{\partial p_1}{\partial E[\alpha]}\\[4pt] \frac{\partial p_2}{\partial E[\alpha]}\end{bmatrix} = -0.01(\mathrm{J}f^p)^{-1}\left[\underbrace{\begin{matrix}0.0686\\ 0.0188\end{matrix}}_{\frac{\partial \mathrm{D}_i}{\partial E[\alpha]}} \begin{matrix}+\\ +\end{matrix} \underbrace{\begin{matrix}0.0398\\ 0.1537\end{matrix}}_{\frac{\partial^2 \mathrm{D}_i}{\partial p_i\,\partial E[\alpha]}} \begin{matrix}p_1\\ p_2\end{matrix}\right] = 0.01\begin{bmatrix}0.6350\\ 0.6037\end{bmatrix} \tag{36-∂p}$$

$$\begin{bmatrix}\frac{\partial p_1}{\partial E[y]}\\[4pt] \frac{\partial p_2}{\partial E[y]}\end{bmatrix} = -0.01(\mathrm{J}f^p)^{-1}\left[\underbrace{\begin{matrix}6.4546\\ -2.9875\end{matrix}}_{\frac{\partial \mathrm{D}_i}{\partial E[y]}} \begin{matrix}+\\ +\end{matrix} \underbrace{\begin{matrix}0.7115\\ 3.1066\end{matrix}}_{\frac{\partial^2 \mathrm{D}_i}{\partial p_i\,\partial E[y]}} \begin{matrix}p_1\\ p_2\end{matrix}\right] = 0.01\begin{bmatrix}23.5877\\ 8.6487\end{bmatrix} \tag{37-∂p}$$

In both instances, equilibrium prices rise, holding locations constant at $x^{\star}$. The dominant impact in both cases is that mass of consumers in the sloped neighborhood declines, reducing incentives to compete-by-price. In the case of $E[y]$, this impact is reinforced by the impact on the flat neighborhood; the the case of $E[\alpha]$ it is dampened. Consequently the combined impact on prices is much larger in the former case. This difference is reflected in the difference between the price level effects in the two cases, when we plug the numbers into expression (33): in (36-$d^2\Pi$), the price level effect is dominated by the location competition effect while in (37-$d^2\Pi$) the former dominates the latter:

$$\underbrace{0.01\begin{bmatrix}-0.0305\\ -0.0359\end{bmatrix}}_{\frac{d^2\Pi_i(x;E[\alpha])}{dx_i\,dE[\alpha]}} = 0.01\left(\overbrace{\underbrace{\begin{bmatrix}0.0077\\ 0.1086\end{bmatrix}}_{\Psi_i\left(\frac{\partial p_1}{\partial E[\alpha]},\frac{\partial p_2}{\partial E[\alpha]}\right)}}^{\text{price level effect}} + \overbrace{\underbrace{\begin{bmatrix}-0.0379\\ -0.1823\end{bmatrix}}_{p_i\frac{\partial^2 \mathrm{D}_i}{\partial x_i\,\partial E[\alpha]}}}^{\text{location competition effect}} + \underbrace{\begin{bmatrix}-0.0045\\ 0.0183\end{bmatrix}}_{p_i\frac{dp_{\neg i}}{dx_i}\frac{\partial^2 \mathrm{D}_i}{\partial p_{\neg i}\,\partial E[\alpha]}} + \underbrace{\begin{bmatrix}0.0042\\ 0.0187\end{bmatrix}}_{\frac{\partial^2 p_{\neg i}}{\partial x_i\,\partial E[\alpha]}\frac{\partial \pi_i}{\partial p_{\neg i}}.}\right) \tag{36-$d^2\Pi$}$$

$$\underbrace{0.01\begin{bmatrix}0.7671\\ 0.8732\end{bmatrix}}_{\frac{d^2\Pi_i(x;E[y])}{dx_i\,dE[y]}} = 0.01\left(\underbrace{\begin{bmatrix}0.9038\\ 2.7281\end{bmatrix}}_{\Psi_i\left(\frac{\partial p_1}{\partial E[y]},\frac{\partial p_2}{\partial E[y]}\right)} + \underbrace{\begin{bmatrix}-0.1732\\ -1.6320\end{bmatrix}}_{p_i\frac{\partial^2 \mathrm{D}_i}{\partial x_i\,\partial E[y]}} + \underbrace{\begin{bmatrix}-0.0205\\ 0.4168\end{bmatrix}}_{p_i\frac{dp_{\neg i}}{dx_i}\frac{\partial^2 \mathrm{D}_i}{\partial p_{\neg i}\,\partial E[y]}} + \underbrace{\begin{bmatrix}0.0574\\ -0.6527\end{bmatrix}}_{\frac{\partial^2 p_{\neg i}}{\partial x_i\,\partial E[y]}\frac{\partial \pi_i}{\partial p_{\neg i}}.}\right) \tag{37-$d^2\Pi$}$$

As a result of this difference, equilibrium locations and prices both decline in response to an increase in $E[\alpha]$:

$$\begin{aligned}
\begin{bmatrix}\frac{dx_1}{dE[\alpha]}\\ \frac{dx_2}{dE[\alpha]}\end{bmatrix} &= -\begin{bmatrix}\frac{\partial^2\Pi_1(x;E[\alpha])}{\partial x_1{}^2} & \frac{\partial^2\Pi_1(x;E[\alpha])}{\partial x_1\,\partial x_2}\\ \frac{\partial^2\Pi_2(x;E[\alpha])}{\partial x_2\,\partial x_1} & \frac{\partial^2\Pi_2(x;E[\alpha])}{\partial x_2^2}\end{bmatrix}^{-1}\begin{bmatrix}\frac{\partial^2\Pi_1(x;E[\alpha])}{\partial x_1\,\partial E[\alpha]}\\ \frac{\partial^2\Pi_2(x;E[\alpha])}{\partial x_2\,\partial E[\alpha]}\end{bmatrix}\\
&= 0.01\begin{bmatrix}36.9228 & 15.0511\\ 0.9442 & 4.8986\end{bmatrix}\begin{bmatrix}-0.0305\\ -0.0359\end{bmatrix} = 0.01\begin{bmatrix}-2.0207\\ -0.6504\end{bmatrix} \qquad (36\text{-}dx)\\
\begin{bmatrix}\frac{dp_1}{dE[\alpha]}\\ \frac{dp_2}{dE[\alpha]}\end{bmatrix} &= \begin{bmatrix}\frac{\partial p_1(x)}{\partial x_1} & \frac{\partial p_1(x)}{\partial x_2}\\ \frac{\partial p_2(x)}{\partial x_1} & \frac{\partial p_2(x)}{\partial x_2}\end{bmatrix}\begin{bmatrix}\frac{dx_1}{dE[\alpha]}\\ \frac{dx_2}{dE[\alpha]}\end{bmatrix} + \begin{bmatrix}\frac{\partial p_1(x)}{\partial E[\alpha]}\\ \frac{\partial p_2(x)}{\partial E[\alpha]}\end{bmatrix}\\
&= 0.01\begin{bmatrix}0.1410 & -0.4521\\ 0.0446 & 0.2788\end{bmatrix}\begin{bmatrix}-2.0207\\ -0.6504\end{bmatrix} + \begin{bmatrix}0.6350\\ 0.6037\end{bmatrix} = 0.01\begin{bmatrix}0.6443\\ 0.3322\end{bmatrix}\\
&\qquad (36\text{-}dp)
\end{aligned}$$

while both increase in response to an increase in $E[y]$:

$$\begin{aligned}
\begin{bmatrix}\frac{dx_1}{dE[y]}\\ \frac{dx_2}{dE[y]}\end{bmatrix} &= 0.01\begin{bmatrix}36.9228 & 15.0511\\ 0.9442 & 4.8986\end{bmatrix}\begin{bmatrix}0.7671\\ 0.8732\end{bmatrix} = 0.01\begin{bmatrix}50.6733\\ 16.1697\end{bmatrix} \qquad (37\text{-}dx)\\
\begin{bmatrix}\frac{dp_1}{dE[y]}\\ \frac{dp_2}{dE[y]}\end{bmatrix} &= 0.01\begin{bmatrix}0.1410 & -0.4521\\ 0.0446 & 0.2788\end{bmatrix}\begin{bmatrix}50.6733\\ 16.1697\end{bmatrix} + \begin{bmatrix}23.5877\\ 8.6487\end{bmatrix} = 0.01\begin{bmatrix}23.4204\\ 15.4170\end{bmatrix}\\
&\qquad (37\text{-}dp)
\end{aligned}$$

6.3. **Effect of changing the second moment of the income-taste distribution.** Our final comparative statics experiment involves increasing the standard deviation, σ, of the joint distribution of incomes and tastes. The results of this experiment are that as the distribution of consumers becomes more dispersed, competition becomes less intense and equilibrium locations and prices both increase. While these results are intuitive, a number of interesting issues arise in the process of moving from one equilibrium to another.

We begin by considering the effect of increasing σ on equilibrium prices, holding locations constant at $x^\star$. Inserting the numbers into (32), we obtain:

$$\begin{bmatrix}\frac{\partial p_1(x;\sigma)}{\partial\sigma}\\ \frac{\partial p_2(x;\sigma)}{\partial\sigma}\end{bmatrix} = -0.01(\mathrm{J}f^p)^{-1}\begin{bmatrix}-0.1064 & + & 0.1875\ p_1\\ \underbrace{-0.1241}_{\frac{\partial \mathrm{D}_i(x;\theta)}{\partial\sigma}} & + & \underbrace{-0.0271}_{\frac{\partial^2 \mathrm{D}_i(x;\theta)}{\partial p_i\,\partial\sigma}}\ p_2\end{bmatrix} = 0.01\begin{bmatrix}1.3666\\ 0.0813\end{bmatrix} \qquad (38\text{-}\partial p)$$

Observe first that while both prices rise, holding locations constant, the increase in p_2 is due to the cross-price effect, i.e., had p_1 remained constant, p_2 would have declined. The reason will be apparent from Fig. 5:

as the distribution of consumers is dispersed, the probability mass assigned to the indicated neighborhood of the interfirm boundary declines, while the mass assigned to a comparable neighborhood of the participation boundary increases. Thus, holding constant the location vector $x^\star$ and prices $p(x^\star)$, there is a reduction in firms' incentives to compete-by-price *with each other*, but, on balance, an *increase* in firm #2's incentive to reduce p_2, in order to gain clientele at the now more densely populated lower end of the income distribution. Reflecting these observations, the sign of $\frac{\partial^2 \mathrm{D}_1(x;\theta)}{\partial p_1\,\partial\sigma}$ in (38-∂p) above is positive, while the sign of $\frac{\partial^2 \mathrm{D}_2(x;\theta)}{\partial p_2\,\partial\sigma}$ is negative. However, as p_1 increases significantly, the cost to p_1 of raising its own price declines, and the net result of all of these effects is positive.

We now turn to the location game. Replacing the θ_k's with σ's, expression (33$'$) becomes:

$$\frac{d^2\Pi_i(x,\sigma)}{dx_i\,d\sigma} = \underbrace{\Psi_i\left(\frac{\partial p_1}{\partial\sigma},\frac{\partial p_2}{\partial\sigma}\right)}_{\substack{\text{price}\\ \text{level}\\ \text{effect}}} + \underbrace{p_i\frac{\partial^2 \mathrm{D}_i}{\partial x_i\,\partial\sigma}}_{\substack{\text{location}\\ \text{competition}\\ \text{effect}}} + \underbrace{\frac{dp_{\neg i}}{dx_i}\,\frac{\partial^2 \mathrm{D}_i}{\partial p_{\neg i}\,\partial\sigma}}_{\substack{\text{crossprice}\\ \text{competition}\\ \text{effect}}} + \underbrace{\frac{\partial^2 p_{\neg i}}{\partial x_i\,\partial\sigma}\,\frac{\partial\pi_i}{\partial p_{\neg i}}}_{\substack{\text{crossprice}\\ \text{sensitivity}\\ \text{effect}}}. \tag{39}$$

Because both $\frac{\partial p_i(x;\theta)}{\partial\sigma}$'s are positive, the price level effect is positive also: intuitively, spreading out the density of consumers reduces competition, increasing prices and hence the net benefit to increasing locations. On the other hand, the location competition effect is negative for both firms, since spreading out the mass of consumers reduces the measure of consumers near the interfirm boundary. For the same reason, the $\frac{\partial^2 \mathrm{D}_i}{\partial p_{\neg i}\,\partial\sigma}$'s are negative also. But since $\frac{dp_2}{dx_1} > 0 > \frac{dp_1}{dx_2}$, the crossprice competition effect is negative for firm #1 and

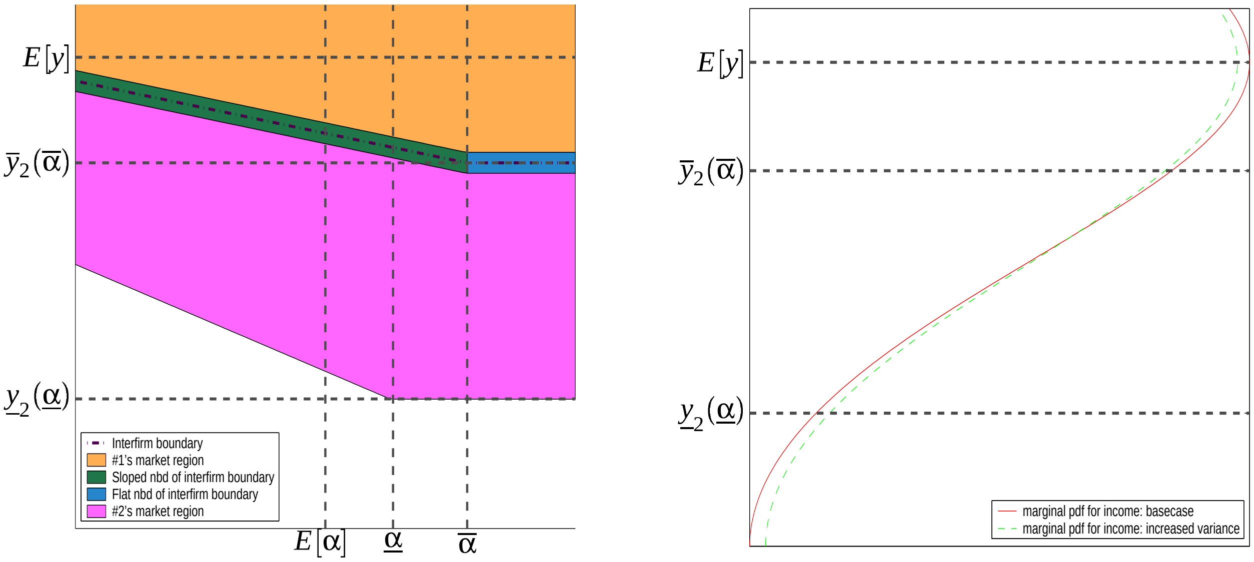

FIGURE 5. Increasing the variance of the consumer distribution

positive for #2. Finally, while the crossprice sensitivity effect is insignificant for firm #1, it is large and negative, though not decisive, for firm #2. As expression (38-$d^2\Pi$) below indicates, the positive impact of the price increases, especially p_1, dominate for both firms, i.e., both $\frac{d^2\Pi_i(x;\sigma)}{dx_i\,d\sigma}$'s are positive:

$$0.01\underbrace{\begin{bmatrix}0.0139\\0.0191\end{bmatrix}}_{\frac{d^2\Pi_i(x;\sigma)}{dx_i\,d\sigma}} = 0.01\left(\underbrace{\begin{bmatrix}0.0704\\0.0911\end{bmatrix}}_{\Psi_i\left(\frac{\partial p_1}{\partial\sigma},\frac{\partial p_2}{\partial\sigma}\right)} + \underbrace{\begin{bmatrix}-0.0521\\-0.0596\end{bmatrix}}_{p_i\frac{\partial^2 \mathrm{D}_i}{\partial x_i\,\partial\sigma}} + \underbrace{\begin{bmatrix}-0.0062\\0.1086\end{bmatrix}}_{p_i\frac{dp_{\neg i}}{dx_i}\frac{\partial^2 \mathrm{D}_i}{\partial p_{\neg i}\,\partial\sigma}} + \underbrace{\begin{bmatrix}0.0018\\-0.1210\end{bmatrix}}_{\frac{\partial^2 p_{\neg i}}{\partial x_i\,\partial\sigma}\frac{\partial\pi_i}{\partial p_{\neg i}}.}\right) \tag{38-$d^2\Pi$}$$

Proceeding as on page 34, we obtain:

$$\begin{bmatrix}\frac{dx_1}{d\sigma}\\\frac{dx_2}{d\sigma}\end{bmatrix} = 0.01\begin{bmatrix}36.9228 & 15.0511\\0.9442 & 4.8986\end{bmatrix}\begin{bmatrix}0.0139\\0.0191\end{bmatrix} = 0.01\begin{bmatrix}0.9383\\0.3140\end{bmatrix} \tag{38-dx}$$

$$\begin{bmatrix}\frac{dp_1}{d\sigma}\\\frac{dp_2}{d\sigma}\end{bmatrix} = 0.01\begin{bmatrix}0.1410 & -0.4521\\0.0446 & 0.2788\end{bmatrix}\begin{bmatrix}0.9383\\0.3140\end{bmatrix} + \begin{bmatrix}1.3666\\0.0813\end{bmatrix} = 0.01\begin{bmatrix}1.3568\\0.2107\end{bmatrix} \tag{38-dp}$$

7. CONCLUSION

REFERENCES

Caplin, A. and Nalebuff, B. (1991). Aggregation and imperfect competition: On the existence of equilibrium, *Econometrica* **59**(1): 25–59.

Cremer, H. and Thisse, J.-F. (1991). Location models of horizontal differentiation: a special case of vertical differentiation, *Journal of Industrial Economics* **39**: 383–390.

d'Aspremont, C., Gabszewicz, J. J. and Thisse, J.-F. (1979). On Hotelling's "stability in competition", *Econometrica* **47**: 1145–1150.

Hotelling, H. (1929). Stability in competition, *Economic Journal* **39**: 41–57.

Mussa, M. and Rosen, S. (1986). Monopoly and product quality, *Journal of Economic Theory* **18**: 301–317.

Tabuchi, T. and Thisse, J.-F. (1995). Asymmetric equilibria in spatial competition, *International Journal of Industrial Organization* **13**: 213–227.

Thaler, R. H. (1985). Mental accounting and consumer choice, *Marketing Science* **4**: 199–214.

Thaler, R. H. (1999). Mental accounting matters, *Journal of Behavioral Decision Making* **12**: 183–206.

Tirole, J. (1993). *The theory of Industrial Organization*, MIT Press, Cambridge, MA.