

# Friction and the Multiplicity of Equilibria

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July 15, 2004

## Abstract

In many models, a decrease in the friction facing mobile factors (e.g., lowering their adjustment costs) increases a coordination problem, leading to more circumstances where there are multiple equilibria. We show that a decrease in friction can decrease coordination problems if, for example, a production externality arises from a changing stock of knowledge or a changing environmental stock. In general, the relation between the amount of friction that mobile factors face and the likelihood of multiple equilibria is non-monotonic.

**Keywords:** C61, J21, O13, R23

**JEL classification numbers:** Indeterminacy, Multiple equilibria, Coordination games, Factor reallocation, Environmental externality, Learning-by-doing, Costs of adjustment

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# 1 Introduction

In some circumstances, the payoff from taking a particular action is higher if many other agents take the same action. In this situation – i.e. where actions are strategic complements – there may be multiple equilibria. For example, agents may decide whether to work in the Agricultural or Manufacturing sector. For some range of labor allocations, an externality causes the benefit of working in a particular sector to increase with the number of workers there. Agents’ decisions depend on their beliefs about what other agents will do, rather than merely on exogenous economic fundamentals. This kind of model has been used to explain why similar countries might follow completely different development paths. A common feature of these models is that an increase in friction, causing slower reallocation of labor, makes the multiplicity of equilibria “less likely”.

We introduce a second stock variable into a standard model, causing the externality to occur with a lag. In this setting, an increase in friction (more costly reallocation of labor) can make multiplicity more likely. This conclusion is at odds with the intuition obtained from previous papers. We first review standard models and then explain why the presence of a second stock variable is both realistic and significant.<sup>1</sup>

In a standard model, the Manufacturing-Agricultural wage differential is an increasing function of the number of workers in Manufacturing, at least for some range of labor allocation. If labor could move between sectors costlessly and instantaneously, agents would play a static coordination game, and there would typically be multiple equilibria. The presence of friction makes their problem dynamic. In a dynamic setting, the existence of multiple equilibria depends not only on the parameters of the model (e.g., the extent of increasing returns to scale and the amount of friction in factor adjustment) but also on the initial allocation of labor.

In Matsuyama (1991), agents decide which sector to work in at the beginning of their life and then remain in that sector. Each worker dies after a random interval and is replaced by a young worker who faces the same kind of decision problem, but possibly with a different initial allocation of labor. An increase in workers’ survival rate increases friction in the model because it means that the labor allocation tends to change more slowly. In Krugman (1991) and Fukao and Benabou (1993), agents live forever, but in order to change sectors they must

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<sup>1</sup>Our paper deals with migration models. However, a large number of apparently dissimilar models of coordination failure have essentially the same ingredients, as Cooper and John (1988) point out. Thus, our conclusions regarding migration models apply to a much wider class of models.

pay a migration cost that increases with the number of other workers who migrate at the same time. An increase in the convexity of adjustment increases friction; i.e., it increases the price of migration, and tends to slow migration.

In both of these models, each stable steady state has a “region of attraction”, defined as a set of initial conditions such that there are equilibrium trajectories leading to that steady state. The “region of multiplicity” (ROM) is the intersection of two or more regions of attraction. From an initial condition in the ROM there exist equilibria that approach different steady states. If the regions of attraction intersect only on their boundaries, the measure of the ROM is 0, so multiplicity is non-generic. A positive measure of the ROM means that multiplicity is generic. (Section 5.1.1 explains why our principal conclusion is not sensitive to the particular measure that we employ.)

Here we explain our usage of two terms that in some papers have different meanings. First, some authors use “multiple equilibria” to mean the existence of two or more stable steady states. We use “multiple equilibria” to mean that there are multiple equilibrium *trajectories* emanating from an initial condition. In a dynamic setting, the equilibrium is a mapping from the initial condition (and model parameters) into an equilibrium trajectory. When there are multiple equilibria, this mapping is a correspondence. The distinction between an equilibrium trajectory and an equilibrium steady state is important in a dynamic model, but irrelevant in static models (where there is no state variable). In a dynamic setting, if there is a unique equilibrium trajectory for each initial condition, then we consider the equilibrium unique, regardless of the number of stable steady states. Second, some authors use “indeterminacy” as a synonym for “multiple equilibria”. However, “indeterminacy” is increasingly used to describe the situation where there is a continuum of equilibrium trajectories; in some cases, all of these trajectories approach a single steady state. (Indeterminacy is a special type of multiplicity of equilibria.) We adopt this meaning of indeterminacy.

We emphasize models – such as the migration models above – where there are two (or more) stable steady states, and where, for some initial conditions, there might be multiple equilibrium trajectories leading to different steady states. In this kind of model, there are two ways to interpret the statement that a parameter change makes multiplicity “less likely”:

Interpretation 1: The change reduces the measure of the set of other parameters for which multiplicity is generic.

Interpretation 2: The change reduces the measure of the ROM, holding fixed

other parameters.

Interpretation 2 means that there are fewer initial conditions for which there exist multiple equilibria.

In both Krugman's and Matsuyama's models, an increase in friction makes multiplicity less likely using the first interpretation. In Krugman's model, an increase in friction also makes multiplicity less likely using the second interpretation. Previous literature, excepting Krugman (1991) and Fukao and Benabou (1993), neglects the relation between parameters of the model and the measure of the ROM, and concentrates on the relation between parameter values and the existence of a ROM with positive measure. That is, the literature stresses Interpretation 1 and ignores Interpretation 2. There are two likely reasons for this emphasis. First, although it is sometimes relatively straightforward to determine conditions under which the ROM has positive measure, the comparative statics of this measure are complicated. Second, intuition (supported by Krugman's model) may have encouraged the idea that the two senses in which a parameter change can make multiplicity "more likely" are essentially the same. Multiplicity of equilibria arises when there are increasing returns to scale, or some other feature that makes the economy non-convex. Greater convexity of adjustment costs (more generally, increased friction) appears to convexify the economy, offsetting the effect of increasing returns to scale. Therefore, it seems natural that more convex adjustment costs make multiplicity less likely. Our results challenge this intuition.

In our variation of a standard model, the externality that favors Manufacturing is an indirect consequence of the labor allocation. There are at least two economic situations in which this variation is important. In the first, labor productivity in Manufacturing increases as a result of learning-by-doing and decays in the absence of production, as in Matsuyama (1992). In this situation, the current wage differential depends on the predetermined stock of knowledge; the current migration decision affects future wage differentials indirectly (by changing the stock of knowledge) rather than directly (by changing the stock of labor in a sector). In the second situation, Manufacturing output creates pollution that damages an environmental stock that determines labor productivity in Agriculture, as in Copeland and Taylor (1999). At a point in time the wage differential depends only on the environmental stock and is therefore independent of the labor allocation. However, increased Manufacturing output lowers future labor productivity in Agriculture, via changes in the environmental stock, thereby changing the future wage differential.

For the sake of concreteness, we emphasize to the second situation. In this model, the state variable is two-dimensional; its components are the current labor allocation and the current size of the environmental stock. Our chief result is that there is a non-monotonic relation between the costs of adjustment for labor and the measure of the ROM. For example, suppose we hold constant the speed of adjustment of the environmental stock. If labor migration is very costly, then the labor allocation will change slowly, if at all. In this case, the wage differential for a long time in the future depends on the current environmental stock, so there is little scope for expectations to affect the equilibrium outcome. Here the ROM has small or zero measure. A decrease in adjustment costs means that labor can move rapidly, raising the possibility that future wage differentials might be strongly related to current decisions. This decrease in labor adjustment cost tends to increase the measure of the ROM, just as in the standard setting. Now suppose that labor adjustment costs are extremely small. In this case, a worker bases her decision (whether to change sectors) almost entirely on the predetermined environmental stock. *She knows that regardless of what other agents do, it will be cheap for her to change sectors in the future, in order to remain in the high-wage sector.* In this case, the measure of the ROM decreases as migration becomes cheaper. With extremely small adjustment costs, it is rational for agents to behave almost myopically for most initial conditions; expectations do not matter (much).

We also find that a decrease in labor adjustment costs does increase the set of other parameter values for which the ROM is positive. In this sense (Interpretation I), lower adjustment costs make multiplicity “more likely”, just as in standard models. Taken together, the two results imply that if labor adjustment costs are extremely small, the measure of the ROM is positive for a wide range of parameter values, but the measure is always extremely small. Multiplicity in this case is generic – but not very likely.

Many dynamic models can be viewed as extensions of static models, obtained by introducing a payoff-relevant state variable that adjusts slowly. For example, Matsuyama’s and Krugman’s migration models are based on a two-sector static multiple-equilibria model with mobile labor. Dynamic extensions are more descriptive than the underlying static models and they are also useful for testing the robustness of the latter. Of course, there is usually more than a single way of making a model dynamic. There is surely some friction in the adjustment of the mobile factor, so introducing this source of dynamics is important. Arguably, there is also friction in the adjustment of the relative marginal productivities in the two sectors, for the kinds

of reasons described above (learning-by-doing or environmental spillovers). *A priori*, it is not obvious which of these two sources of dynamics is empirically more important. Therefore, it is worth noting that they have very different effects on the properties of the dynamic model.

Our result shows the danger of drawing conclusions about the importance of multiplicity based on estimates of structural parameters of the model (i.e., based on Interpretation 1). Parameter estimates might suggest, for example, that there are significant increasing returns to scale (or some other source of non-convexity); that factor adjustment costs are very low; and that a one-state variable model apparently provides a good approximation to the economy (because other state variables adjust rapidly). The conventional wisdom is that in these circumstances the ROM is likely to have positive measure. Our results agree with this conclusion, but also suggest that the measure of the ROM is likely to be very small, and therefore the economy is unlikely to have multiple equilibria.

This conclusion has significant policy implications. An economy that has multiple steady states but a unique equilibrium might evolve in very different ways, depending on the initial condition. However, exogenous shocks or changes in policies have predictable effects, given knowledge of the economic fundamentals. In contrast, if the economy has multiple equilibria, the effect of policy changes depends on agents' beliefs as well as economic fundamentals. *The policy problems in these two cases are qualitatively different.*

The next section mentions additional relevant literature. We then review Krugman's model and present a simplified version of it. The next section extends the standard model by adding a second state variable. We then demonstrate formally the result described above. Since this result is obtained using a very simple model, the reader may question its robustness, and thus its importance. In order to allay this concern, we sketch in the Appendix a more general model that produces the same result.

## 2 Related literature

The relation between costs of adjustment (friction) and the existence of multiplicity is an important theme in dynamic macro economics. For a class of models (many of which involve increasing returns to scale or production complementarities) the equilibrium is indeterminate (as defined above). Benhabib and Farmer (1999) review this literature; recent contributions include Cooper and Haltiwanger (1996), Cooper and Johri (1997), Benhabib, Meng, and

Nishimura (2000), Nishimura and Shimonura (2002), Wen (1998a), Wen (1998b), and Lubik and Schorfheide (2004). A recurring question concerns the specification for which plausible estimates of adjustment costs and returns to scale are consistent with indeterminacy. In these models, lower costs of adjustment (less friction for the mobile factor) mean that indeterminacy is more likely, using Interpretation 1.

Davis and Weinstein (2002) and Davis and Weinstein (2004) find that Japanese data is consistent with increasing returns to scale, but that the data is inconsistent with the existence of multiple stable steady states – a necessary condition for multiplicity of equilibria in this setting. Moro (2003) estimates a multiple equilibrium model of wage inequality; Brock and Durlauf (2001) and Brock and Durlauf (2002) discuss the estimation of discrete choice models with social interactions, a situation that can lead to multiple equilibria. Cooper (2002) reviews the estimation problem under different types of multiplicity.

Recent theoretical papers show that changing an assumption of migration models may eliminate the multiplicity of equilibria. Frankel and Pauzner (2000) show that multiplicity disappears in a variation of Matsuyama’s model where the wage differential is subject to Brownian motion and there exist “dominance regions”.<sup>2</sup> In another variation of this model, Herrendorf, Valentinyi, and Waldman (2000) show that there is a unique equilibrium if agents are sufficiently heterogeneous. We provide another explanation that might either increase or diminish the importance of coordination problems.

### 3 A review

We review Krugman’s continuous time, infinite horizon model and then show that a simplified version has the same qualitative properties. We present this material in order to convince the reader that the simpler setting leads to no loss of insight and also to provide a basis for comparison with a two-state model. The infinite horizon and continuous time are unimportant for understanding the role of friction in the type of coordination problem that we study.

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<sup>2</sup>It might seem that imbedding Krugman’s model in a “global games” setting, as in Carlsson and Van Damme (1993), would eliminate the multiplicity of equilibria. However, in this model there are two types of agents, those currently in Agriculture and those currently in Manufacturing. (In contrast, in Matsuyama’s migration model, new-born agents are ex ante identical.) Karp (1999) shows that the fact that these agents are on “different sides of the market” eliminates a type of monotonicity that is needed to prove uniqueness using the arguments in Morris and Shin (1998). Therefore a “global games” setting may not yield a unique equilibrium to Krugman’s model.

### 3.1 Krugman's model

There are two sectors. Agriculture has constant returns to scale, and Manufacturing has increasing returns to scale that are external to the firm. The stock of labor is normalized to 1 and at time  $t$  the stock of labor in Manufacturing is  $L_t$ . The constant wage in Agriculture is  $\alpha_A$  and the wage in Manufacturing is  $\alpha_M + bL_t$ , where  $b > 0$  determines the extent of increasing returns to scale. The Manufacturing-Agriculture wage differential is  $a + bL_t$ , with  $a \equiv \alpha_M - \alpha_A$ . By assumption,  $a < 0$  and  $a + b > 0$ : if all workers are in the same sector, that is the high-wage sector.

The flow of labor into Manufacturing is  $\dot{L} \equiv \frac{dL}{dt} = u_t$ . The social cost of migration is  $\frac{u_t^2}{2\gamma}$ . Migration services are competitively supplied, so the price of migration (the amount that a migrant at time  $t$  pays in order to switch sectors) is  $\frac{|u_t|}{\gamma}$ . A higher value of  $\gamma$ , the speed of adjustment parameter, means that adjustment costs are lower: there is less friction. An agent who decides to migrate pays the migration cost in the current instant, in order to be in a different sector at the next instant. The instantaneous discount rate is  $r > 0$ .

Krugman shows that there may be multiple rational expectations (perfect foresight) competitive equilibrium (i.e., the ROM has positive measure) if and only if  $\gamma > \frac{r^2}{4b}$ . Thus, a decrease in friction (larger  $\gamma$ ) increases the range of other parameters (here  $r$  and  $b$ ) for which there may be multiple equilibria. Fukao and Benabou (1993) show how to calculate the measure of the ROM, which is non-decreasing in  $\gamma$  (strictly increasing when the measure is between 0 and 1). Thus, for both interpretations given in the Introduction, a decrease in friction makes multiplicity more likely.

### 3.2 A simplified version

In our simplified version of the model, all migration occurs in the first period. The initial stock of labor in Manufacturing is  $L$  and the measure of entrants into Manufacturing is  $u$ , so the amount of labor in Manufacturing in the next period is  $L + u$ . Using the same notation as in the previous subsection, the present value (in the current period) of the Manufacturing-Agriculture wage differential in the next period is  $\beta(a + b(L + u))$ ; the discount factor is  $\beta = e^{-r}$ . As before, the price that an individual pays to move sectors depends on the total number of agents who move; this price is  $\frac{|u|}{\gamma}$ . Agents who migrate incur this cost in the current period.

The present value of the benefit minus the cost of migrating to Manufacture ( $u > 0$ ) or to

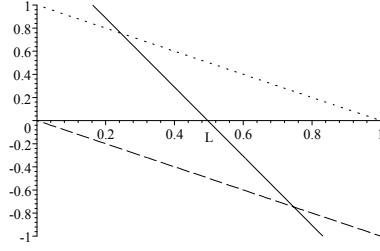


Figure 1: The ROM in a one-period model with  $\beta\gamma G > 1$

Agriculture ( $u < 0$ ) is

$$n(L, u) \equiv \frac{1}{\gamma} (\beta\gamma (a + b(L + u)) - u)$$

and the labor stock constraint is  $-L \leq u \leq 1 - L$ . For  $b\gamma\beta < 1$  actions are strategic substitutes (i.e., the net benefit of migration decreases with the number of other agents migrating); here the equilibrium is unique. Agents play a coordination game (i.e. actions are strategic complements), and there are multiple equilibria, if and only if  $b\gamma\beta > 1$ , i.e. for  $\gamma > \frac{e^r}{b}$ . This inequality has the same characteristics as the condition for multiplicity in the infinite horizon continuous time model,  $\gamma > \frac{r^2}{4b}$ .

Figure 1 graphs the migration constraints (the dotted and dashed lines) and the solution to  $n = 0$  for  $b\gamma\beta > 1$  (the solid line). The  $L$  coordinates of the points of intersection between the graph of  $n = 0$  and the migration constraints define the interval  $[1 - (a + b)\beta\gamma, -\beta a\gamma]$ . The ROM consists of the intersection of this interval and the set of feasible initial conditions,  $[0, 1]$ . For initial conditions in the ROM, points on the solid line are unstable equilibria.<sup>3</sup> At initial conditions inside the ROM there are two stable equilibria: all labor moves to Manufacturing or to Agriculture. The length of this interval is

$$\text{Length of ROM} = \max \{0, \min(1, -\beta a\gamma) - \max(0, 1 - (a + b)\beta\gamma)\}. \quad (1)$$

For both the infinite horizon continuous time and for the static versions of the model, the existence of multiplicity requires a combination of patience, a large externality, and low adjustment costs (large values of  $\beta, b, \gamma$ ). An increase in any of these factors increases the length of the ROM when this is positive and less than 1. Thus, lower adjustment costs makes multiplicity

<sup>3</sup>We use the standard notion of stability. At an interior equilibrium  $n = 0$  and  $\frac{dn}{du} > 0$ . If a small measure of agents “deviate” (e.g., they migrate to Manufacturing when their equilibrium action is to remain in Agriculture), then other agents in would want to follow the deviation. The interior equilibrium is therefore unstable.

“more likely” in both senses described in the Introduction. This conclusion is independent of the particular measure used to assess the likelihood of multiplicity; that is, it is independent of the priors on the initial condition and on  $b$  and  $\beta$ . For example, an increase in  $\gamma$  can cause initial conditions to enter the ROM but never cause initial conditions to leave the ROM. Therefore, there is no loss in generality in using the length of the ROM to measure the likelihood of multiplicity. This measure corresponds to a uniform prior over initial conditions.

## 4 Including an environmental stock

We show how the inclusion of an environmental stock changes both Krugman’s model and our simpler version of that model. The presence of a second state variable means that labor migration has a delayed effect on the wage differential. Therefore, the simple version of the model requires two decision periods, rather than a single decision period as above. The continuous time, infinite horizon version of the model is actually more intuitive, so we begin there.

### 4.1 The two-state continuous time model.

As above, the total stock of labor is fixed at 1;  $L_t$  denotes the amount of labor in Manufacturing, and the flow of labor into Manufacturing is  $\dot{L} = u_t$ ; the social cost of migration is  $\frac{u_t^2}{2}$ , so a person who migrates at time  $t$  pays the price  $\frac{|u_t|}{\gamma}$ . Here there are constant returns to scale in both sectors, but a cross-sector negative externality. Manufacturing output at time  $t$  equals  $0.5L_t$ , so the competitive wage in Manufacturing is the constant 0.5. Labor productivity in Agriculture depends on an environmental stock  $E_t$ , a renewable resource; Agricultural output is  $E_t(1 - L_t)$ , so the competitive wage in Agriculture is  $E_t$ . The Manufacturing-Agricultural wage differential in period  $t$  is  $0.5 - E_t$ .

Each unit of Manufacturing output creates one unit of pollution, which reduces the future environmental stock. The change in the environmental stock is

$$\dot{E} = g(1 - E_t - L_t). \quad (2)$$

In addition to the assumed linearity, this equation imposes two parameter restrictions. We assume that one unit of Manufacturing labor creates one unit of pollution<sup>4</sup>, and we set the

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<sup>4</sup>This assumption represents a genuine restriction rather than merely a choice of units. We chose units of labor

steady state in the absence of pollution at  $E = 1$ . These restrictions merely simplify the exposition. The recovery rate for the environment is  $g > 0$ .

Again, there are two stable steady states. If all labor is in Manufacturing, the environmental steady state is  $E = 0$  and the Manufacturing-Agriculture wage differential is 0.5, so no worker wants to leave the Manufacturing sector. If all labor is in Agriculture, the environmental steady state is  $E = 1$  and the wage differential is  $-0.5$ , so no worker wants to leave Agriculture.

#### 4.1.1 A different interpretation

A reinterpretation of variables leads to a model of learning-by-doing in Manufacturing. Define  $K = 1 - E$  as the stock of knowledge in Manufacturing. Equation (2) implies  $\dot{K} = g(L - K)$ . Knowledge decays at a constant rate  $g$ ; if the Manufacturing sector were to close down, it would not be possible to restart it at the previous level of efficiency because machines and skills get rusty. More activity in the Manufacturing sector (larger  $L$ ) increases knowledge. At a point in time there are constant returns to scale in both sectors; the Manufacturing wage is  $K_t$  and the Agricultural wage is the constant 0.5, so the wage differential is  $K_t - 0.5$ .

This variation reproduces Krugman's model with one important difference. The increasing returns to scale in Manufacturing depend on experience, not on the current size of the Manufacturing labor force. It takes time for learning to be incorporated into greater productivity, and a higher wage. This greater realism comes at the cost of greater complexity. This greater complexity fundamentally changes the insight produced by the one-state variable model.

## 4.2 The two-stage model

Now we present our simplification of this two-state variable model; we revert to our first interpretation, where  $E$  is an environmental stock rather than a stock of knowledge. As above, the wage in Manufacturing is the constant 0.5 and the wage in Agriculture is  $E_t$ . In period  $t + 1$  the environmental stock is

$$E_{t+1} = E_t + G(1 - E_t - L_t), \quad (3)$$

the discrete time analog of equation (2).

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by the normalization that the total stock of labor equals 1, and we chose units of Manufacturing output in writing the production function in the sector as  $0.5L$ .

We will use a model in which migration decisions are made at only two points in time, at  $t = 0$  and at  $t = 1$ , and agents care about wages in only periods  $t = 1$  and  $t = 2$  and migration costs in periods  $t = 0$  and  $t = 1$ . Wages in period  $t = 0$  are predetermined, and therefore have no effect on agents' decisions. At the beginning of period  $t$ ,  $L_t$  and  $E_t$  are predetermined. As in the one-period model, current migration affects the stock of labor in the next period. *Consequently,  $E_{t+1}$  and the wage differential at  $t + 1$  are predetermined at time  $t$ .*

In the initial period, ( $t = 0$ ) the state variables  $L_0, E_0$  are given. Denote the amount of migration to Manufacturing in period 0 as  $u$  (as in the one-period model) and the amount of migration to Manufacturing in period 1 as  $v$ . Negative values mean that migration is into Agriculture. As before, the social cost of migration is quadratic in migration, and the price that an agent (who migrates) pays is  $\frac{|u|}{\gamma}$  in period 0 and  $\frac{|v|}{\gamma}$  in period 1.

If labor is allocated evenly between sectors and the environment is in a steady state, then  $L = 0.5$  and  $E = 0.5$ . In order to simplify notation, we define the state variables as deviations from these values:  $l_t \equiv L_t - 0.5$  and  $e_t \equiv E_t - 0.5$ . By construction,  $e_t$  equals the *negative* of the Manufacturing-Agricultural wage differential in period  $t$ . The state space for the model is the square

$$-0.5 \leq l_t \leq 0.5, \quad -0.5 \leq e_t \leq 0.5, \quad (4)$$

and the equation of motion for the transformed environmental variable (which equals the negative of the wage differential) is

$$e_{t+1} = e_t - G(l_t + e_t). \quad (5)$$

#### 4.2.1 A parameter restriction

In the two-period setting, it might be argued that there is no reason to restrict the magnitude of the parameter  $G$ . However, we need one more parameter restriction in order to present our conclusions in a simple manner. It is reasonable to obtain this restriction by considering the relation between the two-period model and the infinite horizon model.

Equations (2) and (3) both involve a single parameter, either  $g$  or  $G$ . Suppose that we hold  $L_t$  in equation (2) constant for one unit of time and solve that equation, to rewrite it in the same form as equation (3). Comparison of this solution and equation (3) shows that the relation between the two parameters is  $G = 1 - e^{-g}$ . As  $g \rightarrow \infty$  the environmental stock adjusts instantaneously. Rapid adjustment of the environmental stock appears to lead to a model that

is “similar” to the one-state variable model. Rapid adjustment of the environmental stock is an obvious justification for using the one-state model as an approximation. Therefore, this case is of special interest.

Instantaneous adjustment in the continuous time model ( $g = \infty$ ) corresponds, in the discrete time model, to complete adjustment within a single period ( $G = 1$ ). This observation (and the assumption that  $g > 0$ ) suggests that the restriction  $0 < G \leq 1$  is reasonable for our two-period model. In fact, the special case where  $G = 1$  gives particularly sharp results, and we emphasize it below.

However, it might be argued that  $G \leq 1$  is too strong a restriction. This restriction implies that – given a constant trajectory of  $\{L_t\}$  – the solution to equation (3) is monotonic in time. Stability of equation (3) requires only  $G < 2$ , so that might seem a more natural bound. However, this restriction does not permit a simple demonstration of our results. We therefore impose the following parameter restriction

$$G < 1 + \frac{\sqrt{\gamma\beta + 1}}{\sqrt{\gamma\beta}}. \quad (6)$$

This restriction is stronger than stability and weaker than monotonicity in an infinite horizon discrete time model. It includes the case of special interest,  $G = 1$  (complete adjustment within a period), and it eliminates complications that only distract from our message.

## 5 Results

We first present the analysis of the model with two decision periods. We noted that in the one-state model from Section 3.2, the assumption of a uniform prior over initial conditions entails no loss of generality in assessing the likelihood of multiplicity. Matters are slightly more complicated in the two-state model here. For clarity, we begin by assuming that the prior on the initial condition of the two-dimensional state is uniform. This assumption, together with our previous normalization of the state space, means that we can measure the likelihood of multiplicity by calculating the area of the ROM. In Section 5.1.1 we show that our main result does not depend on the assumption of uniform priors. Section 5.1.2 summarizes the chief difference between the one-state and two-state models. We then informally discuss the infinite horizon, continuous time model.

## 5.1 The two-stage model

Our principal results use the following definitions:

$$\begin{aligned} X(e_0) &= \frac{\phi}{\chi} e_0 - 0.5 \frac{\gamma \beta^2 G - 1}{\chi} \\ Y(e_0) &= \frac{\phi}{\chi} e_0 + 0.5 \frac{\gamma \beta^2 G - 1}{\chi} \end{aligned} \quad (7)$$

where

$$\begin{aligned} \chi &= \gamma \beta G (\gamma G \beta + 3 + \beta - \beta G - G) + 1 \\ \phi &= \gamma \beta (\gamma \beta G + 2.0 - 1.0 \gamma \beta G^2 - 3.0 G + G^2 - 2.0 \beta G + \beta + \beta G^2). \end{aligned}$$

The following Proposition summarizes our main results; the Appendix contains the proof.

**Proposition 1** *Suppose that inequality (6) holds in our two-period model. (i)  $\beta^2 G \gamma > 1$  is necessary and sufficient for the ROM to have positive measure. Thus, an increase in  $\gamma$  (i.e., a decrease in friction) increases the range of other parameter values ( $\beta$  and  $G$ ) for which the ROM has positive measure. (ii) The ROM is defined by the following set.*

$$ROM = \{(e, l) : -.5 \leq e \leq .5 \cap -.5 \leq l \leq .5 \cap X(e) \leq l \leq Y(e)\}.$$

(iii) For  $\beta^2 G \gamma > 1$ , the area of the ROM is non-monotonic in  $\gamma$ .

The condition for the ROM to have positive measure,  $\beta^2 G \gamma > 1$ , is essentially the same as for the one-period model (with  $G$  playing a role equivalent to  $b$ ) except that the condition involves  $\beta^2$  rather than  $\beta$ . This difference is due to the fact that migration in period  $t$  affects the wage differential in period  $t+2$  rather than in period  $t+1$ , as was the case in the one-period model.

Figure 2 shows an example of the ROM. The lower line is the graph of  $X$  and the upper line is the graph of  $Y$ , defined in equation (7). The ROM is symmetric around the origin. Restriction (6) and  $\beta^2 G \gamma > 1$  imply that  $\chi > 0$ . (Details available on request.) This fact, equation (7), and the assumption that  $\gamma \beta^2 G > 1$  imply that  $X < Y$ . For any  $e_0$ , the vertical distance between  $X$  and  $Y$  is

$$M \equiv Y - X = \frac{\gamma \beta^2 G - 1}{\chi} > 0. \quad (8)$$

In general, we do not have a simple closed form expression for the measure of the ROM. If  $Y(.5) < .5$  or  $X(.5) > .5$ , then the ROM does not include the NE or SW corners of state space, and the ROM is a parallelogram. If  $Y(.5) < .5$ , the measure of the ROM is simply  $M$ .<sup>5</sup>

<sup>5</sup>If  $X(.5) > .5$  the measure of the ROM is the horizontal rather than the vertical distance between the lines

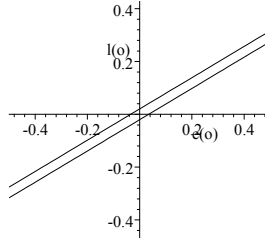


Figure 2: The ROM for  $\beta = .8$ ,  $G = .5$  and  $\gamma = 2.5$

### 5.1.1 Uniform priors

We used the Lebesgue measure, or area, of the ROM to assess the “likelihood” of multiplicity. The problem with this measure is that a change in the magnitude of  $\gamma$  causes the ROM (graphed in Figure 2) to rotate, while also changing the area that it contains.

For example, suppose that an increase in  $\gamma$  causes the area of the ROM to increase, but at the same time causes an open set  $\Sigma$  of state space to leave the ROM. The Lebesgue measure gives equal weight to all points in state space, so it corresponds to a uniform prior over initial conditions. However, we might think that it is particularly likely that the initial condition lies in  $\Sigma$ , in which case a uniform prior is not appropriate. Using a probability measure that reflects our beliefs about the importance of  $\Sigma$ , the increase in  $\gamma$  in our example might decrease the measure of the ROM, even though the area of the ROM increases. This example suggests that our non-monotonicity result might be sensitive to the choice of measure.

The case  $G = 1$  demonstrates that this concern is unwarranted. When  $G = 1$ , our non-monotonicity result is independent of the choice of probability measure. For  $G = 1$  the ROM is flat;  $\phi = 0$ , so the boundaries of the ROM are independent of  $e_0$ . When  $G = 1$ , the current environmental stock has no effect on future wage differentials; therefore the ROM is independent of the current environmental stock. A change in  $\gamma$  or  $\beta$  alters the distance between the boundaries of the ROM, but not their slope. A parameter change that makes the distance between the boundaries larger unambiguously makes the set larger. No points leave the set, and some points enter the set, as the distance between the boundaries increases. Therefore, for  $G = 1$ , the measure of the ROM is non-monotonic with respect to  $\gamma$  regardless of the measure

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$X$  and  $Y$ . If the ROM includes the corner of state space (i.e., if  $Y(.5) > .5 > X(.5)$ ) then in computing its area we need to account for the “missing triangles” at the corners, and the formula for the measure becomes more complicated. It is easy to confirm any of these three configurations are possible, depending on parameter values.

that is used, and the turning point is the same.

### 5.1.2 Comparison of one-state and two-state models

We noted that the presence of a second state variable (the environmental stock or the stock of knowledge) changes the relation between the amount of friction in labor adjustment and the likelihood of multiplicity of equilibria, as reflected by the measure of the ROM. There are two reasons why the special case  $G = 1$  is particularly useful for illustrating this difference. First, as Section 5.1.1 shows, in this situation there is no loss in generality in using a uniform prior for initial conditions (equivalently, the Lebesgue measure of the ROM). Second, the case  $G = 1$  is of special interest because it leads to a model that appears to approximate the one-state variable model. Recall that  $G = 1$  corresponds to complete adjustment of the environmental stock, following changes in  $L$ , within a single period. That is, when  $G = 1$  the second state variable adjusts very rapidly – precisely the situation where we might expect that a one-dimensional model provides a good approximation to the two-dimensional model.

When  $G = 1$  the Lebesgue measure of the ROM is  $\max\{0, M\}$ , with  $M = \frac{\gamma\beta^2 - 1}{\gamma^2\beta^2 - 1}$ ;  $\gamma > \beta^{-2}$  is necessary and sufficient for the ROM to have positive measure when  $G = 1$ . For  $\beta = 1$ ,  $M = \frac{1}{\gamma+1}$  which is strictly *decreasing* in  $\gamma$ . For  $\beta < 1$ ,  $M$  is first increasing in  $\gamma$  (in the neighborhood  $\beta^{-2}$ ) and then decreasing. The measure reaches its maximum at  $\gamma^m \equiv \frac{1}{2\beta^2} (2 + 2\sqrt{1 - \beta^2})$  and thereafter decreases. The maximum point  $\gamma^m$  converges to  $\beta^{-2}$  as  $\beta \rightarrow 1$ . Figure 3 shows

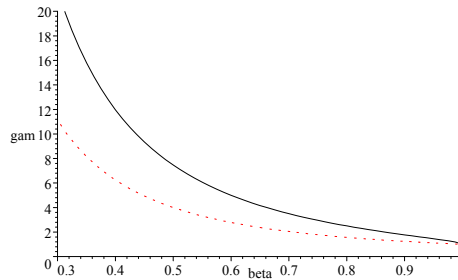


Figure 3: For  $G = 1$ , measure of ROM is 0 below the dotted curve, increasing in  $\gamma$  between curves, and decreasing in  $\gamma$  above solid curve.

the graph of  $\gamma^m$  (the solid curve) and of  $\beta^{-2}$  (the dotted curve). For values of  $\gamma$  below the dotted curve, the measure of the ROM is 0; for values between the two curves, the measure of

the ROM is increasing in  $\gamma$ ; and for values of  $\gamma$  above the solid curve, the measure is decreasing in  $\gamma$ .

The discussion of Proposition 1 noted that  $b$  and  $G$  play analogous roles in determining the existence of a ROM with positive measure, for the models with one state or two state variables. Thus, when  $G = 1$ , a comparison between the two models requires setting  $b = 1$ . In addition, we explained that in writing equation (2) we implicitly imposed parameter restrictions; these restrictions imply that the unstable steady state is at the center of state space, the unit square. In the one-state model, the unstable steady state equals  $\frac{1}{2}$  if and only if  $a = -\frac{b}{2}$ . Finally, set  $\beta = 1$ , so that the necessary and sufficient condition for a positive measure of the ROM is  $\gamma > 1$  in both models. With these restrictions, Figure 4 graphs the measure of the ROM in the two models, for  $\gamma > 1$ . This figure uses equations (1) and (8). The figure shows that  $\gamma$  has the opposite effect on the measure of the ROM in the two models.

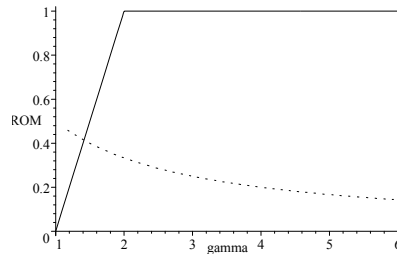


Figure 4: Measure of ROM,  $\beta = G = b = 1$ . One-state model: solid curve. Two-state model: dotted curve

## 5.2 The continuous time, infinite horizon model

We began our discussion of the two-state variable model by describing the continuous time, infinite horizon setting, because the link between that and Krugman's model is very clear. Karp and Paul (2003) analyze the continuous time model and reach the same qualitative conclusions as reported in the previous subsection. This discovery is not particularly surprising, in view of the fact that the one-period version of Krugman's model has the same qualitative properties as the infinite horizon version of that model. Unfortunately, analysis of the continuous time model requires the use of numerical methods. Since the two-stage model can be solved analytically and moreover it provides all of the insight of the continuous time model, we relegate the details

of the latter to a separate paper.

It is worth mentioning the continuous time model so that the reader does not have the impression that the non-monotonicity of the ROM is an artifice of our timing convention in the discrete time model: the assumption that the current migration decision affects the location of an agent in the next period. In the discrete stage model the wage differential in the next period is predetermined in the current period. In this case, it is obvious that there can never be indeterminacy when the cost of adjusting the mobile factor is 0, i.e. if  $\gamma = \infty$ . Here agents have a dominant strategy, to move to the sector that will have the high wage in the next period. In the continuous time model, both stocks change smoothly. In that setting, the non-monotonicity of the ROM is not due to the fact that one stock adjusts “before” a second stock.

The continuous time infinite horizon model is also useful for discussing the issue raised in Section 5.1.1. For both the continuous and the discrete models we find that the ROM (when it exists) becomes flat (in  $(e, l)$  state space) as the speed of adjustment of the environment becomes large ( $G \rightarrow 1$  or  $g \rightarrow \infty$ ). In this case, for all initial values of the environmental stock, there exists an interval of initial values of labor allocation for which there are multiple equilibria. As costs of adjustment become small, this interval becomes small, and the labor stock moves toward a steady state (and therefore away from the ROM) quickly. Suppose that we interrupt the process at an arbitrary point in time, and regard the current state variable as an initial condition for a new equilibrium problem. Since the state moves quickly away from the ROM, it is “likely” that this arbitrary point is outside the ROM. In other words, as the ROM shrinks, it is *not* the case that it shrinks to an area where the state variable spends a lot of time.

Appendix A.2 sketches a non-linear two-state variable model. There we show, under mild assumptions, the generality of our result concerning the non-monotonic relation between the friction of the mobile factor and the measure of the ROM.

## 6 Conclusion

Multiplicity of equilibria can arise in circumstances where agents play a coordination game. This possibility is easy to understand in a static model. Dynamic extensions of these models are more descriptive and provide a check on the robustness of the static models. There is typically more than one plausible source of dynamics, i.e., more than a single state variable that does not adjust instantaneously. Higher dimensional models are difficult to analyze, so

a common practice is to focus on a single state variable. This paper has shown why this simplification may produce misleading results.

Previous dynamic migration models introduce friction in the adjustment of the mobile factor. A key insight is that the multiplicity of equilibria in the underlying static model carries over to the dynamic setting, if the amount of friction is small. However, an increase in friction makes multiplicity less likely. This conclusion is (by now) so well-established that it seems obvious. After all, if it is harder for an agent to take an action, such as moving to a new sector, it seems that there would be fewer circumstances under which her decision would depend on beliefs about what other agents will do.

We have shown that this conclusion can be reversed in a model that is only slightly more general. We used an example in which the externality (that gives rise to the coordination problem) is mediated through a second stock variable. The second stock variable can be interpreted as a stock of knowledge that increases with production and decays in the absence of production, or an environmental stock that affects productivity in another sector. Here, a decrease in the cost of changing sectors – i.e., a decrease in friction – increases the range of parameter values for which multiplicity is generic. In that sense, the model provides no new insight. However, we also showed that the measure of the ROM is non-monotonic in adjustment costs. This measure is 0 if adjustment costs are very high and positive for lower adjustment costs. As the adjustment costs approach 0, the measure of the ROM also approaches 0. This result means that for very low adjustment costs, multiplicity is generic, but occurs on a set of measure approximately 0.

The reason that less friction implies a higher measure of the ROM is familiar from previous papers. However, it is worth repeating the explanation for why this relation can be reversed in the two-state variable model, as the amount of friction becomes small: If the cost of adjustment is very small, then for most initial states the agent's migration decision does not depend on what others do; whatever their actions, it is cheap for an agent to move in future periods in order to remain in the high wage sector.

Our result are empirically important because the distinction between economies with multiple steady states, and economies with multiple equilibria is an important one. The effects of policy may be qualitatively different in the two types of economies. Previous dynamic models suggest that the introduction of dynamics makes less likely, but does not eliminate, the type of coordination problems revealed by static models. Our results go a step further, by showing that

coordination problems may be very unlikely to arise in exactly the circumstance where previous results suggested that coordination problem might be severe.

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## A Appendix:

The Appendix contains the proof of Proposition 1 and the sketch of a general model in which the measure of the ROM is non-monotonic in the amount of friction.

### A.1 Proof of Proposition 1

We construct the equilibrium by working backwards, beginning with the agents' problem in period 1 (the last period during which they can migrate).

Using equation (5) and  $l_1 = l_0 + u$ , we write the present value at  $t = 1$  of being in Manufacturing in period 2, as

$$-\beta e_2 = -\beta (e_1 - G(l_1 + e_1)) = -\beta (-G(2 - G)l_0 + (G - 1)^2 e_0 - Gu) \equiv f(u; e_0, l_0).$$

Our timing conventions imply that this value is predetermined at period 1. The equilibrium for the subgame beginning in period 1 is therefore unique. Agents are indifferent between migrating and staying in their current sector if and only if  $-\beta e_2 = \frac{v}{\gamma}$ , i.e. if  $v = \gamma f(u; l_0, e_0)$ . The speed of adjustment parameter affects the magnitude but not the sign of the quantity  $\gamma f(\cdot)$ , and  $f(\cdot)$  is increasing in  $u$  for all  $G > 0$ .

Taking into account the labor supply constraint, the equilibrium value of  $v$  is

$$v(u) = \left\{ \begin{array}{ll} 0.5 - l_0 - u & \text{if } \gamma f > 0.5 - l_0 - u \\ \gamma f & \text{if } -0.5 - l_0 - u \leq \gamma f \leq 0.5 - l_0 - u \\ -0.5 - l_0 - u & \text{if } \gamma f < -0.5 - l_0 - u \end{array} \right\}. \quad (9)$$

Figure 6 shows an example of the graph of  $v(u)$ , given particular values  $l_0 = 0$  and  $e_0 > 0$ .

The  $u$  coordinate of the left and the right kink in this graph are, respectively

$$\begin{aligned} \text{left kink:} \quad p &\equiv \rho(l_0, e_0) - \frac{.5}{1 + \gamma\beta G} \\ \text{right kink:} \quad q &\equiv \rho(l_0, e_0) + \frac{.5}{1 + \gamma\beta G}, \end{aligned}$$

using the definition

$$\rho(l_0, e_0) \equiv \frac{1}{1 + \gamma\beta G} ((\gamma\beta G(G - 2) - 1)l_0 + \gamma\beta(G - 1)^2 e_0).$$

For all  $l_0$  and  $e_0$ , it is always the case that  $p < q$ . Inequality (6) implies that  $\rho$  is a decreasing function of  $l_0$ , so  $p$  and  $q$  are decreasing functions of  $l_0$  – a fact that we use below.

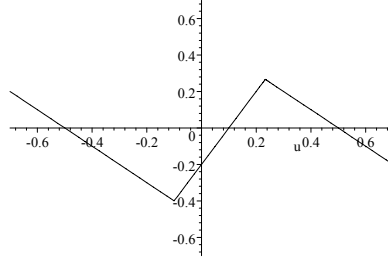


Figure 5: Equilibrium  $v$  as a function of  $u$

Using these definitions and equation (9) implies

$$\frac{dv(u)}{du} = \left\{ \begin{array}{ll} -1 & \text{if } u > q \\ \beta G \gamma & \text{if } p < u < q \\ -1 & \text{if } u < p \end{array} \right\}. \quad (10)$$

Thus, an increase in  $u$  increases the equilibrium  $v$ , provided that  $v$  is interior. In contrast, an increase in  $u$  decreases the equilibrium  $v$  when this variable is on the boundary of the labor supply constraint, as Figure 6 illustrates.

In period 1 an agent is either indifferent between migrating and staying in her current sector (at an interior equilibrium) or she strictly prefers to migrate (at a boundary equilibrium). Agents with rational expectations understand this fact in period 0. Therefore, the benefit of migrating to Manufacturing in period 0 is the present value of the wage differential in period 1 ( $-\beta e_1$ ), plus the present value of migration costs in period 1 ( $\beta \frac{v(u)}{\gamma}$ ).<sup>6</sup> The present value of migrating to Manufacturing in period 0 is therefore

$$\beta \left( -e_1 + \frac{v(u)}{\gamma} \right) = \beta \left( -(1-G)e_0 + Gl_0 + \frac{v(u)}{\gamma} \right).$$

If the value of this expression is negative, its absolute value is the value of migrating to Agriculture. For  $u > 0$  the cost of moving to Manufacturing in period 0 is  $\frac{u}{\gamma}$ ; for  $u < 0$ , the cost of moving to Agriculture is  $\frac{-u}{\gamma}$ .

Define the difference between benefits and costs of moving to Manufacturing in the first period as

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<sup>6</sup>The agent who migrates in period 0 avoids paying the period 1 migration costs. If migration in period 1 is at an interior level, period 1 migration costs equal the present value of the wage differential in period 2.

$$h(u; l_0, e_0) \equiv \beta \left( -(1 - G)e_0 + Gl_0 + \frac{v(u)}{\gamma} \right) - \frac{u}{\gamma}. \quad (11)$$

(Again, if  $h < 0$ , then  $-h$  is the value of moving to Agriculture.) Using equation (10), we have

$$\frac{dh}{du} = \left\{ \begin{array}{ll} \frac{-\beta-1}{\gamma} & \text{if } u > q \\ \frac{\beta^2 G \gamma - 1}{\gamma} & \text{if } p < u < q \\ \frac{-\beta-1}{\gamma} & \text{if } u < p \end{array} \right\}. \quad (12)$$

Period 0 actions are always strategic substitutes for  $u < p$  and for  $u > q$ . For  $q < u < p$  period 0 actions are strategic complements if and only if  $\beta^2 G \gamma > 1$ . When actions are strategic substitutes (for all values of the state variable) the equilibrium is generically unique;  $\beta^2 G \gamma > 1$  is therefore necessary for the ROM to have positive measure, as Part (i) of the Proposition states.

Since we are interested in the measure of the ROM as a function of  $\gamma$ , we hereafter assume that  $\beta^2 G \gamma > 1$ . Given this condition, we want to characterize the ROM, i.e. the region of the  $(e, l)$  plane such that if  $(e_0, l_0)$  is in this region, there are multiple equilibria in period 0.

An interior equilibrium requires that  $h = 0$  and a *stable interior* equilibrium requires in addition that  $\frac{dh}{du} < 0$ , evaluated at the equilibrium. (See footnote 5.) Since we are interested only in stable equilibria, equation (12) means that we can rule out the possibility of interior equilibria where  $p < u < q$ . We are left with three possibilities: (i) The equilibrium is interior with  $0.5 - l_0 > u > q$ , (ii) The equilibrium is interior with  $-0.5 - l_0 < u < p$ , and (iii) The equilibrium is on the boundary, i.e.  $u = -0.5 - l_0$  or  $u = 0.5 - l_0$ .

In order to construct the equilibrium, we determine the values of  $u$  for which  $h(u; l_0, e_0) = 0$  at a stable equilibrium. We first consider the case where  $u \geq q$ ; here, by equation (9),  $v = 0.5 - l_0 - u$ . We substitute  $v = 0.5 - l_0 - u$  into the function  $h(\cdot)$  defined in equation (11), and solve  $h(\cdot) = 0$  to obtain an expression for  $u$  as a function of  $l_0, e_0$ . Denote this function as  $x(l_0, e_0)$ . Next, we consider the case  $u \leq p$ , where  $v = -0.5 - l_0 - u$ . We use this relation in the equation  $h(\cdot) = 0$  and solve for  $u$  to obtain a function that we denote as  $y(l_0, e_0)$ . These functions  $x(\cdot)$  and  $y(\cdot)$  are

$$\begin{aligned} x(l_0, e_0) &\equiv \alpha + \frac{0.5\beta}{\beta+1} \\ y(l_0, e_0) &\equiv \alpha - \frac{0.5\beta}{\beta+1} \end{aligned}, \quad (13)$$

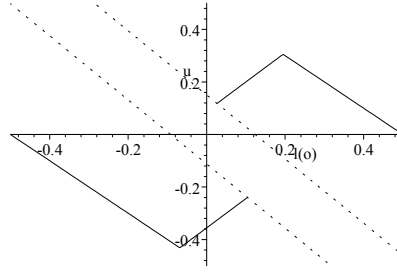


Figure 6: The equilibrium correspondence as a function of  $l_0$ .

using the definition

$$\alpha \equiv \frac{\beta}{1 + \beta} ((\gamma G - 1) l_0 + \gamma (G - 1) e_0).$$

With this notation, we write the equilibrium correspondence:

$$u(l_0, e_0) = \begin{cases} \min \{x, 0.5 - l_0\} & \text{if } x \geq q \\ \max \{y, -0.5 - l_0\} & \text{if } y \leq p \end{cases}. \quad (14)$$

The first line states that if  $x \geq q$ , then a stable equilibrium is  $u = x$ , provided that this value is less than the upper limit of migration,  $0.5 - l_0$ ; otherwise the labor supply constraint is binding, and all labor moves to Manufacturing. The second line has a similar interpretation. Thus, there are two equilibria if the initial condition satisfies both  $q \leq x$  and  $y \leq p$ . Using previous definitions, these two inequalities can be rewritten as

$$0.5 \frac{1 - \gamma \beta^2 G}{(1.0 + \gamma \beta G)(\beta + 1.0)} \leq \alpha - \rho \leq 0.5 \frac{\gamma \beta^2 G - 1}{(1.0 + \gamma \beta G)(\beta + 1.0)} \quad (15)$$

This inequality defines the ROM.

Figure 7 shows the graph of the equilibrium migration correspondence for  $e_0 = 0.2$ ,  $G = 0.7$ ,  $\beta = 0.8$ , and  $\gamma = 5$ . The top solid curve is the graph of  $\min \{x, .5 - l_0\}$  over the interval where  $x \geq q$ . The kink occurs where  $x = .5 - l_0$ . The top dotted line is the graph of  $q$ . The bottom solid curve and dotted line show the graphs of  $\max \{y, -.5 - l_0\}$  and of  $p$ , respectively. The overlap of the two solid curves defines the ROM, given  $e_0 = 0.2$ . If, for example,  $l_0 = 0.05$ , the two equilibrium values of migration are  $u = 0.144$  (a movement to Manufacturing) and  $u = -0.3$  (a movement to Agriculture).

Our assumptions  $\beta^2 G \gamma > 1$  and  $\beta \leq 1$  imply that  $G \gamma > 1$ , so the slope  $x$  and  $y$  (as functions of  $l_0$ ) are always positive, as shown. We noted above that inequality (6) implies that the slope

of  $p$  and  $q$  (graphed as functions of  $l_0$ ) is negative. Therefore, if  $x \geq p$  is satisfied, it holds for large values of  $l_0$ ; if  $y \leq q$  is satisfied, it holds for small  $l_0$ .

The boundaries of the overlap are determined by the solution to  $x = q$  and  $y = p$ . Denote  $X(e_0)$  as the value of  $l_0$  that satisfies  $x = q$ , and denote  $Y(e_0)$  as the value of  $l_0$  that satisfies  $y = p$ . Some calculation yields the formulae in equation (7) of the text. This step establishes Part (ii) of the Proposition.

The vertical distance between the boundaries of the ROM is  $M$ , defined in equation (8). We noted in the text that inequality (6) and the assumption  $\gamma\beta^2G > 1$  imply that  $\chi > 0$ . Therefore, when these two inequalities hold, the ROM has positive measure. This fact establishes sufficiency in Part (i) of the Proposition. The denominator of  $M$  is quadratic in  $\gamma$  and the numerator is linear, so  $M \rightarrow 0$  as  $\gamma \rightarrow \infty$ . Thus, the measure of the ROM approaches 0 as  $\gamma \rightarrow \infty$ . Since the measure is 0 for  $\gamma\beta^2G < 1$ , positive for  $\gamma\beta^2G > 1$  and approaches 0 as  $\gamma \rightarrow \infty$ , it is nonmonotonic in  $\gamma$ , as Part (iii) of the Proposition states.

## A.2 Sketch of a general model

The fact that, the “likelihood” of multiplicity of equilibria is nonmonotonically related to the friction associated with a mobile factor, is very simple and general. However, demonstrating this point requires a model with two state variables. Unfortunately, it is difficult to obtain analytic results using a two-state rational expectations model; therefore, our results in the text use a simplification of an already simple model. This procedure leads to clear results, but the special model has two disadvantages. First, it may leave the reader with the impression that the conclusions require this sort of special setting, and therefore are not robust. Second, the analysis of the simple model requires some tedious calculation, which obscures intuition. To offset these disadvantages, we sketch here a general model that, under mild assumptions, reproduces the results shown formally for the special model.

In the interests of brevity, we do not describe all of the assumptions that lead to the model presented here, or all of its implications. However, it is worth pointing out that here – unlike Krugman’s model and our continuous time infinite horizon variation of that model – we assume that the steady states are interior and are approached asymptotically.

There are two state variables:  $L_t$  is the fraction of labor in Manufacturing;  $E_t$  is a state variable that depends on the history of labor allocation. The  $t$  subscript denotes time. The Manufacturing-Agriculture wage differential,  $\omega(E)$ , depends on  $E$  rather than on  $L$  as in stan-

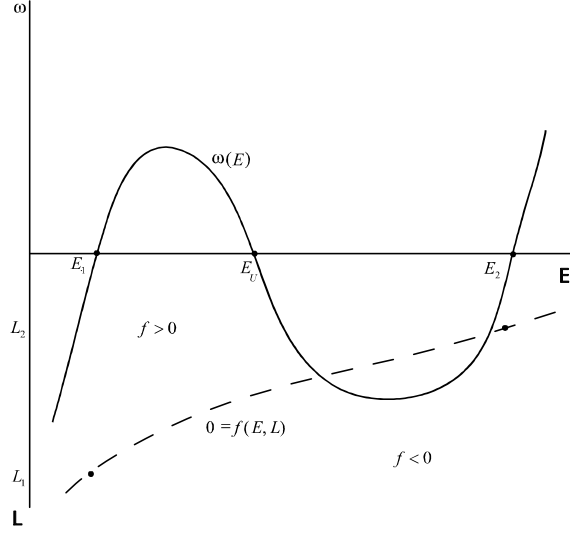


Figure 7: Illustration of general model

standard models. Denote  $\Omega_t$  as the trajectory over  $(t, \infty]$  of  $\omega_\tau \equiv \omega(E_\tau)$  and denote  $\vec{0}_t$  as the trajectory where  $\omega(E_\tau) = 0$  for  $\tau \geq t$ . The dynamics of the state variables are given by

$$\dot{L} = \frac{dL}{dt} = \gamma h(\Omega_t), \text{ with } h(\Omega_t) = 0 \text{ iff } \Omega_t = \vec{0}_t, \quad (16)$$

$$\dot{E} = \frac{dE}{dt} = gf(E, L). \quad (17)$$

Agents' intersectoral migration decisions depend on their beliefs about future wage differentials,  $\Omega_t$ . In a deterministic rational expectations equilibrium, agents' beliefs are correct in equilibrium. The functional  $h(\cdot)$  is determined by the equilibrium condition to agents' problems.<sup>7</sup> The parameter  $\gamma > 0$  is inversely related to the amount of friction (e.g. the costs of migration). The restriction on  $h(\cdot)$  states that migration stops if and only if the the future trajectory of the wage differential is identically 0. The function  $f(\cdot)$  is given exogenously, and  $g > 0$  is a speed of adjustment parameter.

Figure 6 illustrates the situation where  $\omega(E)$  has three roots, and where the implicit function  $\tilde{L}(E)$  given by  $f(E, \tilde{L}(E)) = 0$  is decreasing,  $\tilde{L}'(E) < 0$ ; also  $f > 0$  if and only if  $L < \tilde{L}(E)$ .

<sup>7</sup>For example,  $h(\cdot)$  may be a function of the present discounted value of the future stream of wage differentials, denoted  $q_t$ . Let  $p(\dot{L}_t)$  be the price that an individual pays to migrate at time  $t$ . The equilibrium condition is  $p(\dot{L}_t) = q$  or  $\dot{L}_t = p^{-1}(q) \equiv \gamma h(q)$ . The complete dynamical system of the model consists of equations (16) and (17) and  $\dot{q} = rq_t - \omega_t$ , where  $r$  is the constant discount rate.

The state variable  $E$  depends on the history of labor allocation; by assumption an increase in  $L$  reduces  $E$ , i.e.  $f_L < 0$ .

The non-monotonicity of  $\omega(E)$  has the following interpretation. Large past values of  $L$  are associated with greater Manufacturing output and higher pollution that lowers agricultural productivity; however, larger past values of  $L$  are also associated with less intensive agricultural practices, tending to increase current agricultural productivity. Since these two forces oppose each other,  $\omega$  can be non-monotonic. If most labor had been in Agriculture for most of history, then  $E$  is large; in this case, the soil is nearly exhausted even though there has been little pollution from Manufacturing:  $\omega > 0$ . In the region over which  $\omega'(E) < 0$ , the decrease in agricultural productivity caused by manufacturing pollution exceeds the increased agricultural productivity associated with less intensive farming. The existence of this region leads to the possibility of coordination problems.

There are two stable steady states in this model,  $(E_1, L_1)$  and  $(E_2, L_2)$ , independent of the values of  $\gamma$  and  $g$ . (The requirement that  $\omega = 0$  determines the  $E$  coordinate of a steady state, and the requirement that  $f = 0$  determines the corresponding  $L$  coordinate.) There may or may not be multiple equilibria; that is, the ROM may have positive or 0 measure. In the limit, as  $\gamma \rightarrow \infty$  and  $g \rightarrow \infty$ , we obtain a static model for which the two stable steady states of the dynamic model are stable equilibria. In the static model there is certainly a coordination problem (multiple equilibria). We remarked above on the qualitative difference between policy problems in economies where there are multiple steady states but a unique equilibrium, and economies where there are both multiple steady states and multiple equilibria (for some initial conditions). In view of this difference, it is worth knowing the extent to which the static model might exaggerate the likelihood of multiple equilibria.

We make the following two assumptions:

**Assumption 1** *The equilibrium correspondence (mapping initial conditions and parameter values into trajectories) is continuous in  $\gamma$  and  $g$  for all positive values.*

**Assumption 2** *For  $g = \infty$ , the ROM has positive measure if and only if  $\gamma$  is sufficiently large.*

Assumption 1 implies that the measure of the ROM is continuous in  $\gamma$  and  $g$ . Assumption 2 means that the current model is similar to both Matsuyama (1991) and Krugman (1991), regarding the role of friction of the mobile factor.<sup>8</sup>

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<sup>8</sup>By imposing more structure on the model, e.g. by using the equilibrium condition in the previous footnote,

These assumptions may appear to suggest that the one-state model should provide a good approximation to the two-state model if the omitted state adjusts rapidly. In an important respect, however, the one-state model can be misleading precisely when the omitted state adjusts rapidly. In order to understand why, consider two limiting cases, in each of which the state is one-dimensional.

Case i)  $\gamma = \infty$  and  $g < \infty$ , so that the single state variable is  $E$ . In this case, for  $E \neq E_U$  (the unstable steady state), all labor moves immediately to the high wage sector and the system then moves toward either steady state  $E_1$  or  $E_2$ . The equilibrium is unique; here the measure of the ROM is 0.

Case ii)  $g = \infty$  and  $\gamma < \infty$ , so that the single state variable is  $L$ . By Assumption 2, this case leads to a model with familiar characteristics.

The more interesting case occurs where  $g$  is large but finite and  $\gamma < \infty$ . If  $\gamma$  is large, the ROM has positive measure, by virtue of the two Assumptions. For large  $\gamma$  it is difficult for agents to predict what other agents will do in the future, because migration is cheap ( $\gamma$  is large); this inability is important because the wage differential adjusts quickly to migration ( $g$  is large). Therefore the measure of the ROM is positive. However, as  $\gamma$  approaches  $\infty$ , we move toward Case i, where the measure of the ROM is 0. Given Assumption 1, the measure of the ROM must be decreasing in  $\gamma$  for  $\gamma$  large. For small  $\gamma$ , migration is slow in any equilibrium, so the value of being in a particular sector depends mostly on the predetermined variable  $E$ . For sufficiently small  $\gamma$ , expectations have negligible effect on the equilibrium, so the measure of the ROM is 0. For this model, and for  $g < \infty$  but large, the measure of the ROM is therefore non-monotonic in  $\gamma$ .

It is worth emphasizing that this non-monotonicity arises in the situation where the state variable  $E$  adjusts quickly, precisely the situation where it might seem that little insight is lost by treating it as adjusting instantaneously.

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Assumption 2 can be shown to be an implication rather than an assumption; in the interest of brevity we state it as an assumption.