Appendix (not intended for publication) for

Indeterminacy with Environmental and Labor Dynamics

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This appendix proves Proposition 2 and Remark 2 from the paper. It explains why we have not been able to exclude the possibility that the projection of a trajectory leaves Ω before satisfying the boundary condition. It discusses the numerical algorithm used in the Section 5.

Proof of Proposition 2

Part (i). The function ϕ is quadratic in γ. The positive root of $\phi = 0$ is

$$\begin{aligned} \gamma^* &= \frac{1}{54g}\left(6gr^2 - 4g^3 + 4r^3 - 6g^2 r + 4\Delta\right) \\ \Delta &\equiv \sqrt{6g^2r^4 + 6g^4r^2 + 3gr^5 + 7g^3r^3 + g^6 + 3g^5r + r^6}. \end{aligned}$$

Part (ii). Using the formulae for ϕ,

$$\frac{d\phi}{d\gamma} = \frac{1}{2}g^2\gamma + \frac{1}{27}g^4 - \frac{1}{27}gr^3 + \frac{1}{18}g^3r - \frac{1}{18}g^2r^2.$$

Evaluating this derivative at $\gamma = \gamma^*$ gives

$$\begin{aligned} & \frac{1}{2}g^2\gamma^* + \frac{1}{27}g^4 - \frac{1}{27}gr^3 + \frac{1}{18}g^3r - \frac{1}{18}g^2r^2 \\ = \; & \frac{1}{27}g^4 - \frac{1}{27}gr^3 + \frac{1}{18}g^3r - \frac{1}{18}g^2r^2 + \\ & \frac{1}{108}g\left(6gr^2 - 4g^3 + 4r^3 - 6g^2r + 4\Delta\right) \\ = \; & \frac{1}{27}g\left(g^2 + gr + r^2\right)^{\frac{3}{2}} > 0. \end{aligned}$$

Consequently, $\phi > 0$ for $\gamma > \gamma^*$.

Part (iii). We first establish that $\frac{\partial \gamma^*}{\partial g} < 0$. Using the definition of γ^* we have

$$\begin{aligned} \frac{\partial \gamma^*}{\partial g} &= \frac{1}{27}\frac{N}{g^2\sqrt{\left(g^2+gr+r^2\right)^3}} \\ N &\equiv -4\sqrt{\left(g^2+gr+r^2\right)^3}g^3 - 2\sqrt{\left(g^2+gr+r^2\right)^3}r^3 - \\ & \quad 3\sqrt{\left(g^2+gr+r^2\right)^3}g^2r + 12g^4r^2 - 3gr^5 + 7g^3r^3 + 4g^6 + 9g^5r - 2r^6. \end{aligned}$$

The sign of this derivative depends on the sign of N. This function, evaluated at $g = 0$ is $-4r^6$, so for small g we have $\frac{\partial \gamma^*}{\partial g} < 0$. To complete the argument, we need to show that N never changes signs. The roots of $N = 0$ are

$$\{r=0\}, \{r=-g\}, \left\{r=\left(-\frac{1}{2}+\frac{1}{2}i\sqrt{3}\right)g\right\}, \left\{r=\left(-\frac{1}{2}-\frac{1}{2}i\sqrt{3}\right)g\right\},$$

so there are no real positive roots, and N does not change signs for $r > 0$.

We now establish that $\frac{\partial\gamma^*}{\partial r} > 0$. Using the definition of γ^* we have

$$\begin{aligned}
\frac{\partial\gamma^*}{\partial r} &= \frac{1}{9}\frac{M}{g\sqrt{(g^2+gr+r^2)^3}} \\
M &\equiv 2gr\sqrt{(g^2+gr+r^2)^3} + 2r^2\sqrt{(g^2+gr+r^2)^3} - g^2\sqrt{(g^2+gr+r^2)^3} \\
&\quad +8g^2r^3 + 4g^4r + 5gr^4 + 7g^3r^2 + g^5 + 2r^5.
\end{aligned}$$

The function M evaluated at $g = 0$ equals $4r^5 > 0$. The roots of $M = 0$ are

$$\{r=0\}, \{r=-g\}, \left\{r=\left(-\frac{1}{2}+\frac{1}{2}i\sqrt{3}\right)g\right\}, \left\{r=\left(-\frac{1}{2}-\frac{1}{2}i\sqrt{3}\right)g\right\},$$

so $M > 0$ for all positive real values of r and g.

Part (iv). In view of Parts (ii) and (iii),

$$\phi > 0 \Rightarrow \gamma > \gamma^* \geq \min_g \gamma^* = \lim_{g\to\infty} \gamma^*.$$

A necessary condition for $\phi > 0$ is that γ is sufficiently large, that is, such that $\phi > 0$ in the limit as $g \to \infty$. The leading term (by powers of g) of ϕ is

$$\frac{g^4}{27}\left(\gamma - \frac{r^2}{4}\right),$$

which is positive if and only if $\gamma > \frac{r^2}{4}$.

Proof of Remark 2

We begin by writing the three dimensional system as a single third order differential equation. We use this equation to show how the initial conditions and the boundary conditions determine the unknown constants. Using this formula we show that the iso-T curves are straight lines.

To obtain a third order equation, we first differentiate equation (8) (from the paper) with respect to time,

$$\ddot{q} = r\dot{q} + \dot{e}. \tag{A1}$$

Using (6) and the above equation yields

$$\ddot{q} = r\dot{q} - g\left(e + l\right). \tag{A2}$$

Differentiating (A2) with respect to time leads to

$$\dddot{q} = r\ddot{q} - g\left(\dot{e} + \dot{l}\right).$$

Combining with (7) and (A1), we get

$$\dddot{q} + (g - r)\ddot{q} - rg\dot{q} + \gamma g q = 0. \tag{A3}$$

The general solution of the third order differential equation (A3) is

$$q(t) = C_1 e^{v_1 t} + C_2 e^{v_2 t} + C_3 e^{v_3 t} \tag{A4}$$

where v_1, v_2 and v_3 are the roots of the characteristic polynomial of the dynamic system, C_1, C_2 and C_3 are constants whose values are found using the boundary conditions of the model

Given the closed form solution (A4), we can recover $l(t)$ and $e(t)$ using equations (8) and (A2)

$$\begin{aligned} l(t) &= -\frac{1}{g}\left(\ddot{q} - (r - g)\,\dot{q} - rgq\right) \\ &= d_1(t)C_1 + d_2(t)C_2 + d_3(t)C_3 \end{aligned}$$

$$\begin{aligned} e(t) &= \dot{q} - rq \\ &= f_1(t)C_1 + f_2(t)C_2 + f_3(t)C_3 \end{aligned}$$

where $d_i(t) = -\frac{e^{v_i t}}{g}\left[v_i^2 - (r - g)v_i - rg\right], f_i(t) = (v_i - r)e^{v_i t}$.

Using this general solution, a trajectory with initial condition $w_0 = (l_0, e_0)$, $w_0 \in B(k)$ (for $k = -0.5$ or $k = 0.5)$ is a solution to the system if and only if there exist real $T > 0$ and C_1, C_2, C_3, that satisfy

$$A(T)C = F, \tag{A5}$$

where

$$A(T) = \begin{pmatrix} d_1(0) & d_2(0) & d_3(0) \\ f_1(0) & f_2(0) & f_3(0) \\ d_1(T)e^{v_2 T} & d_2(T)e^{v_2 T} & d_3(T)e^{v_3 T} \\ e^{v_1 T} & e^{v_2 T} & e^{v_3 T} \end{pmatrix};\ C = \begin{pmatrix} C_1 \\ C_2 \\ C_3 \end{pmatrix}$$

$$\text{and } F = \begin{pmatrix} l_0 - 0.5 \\ e_0 - 0.5 \\ k \\ 0 \end{pmatrix}$$

The first two equations of system (A5) reproduce the initial condition $w_0 = (l_0, e_0)$ and the last two equations reproduce the terminal conditions $l_T = k$ and $q(T) = 0$.

Suppose that $w_0 = (l_0, e_0) \in B(k)$ (for $k = -0.5$ or $k = 0.5$) and let T^* be the amount of time it takes to reach $l_T = k$. From above, we know that there exists a vector $C' = (C_1, C_2, C_3)'$ that satisfy

$$\widetilde{A}(T^*)C = \widetilde{F}$$

where $\widetilde{A}(T^*)$ is a 3x3 matrix obtained by removing the first row of matrix $A(T^*)$ and $\widetilde{F}$ is 3x1 vector obtained by removing the first element of vector F. From the above equation, we obtain $C = \widetilde{A}^{-1}(T^*)\widetilde{F}$

Using system (A5) to solve for l_0, we have

$$l_0 = d(0)'\widetilde{A}^{-1}(T^*)\widetilde{F} + 0.5$$

where $d(0)' = (d_1(0), d_2(0), d_3(0))$. Since $\widetilde{A}^{-1}(T^*)\widetilde{B}$ is linear in e_0, the above expression can be rewritten as

$$l_0 = X(T^*) + Y(T^*)e_0 \tag{A6}$$

For a given T^*, equation (A6) defines a straight line; from any point (l_0, e_0) lying on that line, there exists a trajectory starting from this point and reaching $l_T = k$ in T^* units of time.

Do trajectories remain in Ω?

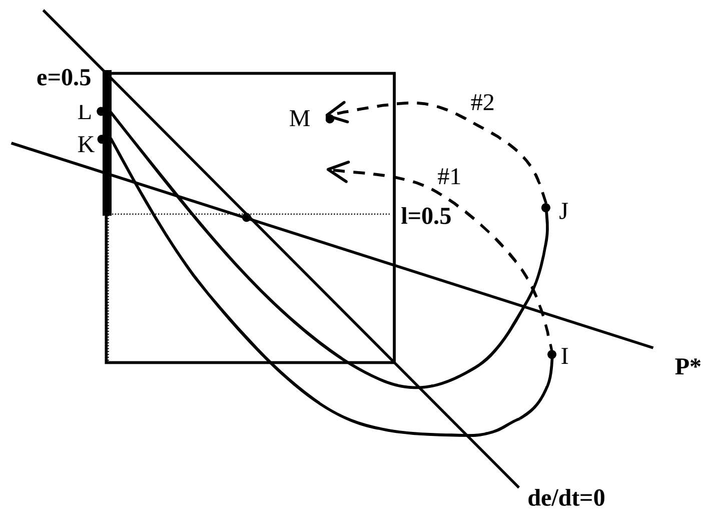

Figure 1: Examples of trajectories leaving Ω

Figure 1 illustrates the difficulty of proving that projections of trajectories onto the (l, e) plane do not leave Ω before satisfying the boundary condition. Let P be the plane that is spanned by the characteristic vectors associated with the unstable roots of F; P is the stable manifold for the reversed time system. Let P^* be the intersection of this plane and the (l, e) plane. The heavy line segment shows the feasible boundary values at T, given specialization in Agriculture (using Proposition 3).

Consider the reversed-time system of our original three-dimensional system, and choose an initial condition $l(0) = -0.5$, $q(0) = 0$, and $0 < e(0) < 0.5$, such as either the point L or K in figure 1. From Proposition 3, the projection (onto the (l, e) plane) of this trajectory (running in reversed time) enters Ω, as shown. The solid part of the projected trajectory represents the portion where $\dot{l} > 0$ (in reversed time), i.e. $q < 0$, and the dashed part represents the portion where the direction is reversed. The two hypothetical trajectories from K and L reverse direction at points I and J, respectively.

A trajectory cannot cross the plane P, since any trajectory that touches this plane remains on it. Consequently, no trajectory can pass through the (e, l) plane on different sides of P^*. This fact means that trajectory #1, through points K and I, cannot occur. However, we cannot exclude the possibility shown by trajectory #2. In particular, we cannot rule out the possibility that the projection of this trajectory reenters Ω and reaches point M.

If such a trajectory exists, then in the normal time system there is a trajectory beginning at point M (with $q > 0$) that satisfies the differential equations (in normal time) and the boundary conditions for specialization in Agriculture. This trajectory would not be an equilibrium, since it leaves Ω. Consequently, satisfaction of the differential equations and the boundary conditions would not imply that $M \in B(-0.5)$. Analysis of the differential system and boundary conditions in this case is not sufficient to reach conclusions about the ROI.

The numerical algorithm

We write the three dimensional system as a single third order differential equation as above. The solution to this equation requires three constant C_i, $i = 1, 2, 3$, and a terminal time T. We express the constant C_i as functions of the terminal time T using the initial conditions e_0, l_0 and the terminal condition $l_T = k \in \{-0.5, 0.5)$. We then use the transversality condition $q(T) = 0$ to obtain an implicit function which gives the equilibrium values of T as a function of e_0, l_0 and l_T:

$$H(T; e_0, l_0, l_T) = 0.$$

If, for a given initial value (e_0, l_0) and boundary value $l_T = -0.5$, there exists a *real and positive* solution T, then there is at least one trajectory converging toward full specialization in Agriculture. We then need to confirm that this trajectory reaches the boundary without exiting Ω. If these conditions are satisfied, then $(e_0, l_0) \in B(-0.5)$. (If (e_0, l_0) is near U and $\phi > 0$ there may be many real, positive roots of H, and many trajectories to the boundary.) The method of determining which points belong to $B(0.5)$ is symmetric.