

## **Estimating a Mixed Strategy: United and American Airlines**

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### **Abstract**

We develop a generalized maximum entropy estimator that can estimate pure and mixed strategies subject to restrictions from game theory. This method avoids distributional assumptions and is consistent and efficient. We demonstrate this method by estimating the mixed strategies of duopolistic airlines.

**KEYWORDS:** Mixed strategies, noncooperative games, oligopoly, maximum entropy, airlines

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## 1. INTRODUCTION

We develop a method for estimating oligopoly strategies subject to restrictions implied by a game-theoretic model. We illustrate our new method by estimating the pricing strategies of American and United Airlines.

Unlike most previous empirical applications, we do not assume that firms use a single pure strategy nor do we make the sort of ad hoc assumptions used in conjectural variations models.<sup>1</sup> Our method allows firms to use either pure or mixed (or distributional) strategies consistent with game theory.

First, we approximate a firm's continuous action space (such as price, quantity, or advertising) with a discrete grid. Then, we estimate the vector of probabilities — the mixed or pure strategies — that a firm chooses an action within each possible interval in the grid. We use these estimated strategies to calculate the Lerner index of market structure.

The main advantage of our method is that it can flexibly estimate firms' strategies subject to restrictions implied by game theory. The restrictions we impose are consistent with a variety of assumptions regarding the information that firms have when making their decisions. Firms may use different pure or mixed strategies in each state of nature. Firms may have private or common knowledge about the state of nature, which is unobserved by the econometrician. For example, a firm may observe a random variable that affects its marginal profit and know the distribution (but not the realization) of the random variable that affects its rival's marginal profit. Each firm may choose a pure strategy in every state of nature and

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<sup>1</sup> Breshnahan (1989) and Perloff (1992) survey conjectural variations and other structural and reduced-form "new empirical industrial organization" studies.

regard its rival's action as a random variable. Alternatively, there may be no exogenous randomness, but the firm uses a mixed strategy. To the econometrician, who does not observe the firm's information or state of nature, the distribution of actions looks like the outcome of a mixed strategy in either case. The econometrician is not able to determine the true information structure of the game. Nevertheless, the equilibrium conditions for a variety of games have the same form, and by imposing these conditions we can estimate strategies that are consistent with theory.

There have been few previous studies that estimated mixed or pure strategies based on a game-theoretic model. These studies (Bjorn and Vuong 1985, Bresnahan and Reiss 1991, and Kooreman 1994) involve discrete action spaces. For example, Bjorn and Vuong and Kooreman estimate mixed strategies in a game involving spouses' joint labor market participation decisions using a maximum likelihood (ML) technique. Our approach differs from these studies in three important ways. First, they assume that there is no exogenous uncertainty. Second, in order to use a ML approach, they assume a specific error distribution and likelihood function. Third, they allow each agent a choice of only two possible actions. Despite this limited number of actions, their ML estimation problems are complex.

Our problem requires that we include a large number of possible actions so as to analyze oligopoly behavior and allow for mixed strategies. To do so using a ML approach would be extremely difficult. Instead, we use a generalized-maximum-entropy (GME) estimator. An important advantage of our GME estimator is its computational simplicity. With it, we can estimate a model with a large number of possible actions while imposing inequality and equality restrictions implied by the equilibrium conditions of the game. In

addition to this practical advantage, the GME estimator does not require strong, arbitrary distributional assumptions, unlike ML estimators.

In the next section, we present a game-theoretic model of firms' behavior. In the third section, we describe a GME approach to estimating this game. The fourth section contains estimates of the strategies of United and American Airlines, and sampling experiments that illustrate the small sample properties of our GME estimator. In the final section, we discuss our results and possible extensions.

## 2. OLIGOPOLY GAME

Our objective is to estimate the strategies of oligopolistic firms using time-series data on prices, quantities, and, when available, variables that condition the demand or cost relations. We assume that two firms,  $i$  and  $j$ , play a static game in each period of the sample. (The generalization to several firms is straightforward.)

Firm  $i$  (and possibly Firm  $j$ ), but not the econometrician, observes the random variable  $\varepsilon^i(t)$  in period  $t$ . For notational simplicity, we suppress the time variable  $t$ . The set of  $K$  possible realizations,  $\{\varepsilon_1, \varepsilon_2, \dots, \varepsilon_K\}$ , is the same every period and for both firms. This assumption does not lead to a loss of generality because the distribution may be different for the two firms. The firms — but not the econometrician — know the distributions of  $\varepsilon_k$ . We consider three possible stochastic structures: (1) Firms face no exogenous randomness ( $K = 1$ ); (2)  $\varepsilon_k$  is private information for Firm  $i$ ; (3)  $\varepsilon_k$  is common-knowledge for the firms. Because the econometrician does not observe  $\varepsilon_k$ , even if the firms use a pure strategy in each period, it appears to the econometrician that they are using a mixed strategy whenever their actions vary over time.

## 2.1 Strategies

The set of  $n$  possible actions for either firm is  $\{x_1, x_2, \dots, x_n\}$ . The assumption that the action space is the same for both firms entails no loss of generality because the profit functions can be specified so that certain actions are never chosen. The notation  $x_s^i$  means that Firm  $i$  chooses action  $x_s$ . We now describe the problem where the random state of nature is private information and then discuss alternative assumptions of a single state of nature or common information.

In determining its own strategy, Firm  $i$  forms a prior,  $\beta_{sk}^i$ , about the probability that Firm  $j$  will pick action  $x_s^j$  when  $i$  observes  $\epsilon_k^i$ . If the firms' private information is correlated, it is reasonable for Firm  $i$  to base its beliefs about  $j$ 's actions on  $\epsilon_k^i$ . If the private information is uncorrelated, Firm  $i$  form priors that are independent of  $\epsilon_k^i$ . We do not, however, assume independence. In state  $k$ , Firm  $i$ 's strategy is  $\underline{\alpha}_k = (\alpha_{k1}^i, \alpha_{k2}^i, \dots, \alpha_{kn}^i)$ , where  $\alpha_{ks}^i$  is the probability that Firm  $i$  chooses action  $x_s^i$ . If Firm  $i$  uses a pure strategy,  $\alpha_{ks}^i$  is one for a particular  $s$  and zero otherwise.

The profit of Firm  $i$  is  $\pi_{rsk}^i = \pi^i(x_r^j, x_s^i, \epsilon_k^i)$ , where  $r$  indexes the strategies of Firm  $j$  and  $s$  indexes the actions of Firm  $i$ . In state  $k$ , Firm  $i$  chooses  $\underline{\alpha}_k$  to maximize expected profits,  $\sum_r \beta_{rk}^i \pi_{rsk}^i$ , where the expectation is taken over the rival's actions. If  $Y_k^i$  is Firm  $i$ 's maximum expected profits when  $\epsilon_k^i$  occurs, then  $L_{sk}^i \equiv \sum_r \beta_{rk}^i \pi_{rsk}^i - Y_k^i$  is Firm  $i$ 's expected loss of using action  $x_s^i$  in  $k$ . Because  $Y_k^i$  is the maximum possible expected profit, the expected loss when Firm  $i$  uses action  $s$  must be nonpositive,

$$(2.1) \quad L_{sk}^i \leq 0.$$

For  $\underline{\alpha}_k$  to be optimal, the product of the expected loss and the corresponding probability must equal zero:

$$(2.2) \quad L_{sk}^i \alpha_{sk}^i = 0.$$

Equation 2.2 says that there is a positive probability that Firm  $i$  will use action  $s$  only if the expected profits when action  $s$  is used are equal to the maximum expected profit.

This problem may have more than one pure or mixed strategy. Our estimation method selects a particular pure or mixed strategy consistent with these restrictions and the data.

## 2.2 Econometric Implications

Our objective is to estimate the firms' strategies subject to the constraints implied by optimization, Equations 2.1 and 2.2. We cannot use these constraints directly, however, because they involve the unobserved random variables  $\varepsilon_k^i$ . By taking expectations, we eliminate these unobserved variables and obtain usable restrictions.

Using the expectations operator  $E_k$ , we define  $\beta_r^i \equiv E_k \beta_{rk}$ ,  $Y^i \equiv E_k Y_k^i$ ,  $\alpha_s^i \equiv E_k \alpha_{sk}^i$ ,  $\pi_{rs}^i \equiv E_k \pi_{rsk}^i$ , and  $E_k L_{sk}^i \equiv L_s^i$ . If we define  $\theta_{sk}^i \equiv L_{sk}^i - (\sum_r \beta_r^i \pi_{rs}^i - Y^i)$  and take expectations, then  $E_k \theta_{sk}^i = \sum_r \text{cov}(\beta_{rk}^i, \pi_{rsk}^i) \equiv \theta_s^i$ . Thus,  $L_s^i \equiv E_k L_{sk}^i = \sum_r \beta_r^i \pi_{rs}^i - Y^i + \theta_s^i$ . Taking expectations with respect to  $k$  of Equation 2.1, we obtain

$$(2.3) \quad \sum_r \beta_r^i \pi_{rs}^i - Y^i + \theta_s^i \leq 0.$$

Taking expectations with respect to  $k$  of Equation 2.2, we find that

$$(2.4) \quad \left( \sum_r \beta_r^i \pi_{rs}^i - Y^i \right) \alpha_s^i + \delta_s^i = 0,$$

where  $\delta_s^i \equiv \theta_s^i \alpha_s^i + \text{cov}(\theta_{sk}^i, \alpha_{sk}^i)$ . We can estimate the observable (unconditional) strategy vectors  $\underline{\alpha}^i$ ,  $i = 1, 2$ , subject to the conditions implied by Firm  $i$ 's optimization problem, Equations 2.3 and 2.4.

For the general case of private information, we cannot determine the sign of  $\theta_s^i$  and  $\delta_s^i$ . However, if Firm  $i$  does not condition its beliefs about Firm  $j$ 's actions on its own private information (as would be reasonable if the private information is uncorrelated), then  $\beta_{rk}^i$  is constant over  $k$ . Here,  $\theta_s^i = 0$  and  $\delta_s^i = \text{cov}(\theta_{sk}^i, \alpha_{sk}^i) = \text{cov}(L_{sk}^i, \alpha_{sk}^i) \geq 0$ . This last relation holds with strict inequality if and only if the number of states in which it is optimal for Firm  $i$  to use action  $x_s^i$ , with positive probability, is greater than 1 (so that  $\alpha_s^i > 0$ ) and less than  $K$  (so that  $L_s^i < 0$ ). If firms have no exogenous uncertainty but use mixed strategies, then  $\theta_s^i = \delta_s^i = 0$ . Thus, private, uncorrelated information implies  $\theta_s^i = 0$  and  $\delta_s^i \geq 0$ , whereas the absence of exogenous uncertainty implies  $\theta_s^i = 0$  and  $\delta_s^i = 0$ .

If the information that is unobserved by the econometrician is common knowledge to the firms, Firm  $i$ 's beliefs and actions may be conditioned on the random variable  $\epsilon_m^j$  that Firm  $j$  faces. If so,  $\beta_{rk}^i$  is replaced by  $\beta_{rkm}^i$ , and  $\alpha_{sk}^i$  is replaced by  $\alpha_{skm}^i$ , but restrictions 2.1 and 2.2 are otherwise unchanged. Taking expectations over  $k$  and  $m$ , we obtain restrictions of the same form as Equations 2.2 and 2.3. Again, in general we cannot sign  $\theta_s^i$  and  $\delta_s^i$ .

We have assumed that the econometrician observes the actions that firms choose, but not the information they use to condition these actions,  $\varepsilon_k^i$ . This assumption simplifies the estimation problem, because it means that the strategies,  $\underline{\alpha}^i$ , are numbers.<sup>2</sup>

We view 2.3 and 2.4 as stochastic restrictions that hold approximately due to an additive error in each equation. We already have additive parameters ( $\theta_s^i$  and  $\delta_s^i$ ), so we are able to estimate the sum of those parameters and any additive error, but we cannot identify the two components. Thus, for notational simplicity, we call the sum of the systematic and random components  $\theta_s^i$  and  $\delta_s^i$  (rather than add new random variables). We also include an additive error,  $\mu_s^i \in [-1, 1]$ , associated with  $\alpha_s^i$ . That is, we replace 2.4 with

$$(2.5) \quad \left( \sum_r \beta_r^i \pi_{rs}^i - Y^i \right) \left( \alpha_s^i + \mu_s^i \right) + \delta_s^i = 0.$$

We have an analogous set of restrictions for Firm  $j$ .

The Nash assumption is that agents' beliefs about their rival's actions are correct so that

$$(2.6) \quad \beta_r^i = \alpha_r^j,$$

for  $i \neq j$ . We henceforth maintain the Nash assumption.

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<sup>2</sup> If the firms' strategies are conditioned on a variable  $\zeta$  that the econometrician observes, the econometrician may need to estimate functions  $\underline{\alpha}^i(\zeta)$  rather than numbers,  $\underline{\alpha}^i$ . Suppose, however, that Firm  $i$ 's profits can be written as  $f^i(\zeta)\pi_{rsk}^i$ , where  $f^i$  is a positive function. For example, Firm  $i$  chooses price  $p^i$ , and faces demand  $D^i(p^i, p^j)f^i(\zeta)$ . Given this multiplicative form,  $f^i(\zeta)$  merely rescales the restrictions 2.3 and 2.4 [we can divide each restriction by  $f^i(\zeta)$ ], and the equilibrium strategies,  $\alpha^i$ , are independent of  $\zeta$ . Throughout the rest of this paper, we assume that any variables such as  $\zeta$  enter the profit functions multiplicatively so that restrictions 2.3 and 2.4 are correct.

If we tried to estimate this model — Equations 2.3, 2.5 and 2.6 — using traditional techniques, we would run into several problems. First, with conventional sampling theory estimation techniques, we would have to specify arbitrarily an error distribution. Second, imposing the various equality and inequality restrictions from our game-theoretic model would be very difficult if not impossible with standard techniques. Third, as the problem is ill posed in small samples (there are more parameters than observations), we would have to impose additional assumptions to make the problem well posed. To avoid these and other estimation and inference problems, we propose an alternative approach.

### 3. GENERALIZED-MAXIMUM-ENTROPY ESTIMATION APPROACH

We use generalized maximum entropy (GME) to estimate the firms' strategies. In this section, we start by briefly describing the traditional maximum entropy (ME) estimation procedure. Then, we present the GME formulation as a method of recovering information from the data consistent with our game. This GME method is closely related to the GME multinomial choice approach in Golan, Judge, and Perloff (1996). Unlike ML estimators, the GME approach does not require explicit distributional assumptions, performs well with small samples, and can incorporate inequality restrictions.

#### *3.1 Background: Classical Maximum Entropy Formulation*

The traditional entropy formulation is described in Shannon (1948), Jaynes (1957a; 1957b), Kullback (1959), Levine (1980), Jaynes (1984), Shore and Johnson (1980), Skilling (1989), Csiszár (1991), and Golan, Judge, and Miller (1996). In this approach, Shannon's (1948) entropy is used to measure the uncertainty (state of knowledge) we have about the

occurrence of a collection of events. Letting  $x$  be a random variable with possible outcomes  $x_s$ ,  $s = 1, 2, \dots, n$ , with probabilities  $\alpha_s$  such that  $\sum_s \alpha_s = 1$ , Shannon (1948) defined the *entropy* of the distribution  $\underline{\alpha} = (\alpha_1, \alpha_2, \dots, \alpha_n)'$ , as

$$(3.1) \quad H \equiv -\sum_s \alpha_s \ln \alpha_s,$$

where  $0 \ln 0 \equiv 0$ . The function  $H$ , which Shannon interprets as a measure of the uncertainty in the mind of someone about to receive a message, reaches a maximum when  $\alpha_1 = \alpha_2 = \dots = \alpha_n = 1/n$ . To recover the unknown probabilities  $\underline{\alpha}$ , Jaynes (1957a; 1957b) proposed maximizing entropy, subject to available data consistency relations, such as moments from the observed data, and adding up constraints.

To use this approach for our game problem, we need to incorporate the data from our sample. Let  $n_s^i$  be the number of times  $x_s^i$  is observed, out of  $T$  total observations. The observed frequency in the sample is  $n_s^* \equiv n_s^i/T$ . [We henceforth suppress the firm superscript for notational simplicity whenever possible.] For each firm, the observed frequency equals the true strategy probability,  $\alpha_s$ , plus an error term:

$$(3.2) \quad n_s^* \equiv \frac{n_s^i}{T} = \alpha_s + e_s,$$

where the noise term  $e_s \in [-1, 1]$ .

The traditional ME approach sets  $e_s$  in Equation (3.2) equal to zero,

$$(3.3) \quad n_s^* = \alpha_s$$

and maximizes the Shannon measure (3.1) subject to Equation 3.3. The solution to this problem is trivial in the sense that the constraint 3.3 completely determines the parameter estimate. This ME estimator is identical to the ML estimator, when the  $x$ 's have a multinomial distribution.

### 3.2 *The Basic Generalized Maximum Entropy Formulation*

The GME formulation, which uses restriction 3.2, is a more general version of the ME formulation, which uses restriction 3.3. We obtain the basic GME estimator by maximizing the sum of the entropy corresponding to the strategy probabilities,  $\alpha$ , and the entropy from the noise,  $e$ , in consistency condition 3.2 subject to that data consistency condition.

In general, the GME objective is a dual-criterion function that depends on a weighted sum of the entropy from both the unknown and unobservable  $\underline{\alpha}$  and  $\underline{e} = (e_1, e_2, \dots, e)'$ . By varying the weights, we can put more weight on estimation (accuracy of the  $\alpha$  coefficients) or prediction (assignment of observations to a category). The ME estimator is a special case of the GME, in which no weight is placed on the noise component, so that the estimation objective is maximized (thus maximizing the likelihood function). As a practical matter, our GME objective weights the  $\underline{\alpha}$  and  $\underline{e}$  entropies equally because we lack any theory that suggests other weights.

The arguments of the entropy measures must be probabilities. The elements of  $\underline{\alpha}$  are probabilities, but the elements of  $\underline{e}$  range over the interval  $[-1, 1]$ . To determine the entropy of  $\underline{e}$ , we reparameterize its elements using probabilities. We start by choosing a set of discrete points, called the support space,  $\underline{v} = [v_1, v_2, \dots, v_M]'$  of dimension  $M \geq 2$ , that are at uniform intervals, symmetric around zero, and span the interval  $[-1, 1]$ . Each error point  $e_s$

has corresponding unknown weights  $\underline{w}_s = [w_{s1}, w_{s2}, \dots, w_{sM}]'$  that have the properties of probabilities:  $0 \leq w_{sm} \leq 1$  and  $\sum_m w_{sm} = 1$ . We reparameterize each error element as

$$e_s = \sum_m v_m w_{sm}.$$

For example, if  $M = 3$ , then  $\underline{v} = (-1, 0, 1)'$ , and there exists  $w_1, w_2$ , and  $w_3$  such that each noise component can be written as  $e_s = w_1(-1) + w_3(1)$ . Given this reparameterization, we can rewrite the GME consistency conditions, Equation 3.2, as

$$(3.4) \quad \underline{n}^* = \underline{\alpha} + \underline{e} = \underline{\alpha} + W \underline{v},$$

where row  $s$  of the matrix  $W$  is the vector of probabilities  $\underline{w}_s$ , and  $\underline{v}$ , the support space, is the same for all  $s$ .

No subjective information on the distribution of probabilities is assumed. It is sufficient to have two points ( $M = 2$ ) in the support of  $\underline{v}$ , which converts the errors from  $[-1, 1]$  into  $[0, 1]$  space. This estimation process recovers  $M - 1$  moments of the distribution of unknown errors, so a larger  $M$  permits the estimation of more moments. Monte-Carlo experiments show a substantial decrease in the mean-square-error (MSE) of estimates when  $M$  increases from 2 to 3. Further increases in  $M$  provides smaller incremental improvement. The estimates hardly change if  $M$  is increased beyond 7 (Golan, Judge, Perloff, 1996; Golan, Judge, Miller, 1996).

If we assume that the actions,  $\underline{x}$ , and the errors,  $\underline{e}$ , are independent and define  $\underline{w} \equiv \text{vec}(W)$ , the GME problem for each firm is

$$(3.5) \quad \max_{\underline{\alpha}, \underline{w}} H(\underline{\alpha}, \underline{w}) = -\underline{\alpha}' \ln \underline{\alpha} - \underline{w}' \ln \underline{w},$$

subject to the GME consistency conditions, Equation 3.4, and the normalization constraints

$$(3.6a,b) \quad \underline{1}' \underline{\alpha} = 1 \quad \underline{1}' \underline{w}_s = 1$$

for  $s = 1, 2, \dots, n$ .

### 3.3 Generalized Maximum Entropy Formulation of the Nash Model

We can also use the GME approach to estimate the strategies subject to the game-theoretic restrictions. Here, we require the estimates to satisfy the optimality conditions, Equations 2.3 and 2.5, and the Nash condition, Equation 2.6.<sup>3</sup> Thus, our objective is to recover the strategies,  $\underline{\alpha}$ , for each firm given the  $T$  observations and our knowledge of the economic generating process. Here we assume that the econometrician knows the parameters of  $\pi_{rs}^i$ . Appendix 1 shows how to estimate these parameters jointly with other unknowns.

Equation 2.3 includes the noise components  $\underline{\theta}^i$ , and Equation 2.5 includes the noise components  $\underline{\mu}^i$  and  $\underline{\delta}^i$ ,  $i = 1, 2$ . Our first step is to reparameterize these six vectors in terms of probabilities. Let  $\underline{v}^d$  be a vector of dimension  $J^d \geq 2$  with corresponding unknown weights  $\underline{\omega}_k^d$  such that

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<sup>3</sup> Equations 2.3 and 2.4 are not standard econometric restrictions, as each includes an additional unknown parameter:  $\theta_s^i$  in Equation 2.3 and  $\delta_s^i$  in Equation 2.4. Therefore, it might appear that the added degree of freedom caused by the new parameter cancels the added information in the restriction. However, when these restrictions are imposed, the parameters  $\theta_s^i$  and  $\delta_s^i$  appear in the criterion function. As a result, imposing these restrictions causes the estimates of all the parameters to change and improves the estimates, as we show below.

$$(3.7) \quad \sum_j \omega_{sj}^d = 1,$$

$$(3.8) \quad \underline{v}^{d'} \underline{\omega}_s^d = d_s,$$

for  $d = \underline{\mu}^i, \underline{\theta}^i, \text{ and } \underline{\delta}^i, i = 1, 2$ . The support spaces  $\underline{v}^d$  are defined to be symmetric around zero for all  $d$ . The natural boundaries for the errors  $\underline{\mu}^i$  and  $\underline{\mu}^j$  are  $[-1, 1]$ . We do not have natural boundaries for  $\underline{\theta}^i$  or  $\underline{\delta}^i$ , so we use the "three-sigma rule" (Pukelsheim, 1994; Miller 1994; Golan, Judge, and Miller 1996) to choose the limits of these support spaces, where sigma is the empirical standard deviation of the discrete action space of prices or quantities.

To simplify the notation, let  $\underline{n}^* = (\underline{n}^{*i}, \underline{n}^{*j})'$ ,  $\underline{\alpha} = (\underline{\alpha}^i, \underline{\alpha}^j)'$ ,  $\underline{w} = (\underline{w}^i, \underline{w}^j)'$ , and  $\underline{\omega} = (\underline{\omega}^{\mu^i}, \underline{\omega}^{\mu^j}, \underline{\omega}^{\theta^i}, \underline{\omega}^{\theta^j}, \underline{\omega}^{\delta^i}, \underline{\omega}^{\delta^j})'$ . As above, we assume independence between the actions and the errors. The GME-Nash problem is

$$(3.9) \quad \underset{\underline{\alpha}, \underline{w}, \underline{\omega}}{\text{Max}} H(\underline{\alpha}, \underline{w}, \underline{\omega}) = -\underline{\alpha}' \ln \underline{\alpha} - \underline{w}' \ln \underline{w} - \underline{\omega}' \ln \underline{\omega}$$

subject to the data consistency conditions 3.4, the necessary economic conditions 2.3 and 2.5, the Nash condition 2.6, and the normalizations for  $\underline{\alpha}$ ,  $\underline{w}$ , and  $\underline{\omega}$ . The errors  $\theta_s^i, \mu_s^i, \text{ and } \delta_s^i$  in 2.3 and 2.5 are defined by Equations 3.8 and 3.9. Solving this problem yields estimates  $\underline{\tilde{\alpha}}, \underline{\tilde{w}}$ , and  $\underline{\tilde{\omega}}$ .

### 3.4 Properties of the Estimators and Normalized Entropy

All three different estimators, the ME-ML, GME, and GME-Nash, are consistent, but they differ in efficiency and information content. The ML estimator is known to be

consistent. Because the ME and ML estimators are identical, as we noted above, the ME estimator is also consistent. Under the assumption that a solution to the GME-Nash estimation problem exists for all samples and given an appropriate choice of the bounds of the error in the data consistency constraint 3.2, we show in Appendix 2 that the GME and GME-Nash estimators are also consistent.

We can compare the different estimators empirically using the normalized entropy (information) measure  $S(\underline{\alpha}) = -(\sum_s \alpha_s \ln \alpha_s)/(\ln n)$ , which measures the extent of uncertainty about the unknown parameters. If there is no uncertainty,  $S(\underline{\alpha}) = 0$ . If there is full ignorance, in the sense that all actions are equally likely,  $S(\underline{\alpha}) = 1$ . All else the same, additional information reduces the uncertainty in the data analyzed, resulting in a lower normalized entropy measure. Thus, to the degree that the constraints 2.3, 2.5, and 2.6 bind, the GME-Nash normalized entropy measure is lower than is the GME measure:  $S(\underline{\tilde{\alpha}}) \leq S(\underline{\check{\alpha}})$ .

#### 4. AIRLINES

We estimate the strategic pricing behavior of American and United Airlines using the ME-ML, GME, and GME-Nash approaches, and compare results. We allow for the possibility that American and United provide differentiated services on a given route and assume that the demand curve facing Firm  $i$  is

$$(4.1) \quad q_i = a_i + b_i p_i + d_i p_j + u_i,$$

where  $a_i$  and  $d_i$  are positive,  $b_i$  is negative, and  $u_i$  is an error term.<sup>4</sup> In Appendix 1, we show how to reparametrize 4.1 so that it can be estimated along with the other parameters in the GME-Nash model.

If firms choose prices, the necessary conditions 2.3 and 2.5 are

$$(4.2) \quad \sum_r \beta_r^i (p_s^i - c^i) q_{rs}^i - Y^i + \theta_s^i \leq 0,$$

$$(4.3) \quad \left[ \sum_r \beta_r^i (p_s^i - c^i) q_{rs}^i - Y^i \right] (\alpha_s^i + \mu_s^i) + \delta_s^i = 0.$$

The Nash condition 2.6 is unchanged.

The data include price and quantity and cost data for 15 quarters (1984:4-1987:4, 1988:2, 1988:4) for various routes between Chicago and other cities.<sup>5</sup> We calculated marginal costs using the formula in Oum, Zhang, and Zhang (1993), and we used the average of these for the parameters  $c^i$ . The nominal data are deflated using the Consumer Price Index.

We restrict our attention to two city pairs, Chicago-Providence and Chicago-Wichita, where United and American Airlines had no (or trivial) competition from other firms. On these two routes, the average marginal cost was lower than observed prices and the estimated

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<sup>4</sup> We experimented with including various additional right-hand-side variables such as measures of income, population, or economic conditions. None of these variables, however, affected the fit of the equation or the parameters  $a_i$ ,  $b_i$ , and  $d_i$  substantially.

<sup>5</sup> The data were generously provided by James A. Brander and Anming Zhang. They used these data in: Brander and Zhang (1990, 1993) and Oum, Zhang, and Zhang (1993). These papers describe the data.

demand curves were consistent with economic theory: The demand curves slope down,  $b_i < 0$ , and the services are substitutes,  $d_i > 0$ .

#### *4.1 The Airline Model Specification*

We first specify the upper and lower bound of the price space. The lower bound is the smallest observed price for both airlines minus 10% and the upper bound is the largest observed price for both airlines plus 10%. We then divide the price space into 20 equal increments.

Because we do not know the true demand curve parameters, we simultaneously estimate linear demand curves for each firm and a price-strategic choice model. We do not estimate  $c^i$  since we have a measure of the average marginal costs.

#### *4.2 Airline Estimates*

In the GME-Nash model, the correlation between the actual and estimated quantities in the demand equations are 0.1 for the American demand equation and 0.2 for the United equation for Providence and 0.5 for both Wichita equations. For Providence, the demand coefficients ( $a_i$ ,  $b_i$ , and  $d_i$ ) are 1,865.8 (with an asymptotic standard error of 373.4), -12.1 (5.1), 4.7 (1.6) for American Airlines and 1,571.7 (456.2), -10.2 (5.6), 4.8 (6.2) for United Airlines. Given these parameters, the own-demand elasticities at the sample mean are -2.6 for American and -2.1 for United, and both the cross-elasticities are 0.95.

In Wichita, the demand coefficient are 637.7 (345.7), -3.7 (4.4), and 2.6 (4.0) for American and 810.4 (238.9), -6.2 (2.8), and 3.6 (3.1) for United. The own-demand elasticities are -1.5 for America and -2.1 for United, and the cross-price elasticities are 0.9 in the

American equation and 1.2 in the United equation. Estimating these demand curves using ordinary least squares yields estimates of the same sign and magnitude.

The estimated strategy parameters,  $\underline{\alpha}$ , are shown in Figure 1a for American Airlines and in Figure 1b for United Airlines for the Providence route. Figures 2a and 2b show the corresponding distributions for Wichita. The ME-ML estimates are the observed frequencies.

The GME distribution is more uniform than that of the ME-ML model because the GME consistency conditions 3.2 allow the estimates to differ from the actual frequency. In attempting to maximize entropy, the GME estimator pushes the probability estimates toward uniformity.

The GME-Nash distribution is smoother than the other two models and has one peak for American and two peaks for United in both cities. The global maximum of the GME-Nash distribution is closer to the average price based on a standard Bertrand model than to a Cournot or Collusive model. [The Cournot — \$216 for American and \$185 for United — and collusive — \$285 and \$223 — means are too large to appear in Figure 2 for Wichita.]

The normalized entropy measure,  $S(\underline{\alpha})$ , for Providence is 0.66 for American and 0.67 for United for the ME-ML model. Consequently, the goodness of fit measures, pseudo- $R^2$  (see Appendix 2), are 0.34 for American and 0.33 for United. The corresponding normalized entropy measures ( $R^2$  measures) are both 0.90 (0.10) for the GME model and 0.61 (0.39) and 0.65 (0.35) for the GME-Nash model. The normalized entropy measures for Wichita are, respectively, 0.73 (0.27) and 0.77 (0.23) for the ME-ML, 0.93 (0.07) and 0.94 (0.6) for the GME, and 0.71 (0.29) and 0.66 (0.33) for the GME-Nash model. The drop in the entropy

measure when we switch from the GME to the GME-Nash shows that the theoretical restrictions contain substantial information.

The estimated expected rents,  $\tilde{Y}$ , are \$420,000 per quarter for American and \$435,000 for United on the Providence route, and \$435,000 for American and \$449,000 for United on the Wichita route. These rent calculations are based on the assumption that the average cost equals the marginal cost. These numbers do not include fixed costs. Unless the fixed costs are large, these numbers suggest that the airlines made positive profits during this period. The estimated expected rents are consistent with the magnitudes of the prices and quantities observed.

For both airlines for both cities, the average value of  $\tilde{\theta}$  is practically zero. The average value of  $\tilde{\delta}$  is positive. For example, in Providence, only 2 out of the 40 values of  $\tilde{\delta}$  were negative. This sign pattern is consistent with firms having private, uncorrelated information. This pattern is inconsistent with the hypothesis that firms use mixed strategies despite the absence of exogenous randomness.

### 4.3 Comparing Estimators

How does our approach compare to traditional methods?<sup>6</sup> For the purposes of comparison, we estimated a traditional conjectural variations (CV) model given our heteroge-

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<sup>6</sup> We cannot directly compare our results to those in Brander and Zhang (1990, 1993) and Oum, Zhang, and Zhang (1993), because they assume that the services of the two airlines are homogeneous, whereas we estimate demand curves based on differentiated services. Moreover, they estimate pure strategy models in two of their papers, where we permit mixed or pure strategies. In their other paper, Brander and Zhang (1993) estimate a supergame (trigger price) model. If there are punishment periods during the sample, our estimates may show two or more peaks in the distribution of  $\underline{\alpha}$ . However, if the firms are using such supergames, we should modify our repeated single-period game model accordingly.

nous demand equations. The CV model consists of four equations: the two demand curves and two optimality (first-order) conditions.<sup>7</sup>

Figures 3a and 3b show how the conjectural variations distribution compares to the GME-Nash and ME-ML for Providence. The CV distribution has multiple peaks, with its global maximum slightly higher than the GME-Nash. The CV distribution is significantly different than the ME-ML for United on the Chicago-Providence route based on a Kolmogorov-Smirnov test. Similarly, for United on the Chicago-Wichita route, the CV distribution differs from the GME-Nash strategy distribution.

The estimated market power of these firms is similar for the different estimators. Table 1 shows how the expected Lerner Index of market power (the difference between price and marginal cost as a percentage of price) for the different estimators. The ME-ML Lerner Index is identical to the index based on the observed data. The GME indexes are virtually the same or slightly lower than the ME-ML indexes. The average GME-Nash and CV estimates are virtually identical and slightly higher than the sample-based index.

Using the demand parameters from the GME-Nash model, we also calculated the average Bertrand, Cournot, and collusive Lerner Indexes. The average Bertrand index is virtually the same as the average GME-Nash and CV indexes. The Cournot and collusive indexes are much higher.

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<sup>7</sup> When we tried to estimate the four equations simultaneously using standard ML techniques, some of the demand parameters took on theoretically incorrect signs. Consequently, we estimated the demand curves and then estimated the optimality conditions treating the estimated demand parameters as exact. Both approaches produced similar estimates of the conjectures. Figures 3a and 3b use the second set of estimates where we used the marginal cost in each period to generate a distribution of estimates.

#### 4.4 Hypothesis Tests

Using our estimates, we can test various market-structure hypotheses. We start by considering two "pretest" hypotheses. We strongly reject the competitive market hypothesis that the expected price equals the expected marginal cost (first row of Table 2) using a standard  $\chi^2$  test.

In contrast, for three out of the four cases we do not reject the hypothesis that the distribution of the marginal cost is the same as the distribution of the prices for three out of four cases (second row of Table 2). That is, the prices and marginal costs "move together" as we expect.

Next, we examine whether the airlines use identical strategies in the GME-Nash model. We fail to reject (at the 0.05 level) the hypothesis that both airlines have the same  $\tilde{\alpha}$  distribution in Providence, but we reject this hypothesis in Wichita.

We then examine the key hypotheses that firms maximize single-period profits and that the Nash restriction holds. We use entropy-ratio tests, which are analogous to likelihood-ratio tests (Appendix 2). To test whether a particular restriction holds, we compare the entropy measures from a restricted estimator and from an unrestricted estimator. We fail to reject that firms maximize single-period expected profits (Equations 2.3 and 2.5). We similarly fail to reject that they use Nash equilibrium strategies to maximize single-period expected profits (Equations 2.6, 2.3, and 2.5). That is, our GME-Nash model is consistent with the data.

Although the data are consistent with our game-theoretical restrictions, these restrictions affect the estimated strategies significantly. Based on a standard  $\chi^2$  test, we reject the

hypothesis in three out of four cases that the estimated  $\underline{\alpha}$  distribution is the same for the GME (which lacks the maximizing and Nash restrictions) and the GME-Nash models. Not surprisingly, we strongly reject the hypothesis that the estimates of the ME-ML model are the same as those for the GME-Nash model.

#### 4.5 Sample Size Sensitivity Experiments

The squared-error loss of each of our three estimators differs as sample size changes. We can demonstrate these properties using sensitivity experiments, where we assume that the estimated demand equations for the Chicago-Wichita route hold with an error term that is distributed  $N(0, 1)$ . We assume that Firm  $i$  has information  $\varepsilon^i$  about its marginal cost and that this information is private and uncorrelated (as is consistent with our estimates), so that Firm  $i$ 's beliefs,  $\beta_r^i$ , do not depend on  $\varepsilon_k^i$ . The marginal cost for each firm in each period is drawn from a normal distribution  $N(60, 5)$ , which closely approximates the distribution of marginal costs for Wichita. We approximate this continuous distribution using a finite grid and use the probabilities associated with the resulting discrete distribution,  $\rho_k$ , to determine the Nash restriction that beliefs are correct in equilibrium. This restriction requires that  $\sum_k \alpha_{rk}^j \rho_k = \beta_r^i$ . We then generate Nash equilibrium strategies  $\underline{\alpha}$  using this restriction and the necessary conditions 2.1 and 2.2. (We establish by means of sensitivity studies that this equilibrium is unique.) We use the resulting equilibrium probabilities  $\underline{\alpha}$  to generate samples of actions by drawing a uniform random number on the unit interval and using that to assign an action for each observation. We generated 200 samples for  $T$  (the number of observations in each sample) = 10, 20 and 40, with  $n$  (the number of possible actions for each firm) = 20.

The GME estimator of  $\alpha$  has smaller variance than the ME-ML estimator.<sup>8</sup> Given that the game-theoretic constraints are correct, the possible solution space for GME-Nash estimate of  $\alpha$  is a subset of the solution space of the GME estimate of  $\alpha$ . Thus, we conjecture that the GME-Nash estimator has a smaller variance than the GME.

The superior finite sample properties of the GME-Nash and GME over the ME-ML are confirmed by our sampling experiments. Table 3 reports the empirical mean square error ( $MSE(\hat{\alpha}^i) = \sum_{s,t} (\hat{\alpha}_{st}^i - \alpha_s^i)^2/200$  (where the index  $t$  denotes the sample) and the correlation coefficient between the estimated and true  $\alpha_s^i$  for each of the models. The table shows two sets of results depending on whether the econometrician knows the demand coefficients or has to estimate them. In the latter case, we generate quantities demanded by adding a  $N(0, 1)$  term to the demand equation.

The ME-ML and GME perform better (in terms of MSE and correlations) as the number of observations increases. The GME-Nash, however, performs well (relative to the other estimators) for a small number of observations, and the GME-Nash estimates do not improve as the number of observations increases beyond 20. This latter result is very attractive if, as usual, one has relatively few time-series observations. Finally, the GME-Nash estimator yields superior estimates even when the demand coefficients are unknown, without assuming knowledge of the error distributions.

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<sup>8</sup> The proof is analogous to the one given in Golan, Judge, and Perloff (1996).

## 5. CONCLUSIONS

Our generalized-maximum-entropy-Nash (GME-Nash) estimator can estimate firms' strategies consistent with game theory and the underlying data generation process. It is free of parametric assumptions about distributions and ad hoc specifications such as those used in conjectural-variations models.

Our simplest approach to estimating strategies is to use the maximum-likelihood (ML) or maximum-entropy (ME) estimators. These approaches produce the same estimates, which are the observed frequencies in the data. These estimators do not make use of demand or cost information and do not impose restrictions based on theory.

We also estimate two GME models. The basic GME estimator allows greater flexibility than the ML-ME estimator, but does not use demand, cost, or game-theoretic information. We show that this GME estimator is more efficient than the ME-ML estimator in terms of mean-square error, correlation, and other measures of variance. The GME-Nash estimator uses all available data, and game-theoretic information. In our sampling experiments, the GME-Nash estimator is more efficient than the basic GME and ME-ML estimators.

On the basis of our hypothesis tests, we find that the static game-theoretic restrictions are consistent with the data. Nonetheless, the restrictions have bite: The estimates of firms' strategies based on the model with theoretical restrictions imposed differ from those based on the unrestricted model.

In Golan, Karp, Perloff (1996), we examine a version of this model where strategies may vary with variables the econometrician observes and allow firms to choose both price

and advertising. In future papers, we plan two other generalizations of our approach. First, we will examine whether a price-choice or quantity-choice model is appropriate. Second, we will extend the model to allow for dynamic games.

Our approach to estimating games can be applied to many problems in addition to oligopoly, such as wars and joint decisions by husbands and wives. To do so only requires replacing profits with some other criterion.

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## Appendix 1: Technical Details

### A1.1 The Solution to Equation (3.5)

The Lagrangean to the GME problem is

$$\begin{aligned}
 (A1.1) \quad L(\underline{\lambda}, \rho, \underline{\eta}) &= - \sum_s \alpha_s \ln \alpha_s - \sum_s \sum_m w_{sm} \ln w_{sm} \\
 &+ \sum_s \lambda_s \left( n_s^* - \alpha_s - \underline{v}' \underline{w}_s \right) + \rho \left( 1 - \underline{1}' \underline{\alpha} \right) \\
 &+ \sum_s \eta_s \left( 1 - \underline{1}' \underline{w}_s \right).
 \end{aligned}$$

where  $\underline{\lambda}$ ,  $\rho$ , and  $\underline{\eta}$  are Lagrange multipliers. Solving this problem, we obtain the GME estimators

$$(A1.2) \quad \check{\alpha}_s = \frac{\exp(-\check{\lambda}_s)}{\sum_j \exp(-\check{\lambda}_j)} \equiv \frac{\exp(-\check{\lambda}_s)}{\Omega(\check{\underline{\lambda}})},$$

$$(A1.3) \quad \check{w}_{sm} = \frac{\exp(-\check{\lambda}_s v_m)}{\sum_m \exp(-\check{\lambda}_s v_m)} \equiv \frac{\exp(-\check{\lambda}_s v_m)}{\Psi_s(\check{\underline{\lambda}})},$$

and

$$(A1.4) \quad \check{\underline{\epsilon}} = \check{W} \underline{v}.$$

The Hessian is negative definite (the first  $n$  elements on the diagonal are  $-1/\alpha_s$ , the rest are  $-1/w_{sm}$ , and the off-diagonal elements are 0) so the solution is globally unique.

Following Agmon et. al (1979), Miller (1994), and Golan et al. (1996), we can reformulate the GME problem as a generalized-likelihood function, which includes the traditional likelihood as a special case:

$$\begin{aligned}
L(\underline{\lambda}) &= -\sum_s \alpha_s(\underline{\lambda}) \ln \alpha_s(\underline{\lambda}) - \sum_s \sum_m w_{sm}(\underline{\lambda}) \ln w_{sm}(\underline{\lambda}) \\
&\quad + \sum_s \lambda_s \left( n_s^* - \alpha_s - \underline{v}' \underline{w}_s \right) \\
\text{(A1.5)} \quad &= -\sum_s \alpha_s(\underline{\lambda}) \left[ -\lambda_s - \ln \Omega(\underline{\lambda}) \right] - \sum_s \sum_m w_{sm}(\underline{\lambda}) \left[ -\lambda_s v_m - \ln \Psi_s(\underline{\lambda}) \right] \\
&\quad + \sum_s \lambda_s \left( n_s^* - \alpha_s - \underline{v}' \underline{w}_s \right) \\
&= \sum_s \lambda_s n_s^* + \ln \Omega(\underline{\lambda}) + \sum_s \ln \left[ \Psi_s(\underline{\lambda}) \right].
\end{aligned}$$

Minimizing Equation 3.11 with respect to  $\underline{\lambda}$  — setting the gradient,  $\Delta L(\underline{\lambda}) = \underline{n}^* - \underline{\alpha} - \underline{e}$ , equal to zero — yields the same estimates as from the original formulation, Equation A.2. One advantage of this dual formulation, Equation A.5, is that it is computationally more efficient.

### A1.2 GME-Nash with Unknown Demand Coefficients

We now show how to simultaneously estimate the profit or demand parameters and the strategy parameters using GME estimation procedures. We need to modify the objective function 3.10 by adding the profit (or demand) functions for each firm as additional constraints. We estimate the demand curves simultaneously with the rest of the model. For example, let the demand curve facing Firm  $i$  be

$$\text{(A1.6)} \quad \underline{q}^i = a^i + b^i \underline{p}^i + d^i \underline{p}^j + \underline{u}^i \equiv X^i \underline{\phi}^i + \underline{u}^i,$$

where  $\underline{q}^i$  is the quantity vector,  $\underline{p}^i$  is the price vector,  $a^i$  and  $d^i$  are positive scalars,  $b^i$  is a negative scalar,  $\underline{u}^i$  is a vector of error terms,  $X^i$  is a matrix, and  $\underline{\phi}^i$  is a vector of parameters.

To use an entropy approach, we need to map the unknown parameters  $\underline{\phi}_i$  and  $\underline{u}^i$  into probability space. Following Golan, Judge, and Miller (1996), we model these unknown parameters as discrete random variables with finite supports. For simplicity, we drop the firm's superscript for the rest of this Appendix.

Let  $\underline{\phi}$  be in the interior of an open, bounded hyperrectangle,  $Z \subset \mathfrak{R}^K$ , and, for each  $\phi_k$ , let there be a discrete random variable  $\underline{z}_k$ , with  $M \geq 2$  possible realizations  $z_{k1}, \dots, z_{kM}$  and corresponding probabilities  $p_{k1}, \dots, p_{kM}$  such that

$$(A1.7) \quad \phi_k = \sum_{m=1}^M p_{km} z_{km}.$$

Letting  $Z$  be the  $M$ -dimensional support for  $\underline{z}_k$ , any  $\underline{\phi} \in Z$  can be expressed as

$$(A1.8) \quad \underline{\phi} = Z\underline{p} = \begin{bmatrix} \underline{z}_1 & \underline{0} & \cdot & \underline{0} \\ \cdot & \underline{z}_2 & \cdot & \underline{0} \\ \cdot & \cdot & \cdot & \cdot \\ \underline{0} & \underline{0} & \cdot & \underline{z}_K \end{bmatrix} \begin{bmatrix} \underline{p}_1 \\ \underline{p}_2 \\ \cdot \\ \underline{p}_k \end{bmatrix},$$

where  $Z$  is a  $(K \times KM)$  matrix and  $\underline{p}$  is a  $KM$ -vector of weights such that  $\underline{p}_K \gg 0$  and  $\underline{p}'_K \underline{1}_M = 1$  for each demand parameter for  $k = 1, 2, 3$ . The upper and lower bounds of  $\underline{z}_k$ ,  $z_{k1}$  and  $z_{kM}$ , are far apart and known to contain  $\phi_k$ . Further, we use our knowledge of the signs of the unknown parameters from economic theory when specifying the support space  $Z$ .

The unknown and unobservable errors,  $u_{it}$ , are treated similarly. For each observation, the associated disturbance,  $u_{it}$ , is modelled as a discrete random variable with realizations  $v_1^u, \dots, v_J^u \in \underline{v}^u$  with corresponding probabilities  $\omega_{t1}^u, \dots, \omega_{tJ}^u$ . That is, each disturbance may be modelled as

$$(A1.9) \quad u_{it} = \sum_{j=1}^J \omega_{tj}^u v_j^u,$$

for each  $t = 1, \dots, T$ . The elements of the vector  $\underline{v}^u$  form an evenly spaced grid that is symmetric around zero.

Given a sample of data  $q_i$ , a simple way to determine the upper and lower bound of  $\underline{v}^u$  is to use the three-sigma rule together with the sample standard deviation  $\sigma_q$ . For example, if  $J = 3$ , then  $\underline{v}^u = (-3\sigma_q, 0, 3\sigma_q)$ .

Having reparametrized the system of demand equation in this manner, the GME-Nash model with unknown demand parameters is

$$(A1.10) \quad \max_{\underline{\alpha}, \underline{w}, \underline{p}, \underline{\omega}} H(\underline{\alpha}, \underline{w}, \underline{p}, \underline{\omega}) = -\underline{\alpha}' \ln \underline{\alpha} - \underline{w}' \ln \underline{w} - \underline{p}' \ln \underline{p} - \underline{\omega}' \ln \underline{\omega},$$

subject to the consistency conditions 3.4, the necessary economic conditions 2.3 and 2.5, the Nash conditions 2.6, the two demand equations for Firms  $i$  and  $j$ , Equations A1.6, and the normalizations for  $\underline{\alpha}$ ,  $\underline{w}$ ,  $\underline{p}$ , and  $\underline{\omega}$ , where  $\underline{\omega} = (\underline{\omega}^{\mu^i}, \underline{\omega}^{\mu^j}, \underline{\omega}^{\theta^i}, \underline{\omega}^{\theta^j}, \underline{\omega}^{\delta^i}, \underline{\omega}^{\delta^j}, \underline{\omega}^{u^i}, \underline{\omega}^{u^j})'$ . The bounds of the error supports for the demand equations  $\pm 3\sigma_q$ .

## Appendix 2: The GME-Nash Estimator

### A2.1 Consistency

Call the GME-Nash estimates of the strategies  $\underline{\tilde{\alpha}}$ , the GME estimates  $\underline{\check{\alpha}}$ , and the ME-ML estimates  $\underline{\hat{\alpha}}$ . We make the following assumptions:

*Assumption 1:* A solution of the GME-Nash estimator  $(\underline{\tilde{\alpha}}, \underline{\tilde{w}}, \underline{\tilde{\omega}})$  exists for any sample size.

*Assumption 2:* The expected value of each error term is zero, its variance is finite, and the error distribution satisfies the Lindberg condition (Davidson and MacKinnon, 1993, p. 135).

*Assumption 3:* The true value of each unknown parameter is in the interior of its support.

We want to prove

*Proposition:* Given assumptions 1-3, and letting *all* the end point of the error support spaces  $\underline{y}$  and  $\underline{y}^d$  be normed by  $\sqrt{T}$ ,  $\text{plim}(\underline{\tilde{\alpha}}) = \text{plim}(\underline{\check{\alpha}}) = \text{plim}(\underline{\hat{\alpha}}) = \underline{\alpha}$ .

When the profit parameters  $\underline{\phi}$  are unknown, the GME-Nash estimates  $\underline{\tilde{\phi}}$  are consistent.

According to this proposition, the GME-Nash estimates,  $\underline{\tilde{\alpha}}$ , GME basic estimates,  $\underline{\check{\alpha}}$ , and the ME-ML estimates,  $\underline{\hat{\alpha}}$ , are equal to each other and to the true strategies in the limit as the sample size becomes infinite,  $T \rightarrow \infty$ . That is, all the estimators are consistent.

*Proof:*

i) The ME-ML estimates are the observed frequencies: As  $T \rightarrow \infty$ , the observed frequencies converge to the population frequencies, so the ME-ML estimates are consistent:  $\text{plim } \hat{\alpha}_T = \alpha$ .

ii) The GME is consistent: Let the end points of the error supports of  $\underline{v}$ ,  $v_1$  and  $v_m$ , be  $-1/\sqrt{T}$  and  $1/\sqrt{T}$  respectively. As  $T \rightarrow \infty$ ,  $\psi_s \rightarrow 1$  for all  $s$  in the dual-GME, Equation A.5. Thus,  $\sum_s \ln \psi_s(\hat{\lambda}) \rightarrow 0$  and  $\text{plim } \check{\alpha}_T = \alpha$ .

iii) The GME-Nash with known profit parameters is consistent: By Assumption 1, after we have added the restrictions 2.3 and 2.5, we still have a solution. The argument in (ii) together with Assumption 2 implies that  $\text{plim } \check{\alpha}_T = \alpha$ .

iv) The GME-Nash with unknown profit parameters is consistent: Given Assumption 3, the GME is a consistent estimator of  $\phi$  in the linear Equation A2.1 (Golan, Judge, and Miller, 1996, Ch. 6; Golan and Judge, 1996; Mittelhammer and Cardell, 1996):  $\text{plim } \check{\phi}_T = \phi$ . By the argument in (iii),  $\text{plim } \check{\alpha}_T = \alpha$ . These asymptotic properties can also be established using the empirical likelihood approach (Owen, 1990; Qin and Lawless, 1994; and Golan and Judge, 1996).

## A2.2 Hypothesis testing

On the basis of the consistency of the estimators, we can define an "entropy ratio statistic" which has a limiting  $\chi^2$  distribution. We use this statistic to test hypotheses. In general, let  $\underline{\lambda}^*$  be the vector of Lagrange multipliers for *all* the model's constraints. Let  $H_M(\underline{\lambda}_0^*)$  be the entropy value of the constrained problem where  $\underline{\lambda}^* = \underline{0}$ , or equivalently all the parameters (strategies as well as demand coefficients) are constrained to be zero (or at the

center of their supports). Thus,  $H_M(\underline{\lambda}_0^*)$  is the maximum value of the joint entropies (objective function). It can be obtained by maximizing Equation (3.9) subject to no constraints (except for the requirement that all distributions are proper). Doing so yields the total entropy value of the three sets of discrete, uniform distributions  $\underline{\alpha}$ ,  $\underline{w}$ , and  $\underline{\omega}$ . Now, let  $H_u(\underline{\lambda}^*)$  be the objective (total entropy) value for the full GME-Nash model — the optimal value of Equation (3.9) — where  $\underline{\lambda}^*$  is the set of estimated values (that is, they are not forced to equal zero).

The *entropy-ratio statistic* for testing the null hypothesis  $H_0$  that all parameters are zero is

$$\mathcal{E}(\text{parameters} = \underline{0}) = 2H_M(\text{parameters} = \underline{0}) - 2H_u(\underline{\tilde{\alpha}}, \underline{\tilde{w}}, \underline{\tilde{\omega}}).$$

Under the mild assumptions we made above (or the assumptions of Owen, 1990 and Qin and Lawless, 1994),  $\mathcal{E}(\text{parameters} = \underline{0}) \rightarrow \chi_K^2$  as  $T \rightarrow \infty$  when  $H_0$  is true and  $K$  is the number of restrictions. The approximate  $\alpha$ -level confidence intervals for the estimates are obtained by setting  $\mathcal{E}(\cdot) \leq C_\alpha$ , where  $C_\alpha$  is chosen so that  $\Pr(\chi_K^2 < C_\alpha) = \alpha$ . Similarly, we can test any other hypothesis of the form  $H_0: \underline{\alpha} = \underline{\alpha}_0$  for all, or any subset, of the parameters. We use these entropy-ratio statistics to test whether the economic and Nash restrictions are consistent with the airline data.

We use the same line of reasoning as above (each constraint, or data point, represents additional potential information that may lower the value of the objective function but can never increase it) to derive a "goodness of fit" measure for our estimator:

$$R^* = 1 - \frac{H_u(\tilde{\lambda}^*)}{H_M(\tilde{\lambda}^* = \underline{0})},$$

where  $R^* = 0$  implies no informational value of the data set, and  $R^* = 1$  implies perfect certainty or perfect in-sample prediction.

The small-sample approximated variances can be computed in a number of ways. We discuss two simplest approaches here. First, for each equation (say the two sets of demand equations), we calculate

$$\hat{\sigma}_i^2 = \frac{1}{T} \sum_t \tilde{u}_{it}^2,$$

where  $\tilde{u}_{it} \equiv \sum_j \tilde{\omega}_{ij}^u v_j^u$  and  $\hat{\text{var}}(\phi_k^i) \equiv \hat{\sigma}_i^2 (X'X)^{-1}$  for each parameter  $\phi_k^i$ . Similarly, for each

set of equations, the relevant  $\sigma^2$  is estimated.

Because our model is a system of a large number of equations, the elements of the asymptotic variance-covariance matrix,  $\Omega$ , for the error terms of the entire system are estimated in the traditional way, taking into account all the data and all the restrictions (Equations 2.3, 2.5, 2.6, and A.6).

Finally, we note the relationship between the entropy objective and the  $\chi^2$  statistic. This relationship is used for comparison of various estimated strategies and various estimated distribution. The cross-entropy measures is defined as

$$(A2.1) \quad I(\underline{\alpha}, \underline{\alpha}^o) = \sum_k \alpha_k \ln(\alpha_k/\alpha_k^o),$$

where  $\underline{\alpha}^o$  is a proper prior probability distribution. Now, let  $\{\alpha_k\}$  be a set of  $K$  observed frequencies (strategies) over a set of  $K$  observed prices. Let the null hypothesis be  $H_0: \underline{\alpha} = \underline{\alpha}^o$ , then

$$\chi_{(k-1)}^2 = \sum_k \frac{1}{\alpha_k^o} (\alpha_k - \alpha_k^o)^2.$$

A second-order approximation of (A2.1) is

$$I(\underline{\alpha}, \underline{\alpha}^o) \cong \frac{1}{2} \sum_k \frac{1}{\alpha_k^o} (\alpha_k - \alpha_k^o)^2,$$

which is the entropy-ratio statistic (for evaluating  $\underline{\alpha}$  versus  $\underline{\alpha}^o$ ) that we previous discussed.

We conclude by noting that two times the entropy-ratio statistic corresponds (at the limit) to  $\chi_{(k-1)}^2$ .

**Table 1: Average Lerner Indexes,  $(p - MC)/p$** 

	<i>Providence</i>		<i>Wichita</i>	
	<i>American</i>	<i>United</i>	<i>American</i>	<i>United</i>
ME-ML: Observed	0.35	0.37	0.62	0.62
GME	0.34	0.35	0.62	0.61
GME-Nash	0.37	0.40	0.64	0.59
Conjectural Variation	0.37	0.40	0.65	0.64
Bertrand	0.37	0.40	0.64	0.59
Cournot	0.40	0.43	0.72	0.68
Collusive	0.45	0.48	0.78	0.74

**Table 2: Hypothesis Tests (Entropy-Ratio Statistics)**

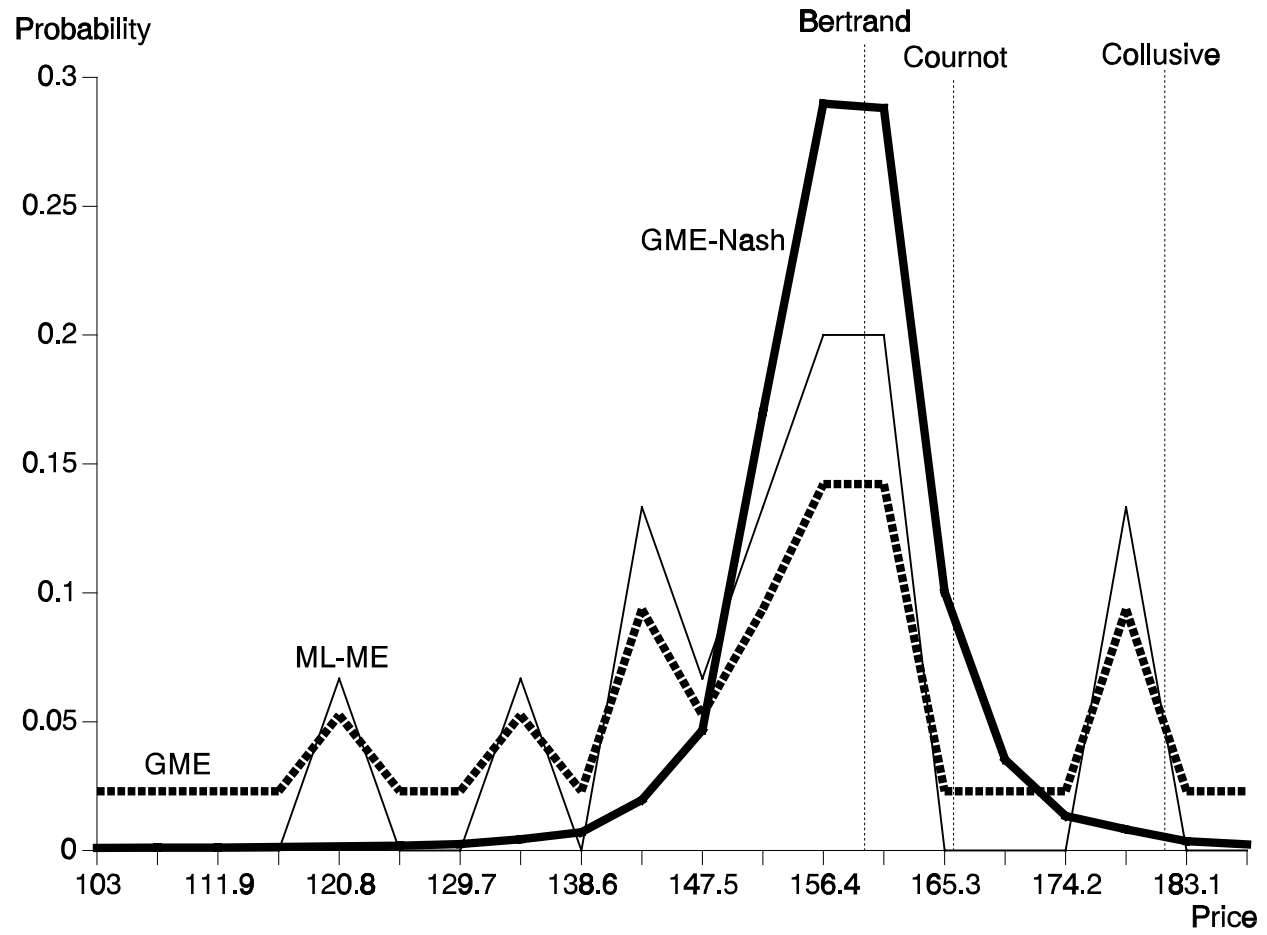
<i>Null Hypothesis</i>	<i>Providence</i>	<i>Wichita</i>
Competitive Market Hypothesis $\sum_i \tilde{\alpha}_i p_i = \overline{MC}$	rejected (at any significance level)	rejected (at any significance level)
Distribution of $MC$ = Distribution of Prices (14 df)	AA: 0.17 fail to reject UA: 0.12 fail to reject	AA: 21.1 fail to reject UA: 29.2 reject
Distribution of $\tilde{\alpha}$ is the same for both airlines (19 df)	15.0 fail to reject	96.1 reject
Maximizing behavior (Eq. 2.3, 2.5) (121 df)	27.0 Fail to reject	40.5 Fail to reject
Nash (Eq. 2.3, 2.5, 2.6)	0.0 Fail to reject	0.7 Fail to reject
Distribution of $\tilde{\alpha}$ is the same for GME and GME-Nash models (19 df)	AA: 79.6 reject UA: 36.4 fail to reject	AA: 55.8 reject UA: 128.0 reject
Distribution of $\tilde{\alpha}$ is the same for ME-ML and GME-Nash models (19 df)	AA: 206.0 reject UA: 308.5 reject	AA: 56.0 reject UA: 128.0 reject

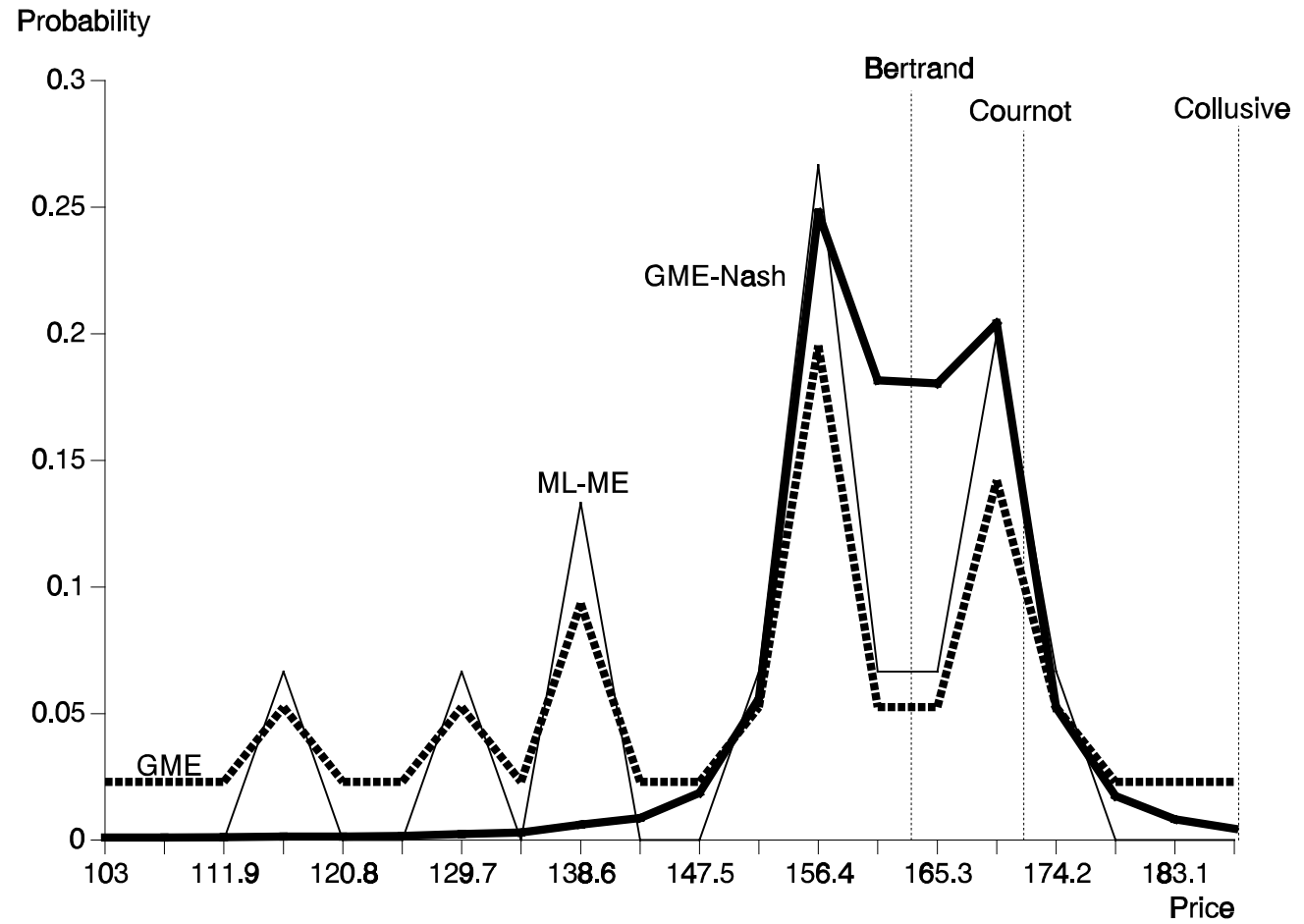
**Table 3: Sample Size Sampling Experiment ( $n = 20$ )**

	$MSE(\underline{\alpha}^1)$	$MSE(\underline{\alpha}^2)$	<i>Correlation, Firm 1</i>	<i>Correlation, Firm 2</i>
<i>T = 10</i>				
ME-ML	.285	.145	.68	.79
GME	.137	.068	.66	.77
GME-Nash <sup>1</sup>	.086	.037	.79	.89
GME-Nash <sup>2</sup>	.110	.060	.66	.76
<i>T = 20</i>				
ME-ML	.263	.104	.69	.84
GME	.132	.050	.66	.81
GME-Nash <sup>2</sup>	.075	.023	.77	.91
<i>T = 40</i>				
ME-ML	.245	.091	.70	.86
GME	.124	.049	.67	.80
GME-Nash <sup>2</sup>	.075	.026	.78	.90

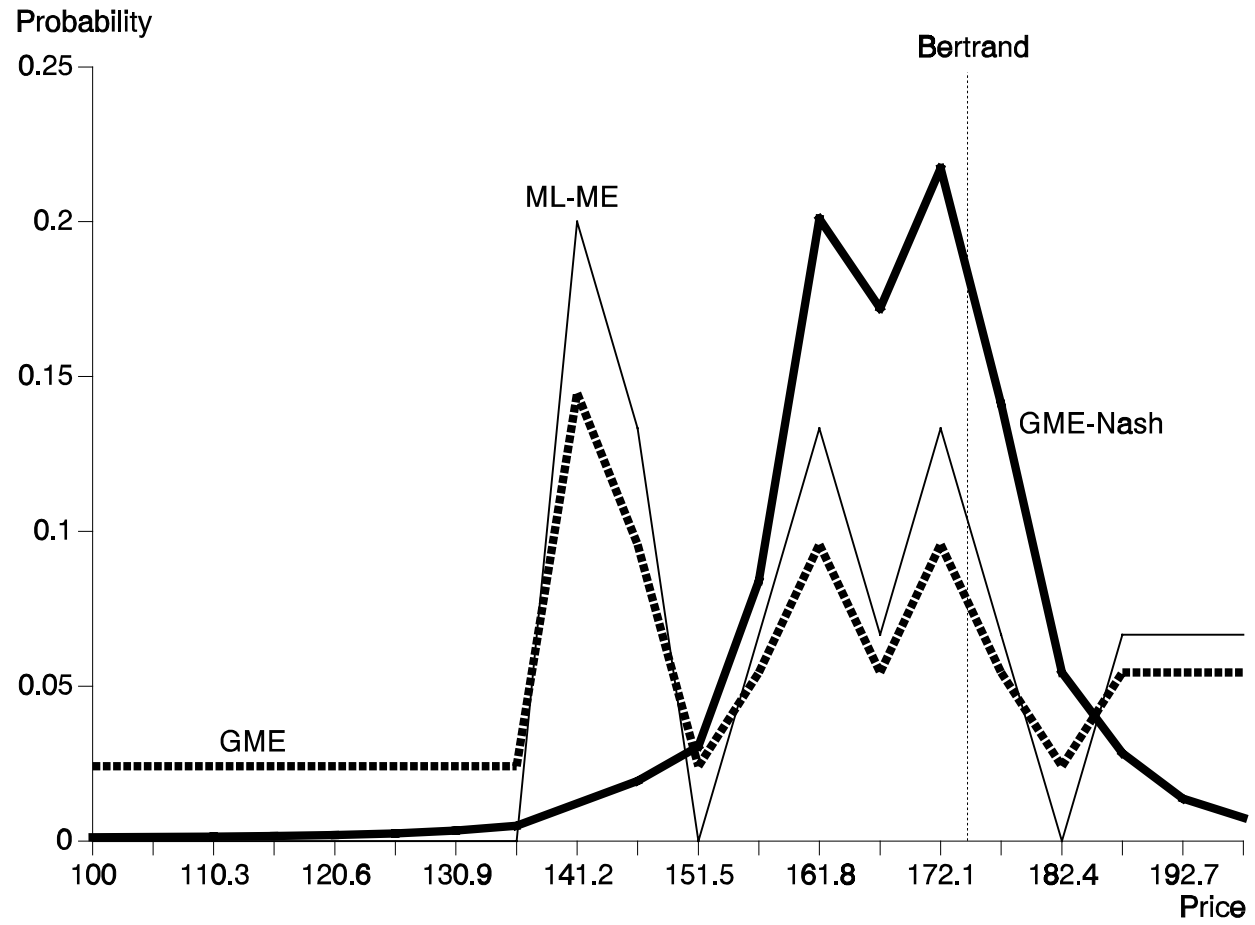
<sup>1</sup> Known demand coefficients.

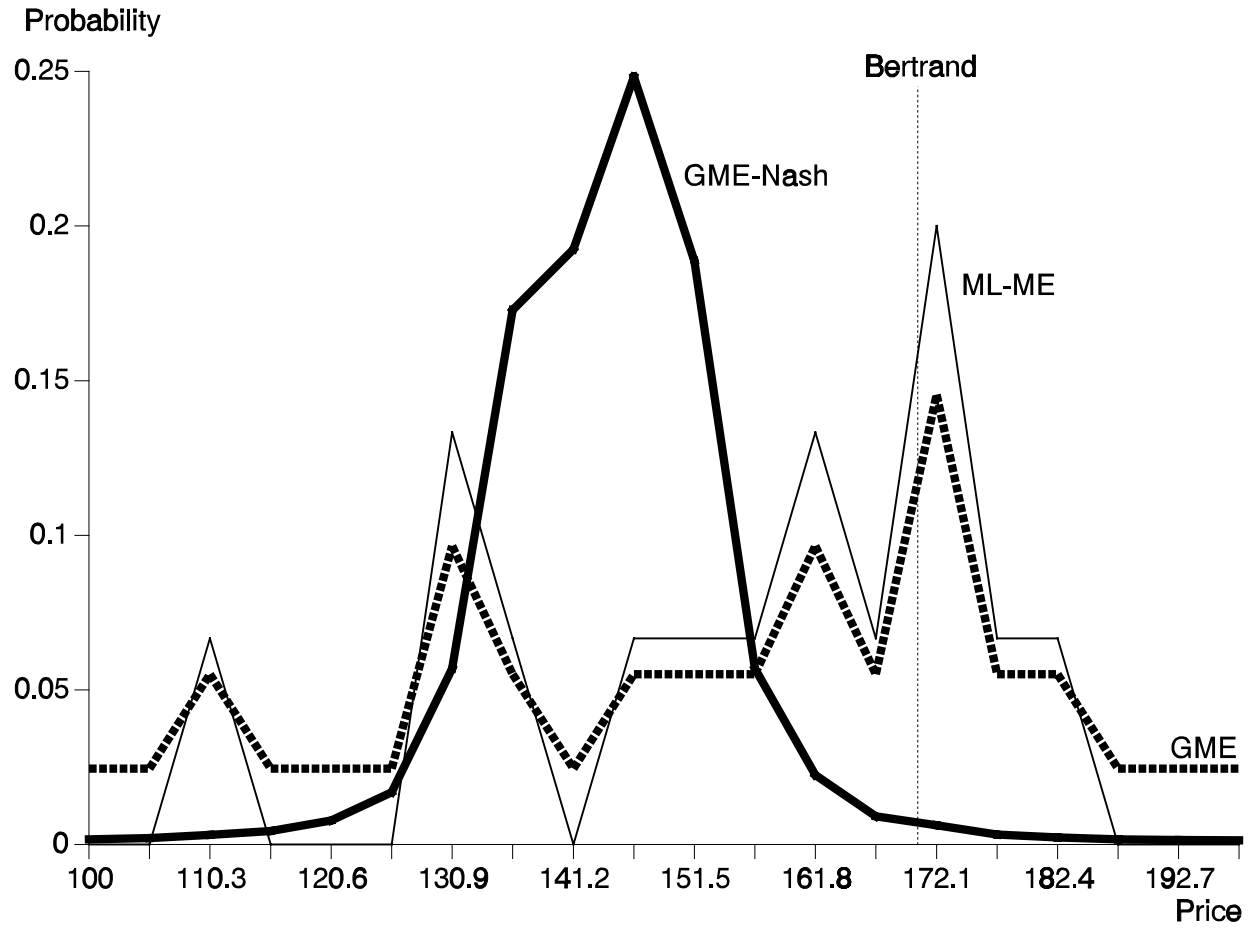
<sup>2</sup> Unknown demand coefficients.

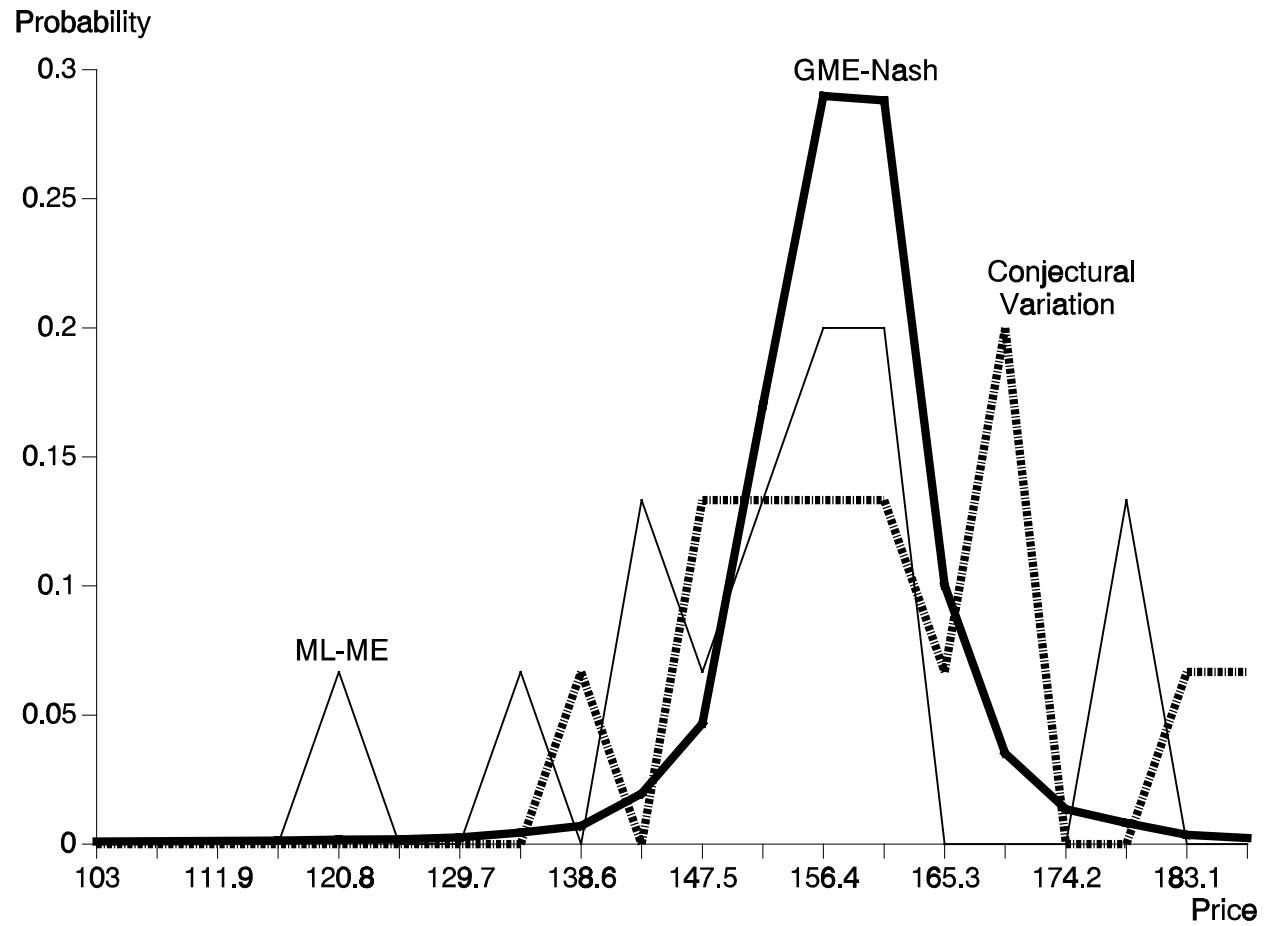
**Figure 1a:** Price Strategy of American Airlines, Chicago-Providence

**Figure 1b:** Price Strategy of United Airlines, Chicago-Providence

**Figure 2a:** Price Strategy of American Airlines, Chicago-Wichita



**Figure 2b:** Price Strategy of United Airlines, Chicago-Wichita

**Figure 3a:** Strategy and Conjectural Variation Models for American Airlines, Chicago-Providence

**Figure 3b:** Strategy and Conjectural Variation Models for United Airlines, Chicago-Providence