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Linear Demand Functions in Theory and Practice*

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Integrability conditions for an incomplete system of linear demand functions are considered. The parameter restrictions consistent with integrability are identified, and the structure of the conditional preference map is obtained. It is found that the conditional preferences for a set of linear demand functions are either quadratic or Leontief from a translated origin. Welfare analysis with linear demand models is also considered. *Journal of Economic Literature* Classification Numbers: 022, 024.

1. INTRODUCTION

One of the most frequently used functional forms in applied economic analysis of demand relationships is the linear model: quantity demanded is a linear function of a subset of prices and of income. Recent examples include [1, 2, 6, 9, 12, 14, 17]. It is quite important for analysts to understand the structural and practical implications of the functional forms chosen to approximate an underlying preference structure. For example, in a system of linear demand functions, a main result of this study is that there is no substitution between any of the goods unless the income effects are all zero. But one would presume that there is at least some degree of substitution between milk and cheese [9, 12, 17], between different types of meat [6, 14], or between types of recreation [1, 2].

The linear functional form is popular because of its simple structure, especially linearity in the parameters. But an analysis of the properties, restrictions, and implications of a set of linear demand equations on the behavior of the individual does not appear to have been previously carried out. The purpose of this paper is to present a characterization of the properties of models where a group of commodities have linear demand functions, to discuss the potential problems associated with these proper-

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ties, and to relate the use of such models to the measurement of welfare effects of changes in the prices in those commodities.

Two recent studies have integrated a single ordinary differential equation to recover the expenditure function for one linear demand function [7, 8]. Also, a useful procedure for the numerical approximation of the income compensation function for any system of demand equations that satisfy the integrability conditions has been very recently developed in [18]. However, the results of these studies do not provide any insight concerning the restrictions on the demand parameters nor the implications on the underlying preference function resulting from the imposition of the integrability conditions for more than one linear demand function.

Lau [13] considered the implications of integrability for a complete, algebraically uniform system of demand functions such that real prices have the form P_i/Y, where P_i is the price of good i and Y is total expenditure, or money income. He found that many commonly used functional forms, including the linear model as a special case, are inconsistent with integrability, or are consistent with integrability only under very restrictive conditions.

With the exception of Epstein [5], very little attention has been paid to incomplete demand systems, or to models that relax Lau's requirements of algebraic uniformity and the use of P_i/Y to obtain zero degree homogeneity. Both of these questions are of interest to the case of linear demands. For example, the adding up condition requires that at most $N-1$ of N goods can have demand functions that are linear in those $N-1$ real prices and real income. Also, one is often primarily concerned with the demands for a group of commodities which form only a subset of the household's budget. One may not care about the consumer's demands for other commodities, or there may be no data on these other commodities. In these circumstances the analyst must deal with an incomplete demand system, and there is no compelling reason to impose uniformity on the demand functions for the other goods, nor is there any clear argument for utilizing income as the deflator to obtain zero degree homogeneity. Indeed, the linear demand model is by construction inconsistent with the use of income as the deflator for all prices.

This paper solves the problem of integrating several linear demand functions to carry out a detailed analysis of the properties of such models. The Slutsky symmetry conditions are used to identify the parameter restrictions that are consistent with integrability, the structure of the expenditure function with respect to the prices for the goods with linear demands is obtained by imposing the symmetry conditions and solving a system of first-order partial differential equations defined by the demand functions, and the structure of the implied conditional preferences is analyzed.

All of the studies cited above utilize consumer's surplus in conjunction with linear demand models to estimate welfare measures from changes in the prices of the goods under study. Consequently, it is of some interest to study the relationship between consumer's surplus and an exact and general measure of welfare change such as equivalent variation [3]. Therefore, the relationship between consumer's surplus and equivalent variation for linear demand models is also considered.

The results of the analysis of the linear demand model may be summarized as follows. There are three possible ways for the parameters to satisfy the conditions necessary for integrability. (1) All income effects are zero and the matrix of cross-price effects for the linear demands is symmetric, negative semidefinite. (2) All income effects are non-zero and have the same sign and the parameters of the linear demand model must satisfy a set of nonlinear constraints. (3) A subset of the linear demands are independent of *all* prices and income, and the remaining linear demand functions have non-zero income effects of the same sign and their parameters satisfy nonlinear constraints analogous to those in case (2).

If the income effects are all zero, then the conditional preference map for the commodities with linear demands is quadratic. If the income effects are all non-zero, then the conditional preference map is Leontief from a translated origin. If the income effects of, say, the first $n^0 < n$ goods are non-zero but the income effects for the remaining $n - n^0$ linear demands are zero, then the conditional preference map for the first n^0 goods is again Leontief from a translated origin, while the remaining $n - n^0$ commodities with linear demands represent a "subsistence" level of extreme necessity, but do not yield any utility beyond the constant consumption level of the perfectly inelastic demand functions. This last case is of little interest except insofar as it represents perhaps the most extreme level of restriction possible implied by the integrability conditions for a set of demand equations. Consequently, it is not considered in the analysis of the use of linear demand models for welfare analysis.

The results that are obtained from the analysis of the use of consumer's surplus for welfare analysis in linear demand models may be summarized as follows: (1) in the case where all income effects are zero and cross-price effects are symmetric, the conditions are met for consumer's surplus, equivalent variation and compensating variation to be equal; (2) if all income effects are non-zero and of the same sign, then consumer's surplus is uniquely defined if and only if the Leontief conditional preference map is homothetic, i.e., all income elasticities are equal for the commodities with linear demands; (3) under homotheticity of the Leontief structure, the conditions of Dixit and Weller [4] are satisfied, hence $CV \leqslant CS \leqslant EV$ if all of the goods with linear demands are normal, and $EV \leqslant CS \leqslant CV$ if all of the goods with linear demands are inferior, where CS is consumer's surplus, CV is compensating variation, and EV is equivalent variation. Therefore, if the linear demand model is specified and restricted so as to satisfy the conditions of integrability, then path independence of the consumer's surplus line integral is a sufficient condition for consumer's surplus to be a reasonable welfare metric. However, the cost of this property is relatively high, and probably unpalatable for most applied situations.

The paper is organized as follows. Section 2 presents a discussion of the approach used for analyzing integrability of incomplete systems of demand equations. Section 3 contains a statement and proof of the main results concerning integrability of a set of linear demand functions. Section 4 discusses some examples of special interest. Section 5 presents the results with respect to welfare analysis and consumer's surplus, and the last section contains a summary and concluding remarks.

2. Integration of Incomplete Demand Systems

We start by introducing a little notation. Let $x=(x_1,..., x_n)$ be the vector of consumption levels for the set of commodities in which the analyst is interested, and let $P=(P_1,..., P_n)$ be the corresponding price vector. Let $z=(z_1,..., z_m)$ be the consumpton levels of all other goods, and $Q=(Q_1,..., Q_m)$ the corresponding price vector. It is convenient to work with normalized prices and income, and many applications employ a general price deflator to represent the cost of other goods. Therefore, let $\pi(Q) \equiv \pi(Q_1,..., Q_m)$ be a known, twice continuously differentiable, positive valued, nondecreasing (strictly increasing in some Q_j), linear homogeneous, and concave function of any nonempty subset of the prices of other goods. The normalized price vectors are $p=(p_1,..., p_n) \equiv (P_1/\pi(Q),..., P_n/\pi(Q))$, $q=(q_1,..., q_m) \equiv (Q_1/\pi(Q),..., Q_m/\pi(Q))$, and normalized income is $y \equiv Y/\pi(Q)$. The ordinary demand system which the analyst estimates is

$$x_i = h^i(p, q, y), \qquad i=1,..., n. \tag{1}$$

In addition to (1), there are demands $z_j = \bar{h}^j(p, q, y)$, $j=1,..., m$, but these are not observed by the analyst. We wish to consider the set of parameters for the incomplete demand system (1) consistent with integrability, and the implied structure of the consumer's underlying conditional preferences for the x's. Since the demands for the z's are not known, it is not possible to recover the complete preference relation, and we are looking for a solution to the normalized Hotelling's Lemma [10, 16],

$$\partial e(p, q, u)/\partial p \equiv g(p, q, u) \equiv h[p, q, e(p, q, u)], \tag{2}$$

where $g(p, q, u)$ is the n-vector of compensated demands for x, $e(p, q, u)$ is the normalized expenditure function, and u is the consumer's level of utility.

As Epstein has pointed out, there are some important differences between the integrability conditions for complete and incomplete demand systems. In this context, it is necessary to distinguish between global and local integrability. Let D be the domain of the demand system $(h, \bar{h})$ and $X \times Z$ the range. Global integrability pertains to the existence of a continuous, monotone, quasi-concave utility function, u, defined over $X \times Z$ such that the values of the demand functions $(h, \bar{h})$ at (p, q, y) maximize $u(x, z)$ subject to $p^{\tau}x + q^{\tau}z \leqslant y$ for any $(p, q, y) \in D$. Local integrability, by contrast, pertains to the existence of a well-behaved utility function which generates $(h, \bar{h})$ in the neighborhood of some point (p^0, q^0, y^0) in the interior of D.

Assume that the demand functions (1) are twice continuously differentiable and define the normalized Slutsky terms

$$s_{ij} \equiv \partial h^i(p, q, y)/\partial p_j + h^j(p, q, y)\, \partial h^i(p, q, y)/\partial y, \qquad i,j = 1,..., n. \tag{3}$$

For complete demand systems, necessary and sufficient conditions for the global integrability of (1) consist of the non-negativity and smoothness of the demand functions, the adding up condition, and symmetry and negative semidefiniteness of the $n \times n$ Slutsky matrix $S \equiv [s_{ij}]$ (Hurwicz and Uzawa [11]). Epstein has shown that for incomplete systems, the sufficient conditions for local integrability of (1) must be strengthened to negative definiteness of S and that expenditure on the first n commodoties must be strictly less than total expenditure, $y > p^{\tau}h(p, q, y)$. In general, however, sufficient conditions for the global integrability of (1) to an expenditure function that is regular with respect to p and q are complex and difficult to apply.

Suppose that the subsystem of demands (1) is integrable. Then upon integrating (2) we obtain a solution for the normalized expenditure function of the form $e(p, q, u) = \varepsilon(p, q, \theta)$, where $\theta \equiv \theta(q, u)$ is the unknown "constant" of integration for the incomplete system (2) which depends in general upon (q, u) but not upon p. Following Hausman [8], $\varepsilon(p, q, \theta)$ is referred to as the "quasi-expenditure function" for the incomplete system (2). It is related to the true expenditure function $E(P, Q, u)$, assuming that it exists, by the identity,

$$E(P, Q, u) \equiv \pi(Q)\, \varepsilon\, [p, q, \theta(q, u)] \equiv \pi(Q)\, e(p, q, u). \tag{4}$$

The quasi-expenditure function contains sufficient information to enable us to calculate exact welfare measures for changes in p and to ascertain the consumer's conditional preferences for the x's. This can be seen as follows.

Since $\theta(q, u)$ is a transformation of u, we may treat θ as a utility index and invert $\varepsilon(p, q, \theta)$ with respect to θ to obtain the "quasi-indirect utility function," $\theta \equiv \Phi(p, q, y)$, where Φ is the inverse of ε with respect to θ. This is related to the true indirect utility function by the identity

$$v(p, q, y) \equiv \Omega[q, \Phi(p, q, y)], \tag{5}$$

where $u = \Omega(q, \theta)$ is the inverse of $\theta(q, u)$ with respect to u. Since $v(p, q, y)$ is montonic in y, it follows that the normalized equivalent variation, ev, for a change in prices from p^0 to p' satisfies

$$\Phi(p^0, q, y + ev) = \Phi(p', q, y). \tag{6}$$

Therefore, ev can be calculated directly from $\Phi(\cdot)$ without complete knowledge of $\Omega(\cdot)$ or $v(\cdot)$.

Now consider obtaining the direct utility function $u(x, z)$ from the true indirect utility function as the solution to

$$u(x, z) \equiv \inf\{v(p, q, y): p^\tau x + q^\tau z \leqslant y, p \geqslant 0, q \geqslant 0\}, \tag{7}$$

where $\inf\{\cdot\}$ is the greatest lower bound. Since $v(p, q, y) \equiv \Omega[q, \Phi(p, q, y)]$, the solution to (7) can be obtained by applying a 2-stage procedure. In the first stage we minimize $v(p, q, y)$ with respect to p for given q. Assuming that $x \gg 0$ and v is strictly increasing in y, this is equivalent to solving

$$\inf\{\Phi(p, q, p^\tau x + q^\tau z): p \geqslant 0\}. \tag{8}$$

For an interior solution to (8) with $p \gg 0$, the necessary conditions are

$$\partial\Phi/\partial p_i + x_i \partial\Phi/\partial y = 0, \qquad i = 1,..., n. \tag{9}$$

Solving (9) for $p(x, q, q^\tau z)$, we obtain the "quasi-direct utility function,"

$$\hat{u}(x, q, q^\tau z) \equiv \Phi[p(x, q, q^\tau z), q, p(x, q, q^\tau z)^\tau x + q^\tau z]. \tag{10}$$

The quasi-direct utility function is related to the true direct utility function, $u(x, z)$, by

$$u(x, z) \equiv \inf\{\Omega[q, \hat{u}(x, q, q^\tau z)]: q \geqslant 0\}, \tag{11}$$

which is the second-stage problem. Applying the Envelope theorem to (11), it follows that

$$(\partial u/\partial x_i)/(\partial u/\partial x_j) = (\partial\hat{u}/\partial x_i)/(\partial\hat{u}/\partial x_j), \qquad i, j = 1,..., n, \tag{12}$$

so that any structural relationship between $\{x_1, ..., x_n\}$ given $\{z_1, ..., z_m\}$ is embodied in $\hat{u}(x, q, q^\tau z)$.

If the demand functions (1) are twice continuously differentiable, then the necessary and sufficient conditions for global integrability of (2) to a quasi-expenditure function $\varepsilon(p, q, \theta)$ that is regular only with respect to p are the same for incomplete systems as for complete systems except that $y > p^\tau x$ for all $(p, q, y) \in D$ is required to allow for positive expenditures on the other commodities. Symmetry and negative semidefiniteness of S guarantees the existence of a solution $\varepsilon(p, q, \theta)$ that is concave in p, while $h^i(p, q, y) > 0$ for all $(p, q, y) \in D$, $i = 1, ..., n$, guarantees that $\varepsilon(p, q, \theta)$ is strictly increasing in p. The existence and regularity of $\varepsilon(p, q, \theta)$ with respect to p is in turn a necessary, though not sufficient, condition for the existence and regularity of the true expenditure function $E(P, Q, u)$. Consequently, the discussion of the integrability of linear demand models focuses on the existence and properties of the quasi-functions $\varepsilon(p, q, \theta)$, $\Phi(p, q, y)$, and $\hat{u}(x, q, q^\tau z)$. Whenever the entire system of demands for the x's and z's is integrable, the structure of the conditional preferences for the x's is contained in the corresponding structure of these functions.

3. Integrability of Linear Demand Models

The linear demand model for an incomplete system of n demands out of $n+m$ commodities that we consider in this section can be written as

$$x_i = \alpha_i(q) + \sum_{j=1}^{n} \beta_{ij} p_j + \mu_i y, \qquad i = 1, ..., n, \tag{13}$$

where each $\alpha_i(\cdot)$ is assumed to be twice continuously differentiable. The normalized Slutsky terms for the n goods with linear demands can be written as

$$s_{ij} = \beta_{ij} + x_j \mu_i, \qquad \forall i, j = 1, ..., n. \tag{14}$$

For an open nonempty subset D of the strictly positive orthant of R^{n+m+1}, assume that $h^i(p, q, y) > 0$ $\forall i = 1, ..., n$ and $y > \sum_{i=1}^{n} p_i h^i(p, q, y)$ $\forall (p, q, y) \in D$. With these preliminaries, we can now state and prove the main result.

THEOREM 1. *The system of demands* (13) *is integrable to a quasi-expenditure function* $\varepsilon(p, q, \theta)$ *that is regular with respect to p on the set D if and only if one of the following holds*:

(1) $\mu_i = 0, \forall i = 1,..., n$ *and* $B \equiv [\beta_{ij}]$ *is symmetric, negative semidefinite, with the functions* $\varepsilon(\cdot)$ and $\Phi(\cdot)$ *defined by*

$$\varepsilon(p, q, \theta) = \sum_{i=1}^{n} \alpha_i(q) p_i + \tfrac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \beta_{ij} p_i p_j + \theta$$

$$\Phi(p, q, y) = y - \sum_{i=1}^{n} \alpha_i(q) p_i - \tfrac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \beta_{ij} p_i p_j;$$

(2) $\operatorname{sgn}(\mu_i) = \operatorname{sgn}(\mu_1) \neq 0$, $\alpha_i(q) \equiv \mu_i[\alpha_1(q) + \beta_{11}/\mu_1 - \beta_{1i}/\mu_i]/\mu_1$, $\beta_{ij} = \beta_{1j}\mu_i/\mu_1$, $\forall i, j = 1,..., n$ *and* $(\beta_{11} + \mu_1 x_1) \leqslant 0\ \forall (p, q, y) \in D$, *with the functions* $\varepsilon(\cdot)$ *and* $\Phi(\cdot)$ *defined by*

$$\varepsilon(p, q, \theta) = \theta \cdot \exp\left\{\sum_{i=1}^{n} \mu_i p_i\right\} - (1/\mu_1)\left(\sum_{i=1}^{n} \beta_{1i} p_i + \alpha_1(q) + \beta_{11}/\mu_1\right)$$

$$\Phi(p, q, y) = \left[y + (1/\mu_1)\left(\sum_{i=1}^{n} \beta_{1i} p_i + \alpha_1(q) + \beta_{11}/\mu_1\right)\right] \cdot \exp\left\{-\sum_{i=1}^{n} \mu_i p_i\right\};$$

(3) $\operatorname{sgn}(\mu_i) = \operatorname{sgn}(\mu_1) \neq 0$, $\alpha_i(q) \equiv \mu_i[\alpha_1(q) + \beta_{11}/\mu_1 - \beta_{1i}/\mu_i]/\mu_1$, $\beta_{ij} = \beta_{1j}\mu_i/\mu_1$, $\forall i = 1,..., n^0 < n$, $\forall j = 1,..., n$, $\alpha_i(q) \equiv -\beta_{1i}/\mu_1 > 0$,

$\beta_{ij} = \mu_i = 0$, $\forall i = n^0 + 1,..., n$, $\forall j = 1,..., n$, *and* $(\beta_{11} + \mu_1 x_1) \leqslant 0\ \forall (p, q, y) \in D$, *with the functions* $\varepsilon(\cdot)$ *and* $\Phi(\cdot)$ *defined by*

$$\varepsilon(p, q, \theta) = \theta \cdot \exp\left\{\sum_{i=1}^{n^0} \mu_i p_i\right\} - (1/\mu_1)\left(\sum_{i=1}^{n} \beta_{1i} p_i + \alpha_1(q) + \beta_{11}/\mu_1\right),$$

$$\Phi(p, q, y) = \left[y + (1/\mu_1)\left(\sum_{i=1}^{n} \beta_{1i} p_i + \alpha_1(q) + \beta_{11}/\mu_1\right)\right] \cdot \exp\left\{-\sum_{i=1}^{n^0} \mu_i p_i\right\}.$$

Proof. Sufficiency is obvious upon application of Roy's identity [22] or Hotelling's lemma, hence only necessity will be proven here. Therefore, suppose the demands (13) are integrable. Expanding the normalized Slutsky terms and imposing symmetry requires

$$\alpha_i(q) \mu_j - \alpha_j(q) \mu_i + \beta_{ji} - \beta_{ij} + \sum_{k=1}^{n} (\beta_{ik}\mu_j - \beta_{jk}\mu_i) p_k = 0 \quad \forall i, j = 1,..., n. \qquad (15)$$

Moreover condition (15) must hold identically throughout D. For any pair i, j with $i \neq j$ there are three relevant possibilities for the income coefficients:

(i) $\mu_i = \mu_j = 0$; (ii) $\mu_i \neq 0$, $\mu_j \neq 0$; (iii) $\mu_i \neq 0$, $\mu_j = 0$.

If (i) holds, then (15) implies that $\beta_{ij} = \beta_{ji}$. If (ii) holds, it follows that

$\beta_{ik}\mu_j = \beta_{jk}\mu_i \forall k$ and $\alpha_j(q)\,\mu_i \equiv \alpha_i(q)\,\mu_j + \beta_{ji} - \beta_{ij}$. Further, by (14) we have the identity $x_j \equiv (x_i\mu_j + \beta_{ji} - \beta_{ij})/\mu_i \forall (p, q, y) \in D$. That is, the demands for x_i and x_j lie on the ray with slope (μ_j/μ_i) through the point $[0, (\beta_{ji} - \beta_{ij})/\mu_i]$ in the (x_i, x_j) plane $\forall (p, q, y) \in D$. By Lemma 1 in the Appendix, this property can hold if and only if the conditional preference map for (x_i, x_j) is Leontief from a translated origin. The *L*-shaped Leontief conditional indifference curves with corners on this ray will not cross only if $\operatorname{sgn}(\mu_i) = \operatorname{sgn}(\mu_j)$. That is, the consumer's preference function will be increasing in both x_i and x_j only if their income effects have the same sign. Now consider (iii). In this case (15) requires $\beta_{jk} = 0 \,\forall k$ and $\alpha_j(q) \equiv -\beta_{ij}/\mu_i > 0$ only if $\operatorname{sgn}(\mu_i) = -\operatorname{sgn}(\beta_{ij})$. Therefore, if some but not all of the income effects are zero, then those demands will be perfectly inelastic, and independent of *all* prices and income. If the income effects are non-zero for more than one good, then the restrictions for case (ii) applies to those goods, and the case of non-zero, symmetric cross price effects with zero income effects is only relevant to the situation, where all income effects are zero. If some but not all of the income effects are zero, redefine the indices if necessary so that the first n^0 goods have non-zero income coefficients and the remaining $n - n^0$ goods all have zero income coefficients. Then the parameter restrictions in (3) follow, except that it remains to be shown that $s_{11} = (\beta_{11} + \mu_1 x_1) \leqslant 0$ is the only other necessary condition for regularity of the quasi-expenditure function. If all of the income coefficients are zero, then the parameter restrictions of (1) follow with $S = B$ implying symmetry and negative semidefiniteness of B. Finally, if all income coefficients are non-zero, then they must clearly satisfy conditions (2), except that it remains to be proven for this case also that $(\beta_{11} + \mu_1 x_1) \leqslant 0$ is the remaining necessary condition for regularity of $\varepsilon(p, q, \theta)$.

When all income effects are zero, the system of partial differential equations arising from Hotelling's lemma is

$$\partial e(p, q, u)/\partial p_i = \alpha_i(q) + \sum_{j=1}^{n} \beta_{ij}\, p_j, \qquad i = 1, ..., n. \tag{16}$$

Integrating with respect to p_1 gives

$$\begin{aligned}&\varepsilon[p_1; p_2, ..., p_n, q, \theta_1(p_2, ..., p_n, q, \theta)]\\ &\quad = \alpha_1(q)\, p_1 + \tfrac{1}{2}\beta_{11}\, p_1^2 + \sum_{j=2}^{n} \beta_{1j}\, p_1\, p_j + \theta_1(p_2, ..., p_n, q, \theta).\end{aligned}$$

Partially differentiating this expression with respect to p_2, setting the result equal to $h^2[p, q, e(p, q, u)]$ and utilizing the symmetry condition $\beta_{12} = \beta_{21}$ gives

$$\partial\theta_1(p_2,..., p_n, q, \theta)/\partial p_2 = \alpha_2(q) + \sum_{j=2}^{n} \beta_{2j} p_j.$$

Direct integration of this expression then gives the result

$$\begin{aligned} &\varepsilon[p_1, p_2; p_3,..., p_n, q, \theta_2(p_3,..., p_n, q, \theta)] \\ &\quad = \alpha_1(q) p_1 + \alpha_2(q) p_2 + \tfrac{1}{2}(\beta_{11} p_1^2 + 2\beta_{12} p_1 p_2 + \beta_{22} p_2^2) \\ &\quad + \sum_{i=1}^{2} \sum_{j=3}^{n} \beta_{ij} p_i p_j + \theta_2(p_3,..., p_n, q, \theta). \end{aligned}$$

Continuing in this fashion through $i = 3,..., n$, we obtain the quasi-expenditure function for the case where all income effects are zero and the matrix of cross price effects is symmetric negative semidefinite as

$$\varepsilon(p, q, \theta) = \sum_{i=1}^{n} \alpha_i(q) p_i + \tfrac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \beta_{ij} p_i p_j + \theta. \tag{17}$$

Setting $y = \varepsilon(p, q, \theta)$, the inverse of $\varepsilon(p, q, \theta)$ with respect to θ is given by

$$\Phi(p, q, y) = y - \sum_{i=1}^{n} \alpha_i(q) p_i - \tfrac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \beta_{ij} p_i p_j. \tag{18}$$

This establishes the validity of (1).

When all income effects are non-zero and of the same sign, the system of partial differential equations can be written as

$$\begin{aligned} \partial e(p, q, u)/\partial p_i = \mu_i \Bigg[&\alpha_1(q) + \beta_{11}/\mu_i - \beta_{1i}/\mu_1 \\ &+ \sum_{j=1}^{n} \beta_{1j} p_j + \mu_1 e(p, q, u) \Bigg] \Bigg/ \mu_1, \end{aligned} \tag{19}$$

where $i = 1,..., n$. In this case, the integrating factor $\exp\{-\sum_{j=1}^{n} \mu_j p_j\}$ makes the system exact, and integration with respect to p_1 by applying integration by parts on the right-hand side yields the result

$$\varepsilon(p, q, \theta) = \theta \cdot \exp\left\{\sum_{i=1}^{n} \mu_i p_i\right\} - (1/\mu_1)\left(\sum_{i=1}^{n} \beta_{1i} p_i + \alpha_1(q) + \beta_{11}/\mu_{11}\right), \tag{20}$$

and upon setting $y = \varepsilon(p, q, \theta)$ and inverting this expression with respect to θ gives the quasi-indirect utility function as

$$\Phi(p, q, y) = \left[y + (1/\mu_1)\left(\sum_{i=1}^{n} \beta_{1i} p_i + \alpha_1(q) + \beta_{11}/\mu_1 \right) \right] \cdot \exp\left\{ - \sum_{i=1}^{n} \mu_i p_i \right\}. \tag{21}$$

Upon differentiating the quasi-expenditure function with respect to p twice, we obtain the Slutsky matrix

$$\begin{aligned} S &= \theta \cdot \exp\{\mu^\tau p\}\, \mu\mu^\tau \\ &= (1/\mu_1^2)[\mu_1(\alpha_1(q) + B_{1.}\, p + \mu_1 e(p, q, u)) + \beta_{11}]\, \mu\mu^T \\ &= (1/\mu_1^2)(\beta_{11} + \mu_1 x_1)\, \mu\mu^\tau \end{aligned} \tag{22}$$

where $\mu \equiv [\mu_1, ..., \mu_n]^\tau$, and $B_{1.} \equiv [\beta_{11}, ..., \beta_{1n}]$. So S is negative semidefinite if and only if $(\beta_{11} + \mu_1 x_1) \leqslant 0\ \forall (p, q, y) \in D$, completing the proof of case (2). The only other set of parameter values consistent with the Slutsky symmetry restrictions are those defined by (3) in the statement of Theorem 1. In this case, the system of partial differential equations is given by

$$\begin{aligned} \partial e(p, q, u)/\partial p_i &= \mu_i \left[\alpha_i(q) + \beta_{11}/\mu_i - \beta_{1i}/\mu_1 + \sum_{j=1}^{n} \beta_{1j} p_j + \mu_1 e(p, q, u) \right] \Big/ \mu_1, \\ &\qquad\qquad i = 1, ..., n^0, \\ &= -\beta_{1i}/\mu_1, \qquad i = n^0 + 1, ..., n, \end{aligned} \tag{23}$$

and the integrating factor $\exp\{-\sum_{i=1}^{n^0} \mu_i p_i\}$ makes the system of partial differential equations exact. Integration by parts with respect to p_1 yields the result

$$\varepsilon(p, q, \theta) = \theta \cdot \exp\left\{ \sum_{i=1}^{n^0} \mu_i p_i \right\} - (1/\mu_1)\left(\sum_{i=1}^{n} \beta_{1i} p_i + \alpha_1(q) + \beta_{11}/\mu_1 \right). \tag{24}$$

Again setting $y = \varepsilon(p, q, \theta)$, and inverting the quasi-expenditure function with respect to θ, we have

$$\Phi(p, q, y) = \left[y + (1/\mu_1)\left(\sum_{i=1}^{n} \beta_{1i} p_i + \alpha_1(q) + \beta_{11}/\mu_1 \right) \right] \cdot \exp\left\{ - \sum_{i=1}^{n^0} \mu_i p_i \right\}. \tag{25}$$

For this case, the Slutsky matrix for the x's has the form

$$S = \begin{bmatrix} (1/\mu_1^2)(\beta_{11} + \mu_1 x_1)\, \mu_{(n^0)} \mu_{(n^0)}^\tau & 0 \\ 0 & 0 \end{bmatrix} \tag{26}$$

where $\mu_{(n^0)} \equiv [\mu_1, \ldots, \mu_{n^0}]^\tau$. Hence S is negative semidefinite if and only if $(\beta_{11} + \mu_1 x_1) \leqslant 0 \; \forall (p, q, y) \in D$. Q.E.D.

Now consider the question of recovering the quasi-direct utility function and the conditional preference relation for the x's. For the case of zero income effects, the demand equations can be written in matrix notation as

$$x = \alpha(q) + Bp. \tag{27}$$

From this it is clear that a necessary and sufficient condition for the quantity dependent demands to be invertible is that B be nonsingular, hence negative definite. When this is true, the price dependent demands $p(x, q, q^\tau z)$ may be written

$$p(x, q, q^\tau z) = B^{-1}[x - \alpha(q)]. \tag{28}$$

Combining this with the budget constraint, $y = p^\tau x + q^\tau z$, and the quasi-indirect utility function, we obtain the quasi-direct utility function

$$\hat{u}(x, q, q^\tau z) = \tfrac{1}{2}[x - \alpha(q)]^\tau B^{-1}[x - \alpha(q)] + q^\tau z. \tag{29}$$

When all of the income effects are non-zero, we have the conditions

$$x_i \equiv \mu_i(x_1 + \beta_{11}/\mu_1 - \beta_{1i}/\mu_i)/\mu_1, \qquad i = 1, \ldots, n, \tag{30}$$

as well as the budget equation. Upon substituting each of these into the quasi-indirect utility function we obtain

$$\begin{aligned} \Phi(p, q, p^\tau x + q^\tau z) = (1/\mu_1)[\alpha_1(q) + \beta_{11}/\mu_1 + \mu^\tau p(x_1 + \beta_{11}/\mu_1) \\ + \mu_1 q^\tau z] \cdot \exp\{-\mu^\tau p\}. \end{aligned} \tag{31}$$

Note that the Leontief conditional preferences imply that the consumer reacts to the single composite price $\mu^\tau p/\mu_1 > 0$. Therefore, minimize the quasi-indirect utility function with respect to this single variable, with necessary condition

$$(\beta_{11} + \mu_1 x_1)(\mu^\tau p/\mu_1) = (x_1 - \alpha_1(q) - \mu_1 q^\tau z). \tag{32}$$

In order that this expression can be solved for $\mu^\tau p/\mu_1$ it is necessary that $(\beta_{11} + \mu_1 x_1) < 0$, that is the expenditure function must be strictly concave in the composite real price $\mu^\tau p/\mu_1$, since

$$\partial^2 \in (p, q, \theta)/\partial(\mu^\tau p/\mu_1)^2 = \mu_1^2 \theta \cdot \exp\{\mu^\tau p\} = \beta_{11} + \mu_1 x_1. \tag{33}$$

Also, a positive composite real price implies $x_1 < \alpha_1(q) + \mu_1 q^\tau z$. Therefore, assume both strict inequalities hold, substitute the solution into the quasi-indirect utility function, and replace x_1 with the Leontief function

$$T(x) = \min\{x_1, \mu_1(x_2 + \beta_{12}/\mu_1 - \beta_{11}/\mu_2)/\mu_2, \ldots, \mu_1(x_n + \beta_{1n}/\mu_1 - \beta_{11}/\mu_n)/\mu_n\} \tag{34}$$

to obtain the quasi-direct utility function as

$$\hat{u}(x, q, q^\tau z) = (1/\mu_1^2)[\beta_{11} + \mu_1 T(x)] \cdot \exp\{\mu_1[\alpha_1(q) + \mu_1 q^\tau z - T(x)]/ \times [\beta_{11} + \mu_1 T(x)]\}. \tag{35}$$

In the case where the demands for the last $n - n^0$ x's are independent of all prices and income, the quasi-indirect utility function has the same form as (31) except that when the constraints $x_i \equiv -\beta_{1i}/\mu_1$ for $i = n^0 + 1, \ldots, n$ are substituted into $\Phi(p, q, p^\tau x + q^\tau z)$, the last $n - n^0$ price variables drop out of the resulting expression. This is because the quasi-indirect utility function is linear in y and in $[p_{n^0+1}, \ldots, p_n]$, and may be interpreted in the following way. The commodities $[x_{n^0+1}, \ldots, x_n]$ are absolute necessities in the quantities given by the levels of the perfectly inelastic demands, but do not provide any additional satisfaction beyond that level of consumption. Therefore, in order to "survive," the consumer's real income must be such that $y \geqslant -\sum_{n^0+1}^{n} \beta_{1i} p_i/\mu_1$, and all income above this level is allocated to the goods $[x_1, \ldots, x_{n^0}, z]$. Since the quasi-direct utility function (35) is less than zero throughout the region of regular behavior with respect to x, we can define the quasi-direct utility function that motivates the linear demand model with n^0 non-zero income coefficients and $n - n^0$ zero income coefficients by

$$\begin{aligned}
\hat{u}(x, q, q^\tau z) &= -\infty \qquad \text{if } \min\{x_{n^0+1} + \beta_{1n^0+1}/\mu_1, \ldots, x_n + \beta_{1n}/\mu_1\} < 0, \\
&= (1/\mu_1^2)[\beta_{11} + \mu_1 \hat{T}(x_{(n^0)})] \\
&\quad \times \exp\{\mu_1[\alpha_1(q) + \mu_1 q^\tau z - \hat{T}(x_{(n^0)})]/[\beta_{11} + \mu_1 \hat{T}(x_{(n^0)})]\}, \\
&\quad \text{if } \min\{x_{n^0+1} + \beta_{1n^0+1}/\mu_1, \ldots, x_n + \beta_{1n}/\mu_1\} \geqslant 0,
\end{aligned} \tag{36}$$

with

$$\hat{T}(x_{(n^0)}) = \min\{x_1, \mu_1(x_2 + \beta_{12}/\mu_1 - \beta_{11}/\mu_2)/\mu_2, \ldots, \mu_1(x_{n^0} + \beta_{1n^0}/\mu_1 - \beta_{11}/\mu_{n^0})/\mu_{n^0}\}. \tag{37}$$

Since the preference function does not explicitly depend upon the last $n^0 - n$ x's, this clearly is not a very interesting case.

As a result of this development we have the following characterization of the conditional preferences for linear demand models.

THEOREM 2. *If the linear demand system* (13) *is integrable to a regular quasi-expenditure function* $\varepsilon(p, q, \theta)$ *on the set* D *and the demand functions can be inverted to obtain the functions* $p(x, q, q^{\tau}z)$, *then*

(1) *if all income coefficients are zero, then the conditional preference map for the* x*'s is quadratic and given by* (29);

(2) *if all income coefficients are non-zero, then the conditional preference map for the* x*'s is Leontief from a translated origin of the form* (34);

(3) *if* $n^0 < n$ *income coefficients are non-zero and the remaining* $n - n^0$ *are zero, then the conditional preference map for the* n^0 *goods with non-zero income coefficients is Leontief from a translated origin, while the* $n - n^0$ *goods with zero income coefficients are absolute necessities from which the consumer derives no direct utility as in* (36).

4. EXAMPLES

In this section, some examples of special interest are presented and their properties discussed. The income coefficients in these examples are assumed to be either all zero or else all non-zero.

1. Let $n = 1$, so that we are dealing with a single linear demand equation of the form

$$x = \alpha(q) + \beta p + \mu y. \tag{38}$$

Then there are no symmetry conditions to consider, and the necessary and sufficient conditions for integrability to a quasi-expenditure function that is increasing and concave in p are: (i) $\alpha(q) + \beta p + \mu y \geqslant 0$, and (ii) $\beta + \mu x \leqslant 0$ for all $(p, q, y) \in D$. In particular, Hausman has argued that this is a flexible functional form over the set of prices and income levels such that conditions (i) and (ii) hold with strict inequality. The functional expressions for the quasi-functions can be obtained directly from Theorems 1 and 2 in both the zero and non-zero income coefficient cases by simply setting $n = 1$.

2. In this example, let $m = 1$ so that $\pi(Q) \equiv Q$, and there is only one other good. Then $q \equiv 1$, and the linear demands have the form

$$x_i = \alpha_i + \sum_{j=1}^{n} \beta_{ij} p_j + \mu_i y, \qquad i = 1, \ldots, n. \tag{39}$$

This is a complete system, since the demand for the other good z is defined by the budget equation, i.e., $z = y - p^{\tau}x$. Consequently, the quasi- and true expenditure, indirect utility and direct utility functions are identical and given by the expressions of Theorems 1 and 2 with $q^{\tau}z$ and $\alpha_i(q)$ replaced by z and α_i, respectively. Therefore, the conditions of Theorem 1 are sufficient for integrability, and strict concavity of the expenditure function with respect to p in the case of zero income effects and with respect to the composite price of the Leontief bundle in the case of non-zero income effects are necessary and sufficient for inversion of the demands to obtain the direct utility function.

3. In this example we assume that $m > 1$, but the linear demands have the same form as in example 2. Then the quasi-functions have the form $\varepsilon(p, \theta)$, $\Phi(p, y)$, and $\hat{u}(f(x), q^{\tau}z)$, where $f(x)$ is quadratic if the income coefficients are zero and translated Leontief if the income coefficients are non-zero, and the x's are separable from the z's in the consumer's preference relation. Under these circumstances, the conditions of Theorem 1 and negative definiteness of the $(n \times n)$ Slutsky matrix B in the case of zero income effects, and the strict inequality $(\beta_{11} + \mu_1 x_1) < 0$ $\forall (p, q, y) \in D$ in the case of non-zero income effects, are sufficient conditions for rationalizing the demands with an expenditure function that is jointly regular in (P, Q). To see this, define the expenditure function by $E(P, Q, u) = \pi(Q)\, \varepsilon(P/\pi(Q), u)$. Then with this definition, we have

(a) $E_P = \varepsilon_p$

(b) $E_Q = (\varepsilon - \varepsilon_{p'}\, p)\, \pi_Q$

(c) $E_{PP'} = \varepsilon_{pp'}/\pi$

(d) $E_{PQ'} = -\varepsilon_{pp'}\, p\pi_{Q'}/\pi$

(e) $E_{QQ'} = (\varepsilon - \varepsilon_{p'}\, p)\, \pi_{QQ'} + p'\varepsilon_{pp'}\, p\pi_Q\pi_{Q'}/\pi$

where subscripts denote partial derivatives and $'$ indicates matrix transposition (for this example only). Our assumptions concerning $\pi(Q)$ and the conditions of Theorem 1 guarantee that $E(P, Q, u)$ is increasing in (P, Q). Since $\pi(Q)$ is linear homogeneous, it follows that $E_{P'}P + E_{Q'}Q = \varepsilon_{p'}\, pQ + (\varepsilon - \varepsilon_{p'}\, p)\, \pi_{Q'}Q = E$. Hence, the Hessian matrix H must be negative semidefinite for $E(P, Q, u)$ to be a regular expenditure function. Take the quadratic form

$$[r's']\, H[r's']' = (1/\pi)\,[r'\varepsilon_{pp'}r - 2r'\varepsilon_{pp'}\, ps'\pi_Q + \pi(\varepsilon - \varepsilon_{p'}\, p)\, s'\pi_{QQ'}s + p'\varepsilon_{pp'}\, p(s'\pi_Q)^2].$$

Since π is assumed concave and $\varepsilon_{pp'}$ is negative semidefinite, if either $r = 0$ or $s = 0$ the quadratic form is nonpositive. Therefore, consider $s \neq 0$. In the case of zero income effects, $\varepsilon_{pp'} = B$ is negative definite, hence there is a unique maximum with respect to r to the above expression, which is defined by $r = ps'\pi_Q$. Upon substitution of this for r, we have $(\varepsilon - \varepsilon_{p'} p) s'\pi_{QQ'} s \leqslant 0$ by the concavity of $\pi(Q)$. In the case of non-zero income effects, $\varepsilon_{pp'}$ has rank one. However, suppose that $r'\varepsilon_{pp'} = 0$, then again we have a non-positive expression by the concavity of $\pi(Q)$. Since $\varepsilon_{pp'} = (1/\mu_1^2)(\beta_{11} + \mu_1 x_1) \mu\mu'$, and all income effects have the same sign, it follows that for $p > 0$, $p'\mu \neq 0$, and $p'\varepsilon_{pp'} p < 0$. If we maximize the quadratic form with respect to $r'\mu$, the result is $r'\mu = p'\mu s'\pi_{Q'}$ and the reduced expression is $(\varepsilon - \varepsilon_{p'} p) s'\pi_{QQ'} s \leqslant 0$. Therefore, strict regularity in this generalized sense for the quasi-expenditure function combined with separability of the conditional preferences is sufficient for global integrability to a true expenditure function. This is a useful result since separability is often a plausible way to reduce the number of unknown parameters, and the above argument clearly shows that it can be applied to other incomplete demand systems.

5. Linear Demand Models and Welfare Measurement

One of the most common uses of linear demand models is to estimate welfare effects of a price change for the commodities of interest. This section presents the correct formulas for equivalent variaton for a system of linear demands, and briefly discusses the relationship between this measure and the most common estimate, consumer's surplus. Throughout this section, the income coefficients are assumed to be either all zero or all non-zero and of the same sign.

Suppose that the prices for the commodities with linear demands change from $p^0 = [p_1^0, ..., p_n^0]^\tau$ to $p' = [p'_1, ..., p'_n]^\tau$, with q and y held constant. If the demand system satisfies the conditions of integrability, then combining (6) with (18) gives the normalized equivalent variation with zero income effects as

$$\mathrm{ev} = \alpha(q)^\tau (p^0 - p') + (p^{0\tau} B p^0 - p'^\tau B p')/2, \tag{40}$$

while if income effects are non-zero (6) and (21) give,

$$\mathrm{ev} = \left[y + (1/\mu_1) \left(\sum_{i=1}^{n} \beta_{1i} p'_i + \alpha_1(q) + \beta_{11}/\mu_1 \right) \right] \cdot \exp\{\mu^\tau (p^0 - p')\} - \left[y + (1/\mu_1) \left(\sum_{i=1}^{n} \beta_{1i} p_i^0 + \alpha_1(q) + \beta_{11}/\mu_1 \right) \right]. \tag{41}$$

When the income effects are zero and the cross-price effects are symmetric, it is clear that normalized consumer's surplus *cs* equals normalized equivalent variation *ev* as well as compensating variation *cv*. On the other hand, if the income effects are non-zero, consumer's surplus is unique if and only if the Leontief conditional preference map is homothetic. This implies an additional set of $n-1$ restrictions on the price effects, i.e., $\beta_{1i}=\beta_{11}\mu_i/\mu_1$ for all $i=1,\ldots,n$. Consequently, there is only one independent intercept term $\alpha_1(q)$, one independent price term, β_{11}, and n independent income terms, μ_i, $i=1,\ldots,n$, which all must have the same sign.

Dixit and Weller [4] have shown that compensating variation and equivalent variation bound consumer's surplus under homotheticity of the conditional preferences. Therefore, the normalized consumer's surplus

$$cs=[\alpha_1(q)+\beta_{11}\mu^\tau p^0/\mu_1+\mu_1 y]\,\mu^\tau(p^0-p')/\mu_1-\tfrac{1}{2}\beta_{11}[\mu^\tau(p^0-p')/\mu_1]^2, \quad (42)$$

satisfies the following set of inequalities:

(a) $cv\leqslant cs\leqslant ev$ if $\mu_i>0\ \forall i=1,\ldots,n$; and

(b) $ev\leqslant cs\leqslant cv$ if $\mu_i<0\ \forall i=1,\ldots,n$.

That is, if the goods with linear demands have all zero income effects and symmetric price coefficients, or else are perfect complements with a homothetic Leontief structure, then consumer's surplus is a reasonable welfare metric for changes in the prices of those goods. However, this property, especially in the case of non-zero income effects, commands a dear price.

6. Summary

In this paper, the conditions for a system of linear demand functions to be rationalized by an underlying set of well-behaved preferences have been identified. The conditions appear to be quite restrictive, especially if the linear demand functions have non-zero income effects. There are in fact only two cases of interest possible. Either all income effects are zero and the matrix of price coefficients symmetric, negative semidefinite with a quadratic conditional preference function, or else all income effects are non-zero and of the same sign with conditional preferences for the commodities of interest characterized by a translated Leontief relationship. If some income coefficients are zero and some not, then the demands with zero income effects are also independent of all prices as well and do not enter the consumer's preference function directly. Rather, they represent in a sense extreme necessities which are consumed in fixed quantities, while the goods with non-zero income effects are again characterized by a Leontief conditional preference map with a translated origin.

Appendix

The purpose of this Appendix is to state and prove a complete dual characterization of nonhomothetic Leontief conditional preferences for an n-vector x of commodities when there are a total of $n+m$ goods, with the m-vector of other goods denoted by z. Let the direct utility function, $u(x, z)$, be defined by

$$u(x, z) = \mathrm{U}\left[\min\{(x_1 - \beta_1)/\mu_1, \ldots, (x_n - \beta_n)/\mu_n\}, z\right], \tag{a}$$

with $\mu_i > 0$, $i = 1, \ldots, n$. Consider an expenditure function, $e(P, Q, u)$, that has the following functional form

$$e(P, Q, u) = f(\mu^\tau P/\mu_1, Q, u) + \beta^\tau P, \tag{b}$$

and an indirect utility function, $v(P, Q, Y)$, with the form

$$v(P, Q, Y) = V(\mu^\tau P/\mu_1, Q, Y - \beta^\tau P). \tag{c}$$

Assume that $\mathrm{U}(\cdot)$ is a twice continuously differentiable, increasing and strictly quasi-concave function of $m+1$ variables; that $f(\cdot)$ is twice continuously differentiable, linear homogeneous and concave in $(\mu^\tau P/\mu_1, Q)$, with $e(P, Q, u)$ increasing in (P, Q, u), linear homogeneous and concave in (P, Q); and that $V(\cdot)$ is twice continuously differentiable, quasi-convex and decreasing in (P, Q), increasing in Y, and strictly quasi-convex in $(\mu^\tau P/\mu_1, Q)$ when $Y = P^\tau x + Q^\tau z$. Then we have the following duality relationship between $u(x, z)$, $e(P, Q, u)$ and $v(P, Q, Y)$.

Lemma 1. (a)$\Leftrightarrow$(b)$\Leftrightarrow$(c).

Proof. (a)$\Rightarrow$(b). By definition $e(P, Q, u) \equiv \min\{P^\tau x + Q^\tau z: u(x, z) \geqslant u\}$. At the minimum it is clear from (a) that $x_i = \beta_i + \mu_i(x_1 - \beta_1)/\mu_1$, $i = 1, \ldots, n$; hence $p^\tau x = p^\tau \beta + p^\tau \mu(x_1 - \beta_1)/\mu_1$. Letting $\hat{x} = x_1 - \beta_1$, it follows that

$$\begin{aligned} e(P, Q, u) &\equiv \min\{\beta^\tau P + (\mu^\tau P/\mu_1)\,\hat{x} + Q^\tau z: u(x, z) \geqslant u\} \\ &\equiv \beta^\tau P + \min\{(\mu^\tau P/\mu_1)\,\hat{x} + Q^\tau z: u(x, z) \geqslant u\} \\ &\equiv \beta^\tau P + f(\mu^\tau P/\mu_1, Q, u). \end{aligned}$$

(b)$\Leftrightarrow$(c). This follows immediately from the property that $v(P, Q, Y)$ is the inverse of $e(P, Q, u)$ with respect to u at $Y = e(P, Q, u)$, and conversely, $e(P, Q, u)$ is the inverse of $v(P, Q, Y)$ with respect to Y at $u = v(P, Q, Y)$.

(b) $\Rightarrow$ (a). By Hotelling's lemma, we have

$$\partial e(P, Q, u)/\partial P_i \equiv g^i(P, Q, u) \equiv h^i[P, Q, e(P, Q, u)], \qquad i = 1, \ldots, n,$$

where $g^i(P, Q, u)$ is the Hicksian compensated demand and $h^i(P, Q, Y)$ is the ordinary demand for good i, respectively. Evaluating this expression when $e(P, Q, u)$ has the form given by (b) gives

$$g^i(P, Q, u) \equiv h^i[P, Q, e(P, Q, u)] \equiv (\mu_i/\mu_1)\, \partial f(\mu^\tau P/\mu_1, Q, u)/\partial(\mu^\tau P/\mu_1) + \beta_i,$$

$i = 1, \ldots, n$. Rearranging terms, this implies that the demands for the x's satisfy the identities

$$(x_1 - \beta_1)/\mu_1 \equiv (x_2 - \beta_2)/\mu_2 \equiv \cdots \equiv (x_n - \beta_n)/\mu_n \qquad \forall (P, Q, Y) \in D,$$

where D is the domain of the demand functions. Since this restriction to a 1-dimensional ray in the n-dimensional positive othant of x space is independent of all prices and income (P, Q, Y) and all levels of utility u the preference structure (a) follows. Q.E.D.

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