The Dual Structure of Incomplete Demand Systems

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Integrability of incomplete demand systems is discussed. The concepts of weak integrability, quasi-expenditure function, quasi-indirect utility function, and quasi-utility function are defined. Their relationships to the expenditure function, indirect utility function, and utility function are developed. The dual structure of the quasi-functions permits exact welfare analysis and reveals the conditional preference structure for the commodities of interest. New results relating the uniqueness and exactness of consumer's surplus to the structure of the expenditure and indirect utility functions are obtained.

Key words: demand systems, dual structure, integration.

The integration of demand systems continues to be of considerable interest to applied economists. There are good reasons for this interest. First, integrability conditions such as Slutsky symmetry provide useful parameter restrictions which may be incorporated into the estimation of the system. Second, recovering the underlying preference structure identifies the degree of flexibility of consumer preferences implied by a demand model. Empirical rejection of the integrability conditions can be due to a restrictive maintained hypothesis embedded in the functional form of the demand system. Thus, the integrability conditions can be used as a specification test in demand modeling. Finally, knowing the direct or indirect utility function permits the calculation of exact measures for the welfare effects of changes in prices and income.

Two basic approaches are followed to generate formulas for demand systems. One approach specifies a direct or indirect utility function and derives the demand functions by maximizing the direct utility function subject to a budget constraint or by applying Roy's identity to the indirect utility function. The other approach specifies demand functions directly. The advantage of the latter approach is its simplicity. The disadvantage is that one needs to check for the integrability of the demand system (Lau).

In applied research, incomplete demand models are the rule rather than the exception. In most cases one is concerned with demands for a group of commodities that form a subset of the household's budget. One may not care about the consumer's demands for other commodities, or, even if one does, there may be no data on the consumption of these other commodities. In these circumstances the analyst must deal with an incomplete demand system.

Just as quantity data may be unavailable for commodities not of interest, price data may be incomplete as well. Often price information for the commodities that are not of direct concern is only available as an aggregated price index. A theory of the integrability of incomplete demand systems can guide researchers in the use of this information.

Incomplete demand systems allow a more general class of functional forms than complete demand models. This added generality arises because the adding-up condition is not an equality restriction but rather an inequality restriction on the total expenditure for the

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This is a contribution of the Montana Agricultural Experiment Station, Journal Series No. J-2216.

Comments from John Antle, Bruce Beattie, Randy Rucker, Wally Thurman, Myles Watts, an anonymous reviewer, and participants at the 1986 WEA meeting, the 1987 TIMS/ORSA meeting, and the natural resource economics workshop at NCSU are greatly appreciated. Any remaining errors are the responsibility of the authors.

goods of interest. For example, a complete demand system cannot be linear in all prices and income, but an incomplete system can be linear in the prices of the goods of interest and in total expenditure and still satisfy the conditions for integrability (LaFrance).

This paper addresses the integrability of a directly specified incomplete demand system. The concept of integrability employed is called weak integrability. Weak integrability is shown to be sufficient to permit the usual tasks of applied economic analysis. It is shown that (*a*) the dual relationships between the recoverable parts of the expenditure, indirect utility, and direct utility functions are analogous to the dual relationships for complete demand systems; (*b*) exact welfare measures can be calculated from weakly integrable incomplete demand systems; and (*c*) the conditional preference structure for the central commodities can be recovered from the incomplete demand system. Also, new results are presented relating uniqueness and exactness of consumer's surplus to the structure of the expenditure and indirect utility functions, and it is demonstrated that the necessary and sufficient condition for measuring the welfare effects of a change in a nonmarket parameter through an incomplete system of demand functions is not a testable hypothesis.

The paper is organized as follows. First the theory of weak integrability for an incomplete demand system is developed. This concept is then applied to practical concerns of welfare measurement in incomplete demand systems. The last section summarizes and concludes the study. Proofs of the theorems are found in the appendix.

Integration of Incomplete Demand Systems

Let $\mathbf{x} = [x_1, \ldots, x_n]'$ be the vector of consumption levels for the commodities of interest and $\mathbf{p} = [p_1, \ldots, p_n]'$ be the corresponding price vector; let $\mathbf{z} = [z_1, \ldots, z_m]'$ be the vector of consumption levels of all other commodities and $\mathbf{q} = [q_1, \ldots, q_m]'$ be the corresponding price vector; let total expenditure be y. Note that scalars are lower case letters, x; vectors are boldfaced lower case letters, $\mathbf{y}$; matrices are boldfaced upper case letters, $\mathbf{A}$; sets are upper case script, $\mathscr{B}$; R is the real number line; R^n_+ is the nonnegative orthant of Euclidean n-space and R^n_{++} is the strictly positive orthant; C^n is the set of functions with continuous nth order derivatives; $\mathbf{x} \geq \mathbf{0}$ indicates $x_i \geq 0$ for all i; $\mathbf{x} \gg \mathbf{0}$ indicates $x_i > 0$ for all i; $\in$ indicates an element of a set; $\subset$ indicates a subset; $\mathscr{A} \times \mathscr{B}$ denotes the product space of the sets $\mathscr{A}$ and $\mathscr{B}$, $\mathscr{A} \times \mathscr{B} \equiv \{(x, y): x \in \mathscr{A}, y \in \mathscr{B}\}$. The observed demand functions are given by

$$(1) \qquad \mathbf{x} = \mathbf{h}(\mathbf{p}, \mathbf{q}, y).$$

The demand functions (1) are assumed twice continuously differentiable (C^2).

In addition, there is a set of demand functions $\mathbf{z} = \hat{\mathbf{h}}(\mathbf{p}, \mathbf{q}, y)$, but these are not observed and they do not necessarily have the same functional form as the demands for $\mathbf{x}$. When $m = 1$, the demand function for z_1 can be derived from (1) by exploiting the adding up condition,

$$(2) \quad z_1 \equiv \hat{h}^1(\mathbf{p}, q_1, y) \equiv [y - \mathbf{p}'\mathbf{h}(\mathbf{p}, q_1, y)]/q_1;$$

and (1) and (2) constitute a complete demand system. If $m > 1$, then (1) is an incomplete demand system; and, because the demands for the elements of $\mathbf{z}$ are not known, it is not possible to recover the complete preference relation.

If the demands in (1) are integrable, then they satisfy Hotelling's lemma,

$$(3) \quad \partial e(\mathbf{p}, \mathbf{q}, u)/\partial \mathbf{p} \equiv \mathbf{g}(\mathbf{p}, \mathbf{q}, u) \equiv \mathbf{h}[\mathbf{p}, \mathbf{q}, e(\mathbf{p}, \mathbf{q}, u)],$$

where $\mathbf{g}(\mathbf{p}, \mathbf{q}, u)$ is the n-vector of compensated demands for the elements of $\mathbf{x}$, $e(\mathbf{p}, \mathbf{q}, u)$ is the expenditure function, and u is the consumer's level of utility. But (3) is a system of n-partial differential equations, while $e(\mathbf{p}, \mathbf{q}, u)$ is a function of $n + m + 1$ variables. The complete solution for $e(\mathbf{p}, \mathbf{q}, u)$ is, therefore, not obtainable from (3).

It is well known that maximizing an increasing, quasiconcave utility function, $u(\mathbf{x}, \mathbf{z})$, subject to $\mathbf{x} \geq \mathbf{0}$, $\mathbf{z} \geq \mathbf{0}$, and the budget constraint, $\mathbf{p}'\mathbf{x} + \mathbf{q}'\mathbf{z} \leq y$, is equivalent to the following properties for a complete system of demand functions: (*a*) the demand functions are 0° homogenous in prices and income; (*b*) the demand functions are positive valued; (*c*) total expenditure is exhausted by the sum of the expenditures on the individual demands; and (*d*) the matrix of Slutsky substitution terms is symmetric, negative semidefinite. Properties (*a*)–(*d*) are also equivalent to the existence of an expenditure function, $e(\mathbf{p}, \mathbf{q}, u)$, that is continuous and increasing in $(\mathbf{p}, \mathbf{q}, u)$, 1°

homogenous and concave in $(\mathbf{p}, \mathbf{q})$, and satisfies Hotelling's lemma (Hurwicz and Uzawa). For an incomplete demand system, (a)–(d) imply that (a') the demands in (1) are 0° homogenous in prices and income; (b') the demands in (1) are positive valued; (c') income is greater than the total expenditure on the demands in (1); and (d') the $(n \times n)$ submatrix of Slutsky substitution terms for the demands in (1) is symmetric and negative semidefinite.

Denote the set of all complete demand models satisfying (a)–(d) by $\mathscr{A}$, and the set of all complete demand models with the subset of demands in (1) satisfying (a')–(d') by $\mathscr{B}$. It follows that $\mathscr{A} \subset \mathscr{B}$ and not $(\mathscr{B}) \subset$ not $(\mathscr{A})$. A rejection of $\mathscr{B}$ is also a rejection of $\mathscr{A}$. Furthermore, (a')–(d') is a complete listing of the properties implied by the utility maximization hypothesis on the incomplete demand model. That is, utility maximization implies a specific set of refutable hypotheses. For complete demand models these are (a)–(d), for incomplete demand models they are (a')–(d').

We are naturally led to consider the implications of properties (a')–(d') for incomplete demand models. Specifically, we demonstrate that properties (a')–(d') are equivalent to the existence of a function, $\epsilon[\mathbf{p}, \mathbf{q}, \theta(\mathbf{q}, u)]$, that is increasing in $(\mathbf{p}, \theta)$, 1° homogenous in $(\mathbf{p}, \mathbf{q})$, concave in $\mathbf{p}$, and satisfies Hotelling's lemma (3). We also show that the properties of $\epsilon[\mathbf{p}, \mathbf{q}, \theta(\mathbf{q}, u)]$ are equivalent to the existence of a function, $\omega(\mathbf{x}, s, \mathbf{q})$, that is increasing, and quasiconcave in $(\mathbf{x}, s)$, where $s = y - \mathbf{p}'\mathbf{x}$ is a composite commodity representing expenditure on all other goods. This is the weakest concept of integrability for incomplete demand models that also exhausts the implications of utility maximization.

Epstein (1982) considers integrability of incomplete demand models in terms of a solution to (3) which gives an expenditure function that is increasing, 1° homogenous, and concave in all prices $(\mathbf{p}, \mathbf{q})$. In this context, he shows that there are some important differences between the integrability of complete and incomplete demand systems as well as an important distinction between what he defines as local integrability and global integrability of incomplete demand systems.

Let $\mathbf{\Omega} \subset R^{n+m+1}_{++}$ be the domain of $(\mathbf{h}, \hat{\mathbf{h}})$ and $\mathscr{X} \times \mathscr{Z} \subset R^{n+m}_{++}$ be the range. The open interior of $\mathbf{\Omega}$, denoted $\mathbf{\Omega}^\circ$, is assumed nonempty. Global integrability is defined as the existence of an increasing, quasiconcave utility function, $u(\mathbf{x}, \mathbf{z})$, defined on $\mathscr{X} \times \mathscr{Z}$ such that $(\mathbf{h}, \hat{\mathbf{h}})$ maximizes $u(\mathbf{x}, \mathbf{z})$ subject to $\mathbf{x} \geq \mathbf{0}$, $z \geq \mathbf{0}$, and $\mathbf{p}'\mathbf{x} + \mathbf{q}'\mathbf{z} \leq y$, for each $(\mathbf{p}, \mathbf{q}, y) \in \mathbf{\Omega}$. Local integrability is defined as the existence of a well-behaved utility function that generates $(\mathbf{h}, \hat{\mathbf{h}})$ in an open neighborhood $\mathcal{N}$ of the point $(\mathbf{p}^\circ, \mathbf{q}^\circ, y^\circ) \in \mathbf{\Omega}^\circ$.

For an incomplete demand system, Epstein (1982) shows that for (1) to be locally integrable, conditions (a')–(d') must be strengthened to include negative definiteness of the $n \times n$ matrix $\mathbf{S} = [s_{ij}]$, where the s_{ij} are the Slutsky substitution terms defined as

$$s_{ij} \equiv \partial h^i(\mathbf{p}, \mathbf{q}, y)/\partial p_j + h^j(\mathbf{p}, \mathbf{q}, y)\partial h^i(\mathbf{p}, \mathbf{q}, y)/\partial y, \quad i, j = 1, \ldots, n. \tag{4}$$

In general, sufficient conditions for global integrability of an incomplete demand system are quite complex. We elaborate on this point below, but the root of the problem is that there is insufficient information about the structure of the expenditure function with respect to the prices of the other goods, $\mathbf{q}$, contained in $\mathbf{h}(\mathbf{p}, \mathbf{q}, y)$. We are integrating a subset of n demand functions to recover an expenditure function defined over $n + m$ prices and the utility index, so the solution includes a constant of integration which is an unspecified function over $m + 1$ variables, $\theta(\mathbf{q}, u)$.

Epstein (1982) shows, however, a situation where local and global integrability for incomplete demand systems are essentially equivalent. Normalizing all prices by q_1 to obtain zero degree homogeneity, he demonstrates that if $\mathbf{h}(\mathbf{p}, \mathbf{q}, y)$ is otherwise independent of $\mathbf{q}$, then we can construct an expenditure function which is well behaved in $(\mathbf{p}, \mathbf{q})$ from the integration of $\mathbf{h}(\mathbf{p}, \mathbf{q}, y)$ alone. We present a generalization of Epstein's result, employing an arbitrary price index for the other goods, $\pi(\mathbf{q})$, to normalize $\mathbf{p}$ and y. This deflator function $\pi(\mathbf{q})$ is defined over any nonempty subset of the elements of $\mathbf{q}$. Some examples are $\pi(\mathbf{q}) \equiv q_1$, $\pi(\mathbf{q}) \equiv \Sigma_{j=1}^{m}\alpha_j q_j$, and $\pi(\mathbf{q}) \equiv \Pi_{j=1}^{m} q_j^{\alpha_j}$ with $\alpha_j \geq 0$ and $\Sigma_{j=1}^{m}\alpha_j = 1$.

THEOREM 1 (Epstein) *If the demand functions (1) satisfy the following conditions for all* $(\mathbf{p}, \mathbf{q}, y) \in \mathbf{\Omega}$:

(1.1) $\mathbf{h} \in C^2$;

(1.2) $\mathbf{h}$ is 0° homogenous in $(\mathbf{p}, \mathbf{q}, y)$;

(1.3) $\mathbf{h}(\mathbf{p}, \mathbf{q}, y) \geq \mathbf{0}$;

(1.4) $\mathbf{p}'\mathbf{h}(\mathbf{p}, \mathbf{q}, y) < y$;

(1.5) S is symmetric, negative definite;

(1.6) *there is a price index* π: $R^m_{++} \rightarrow R_+$ *such that* $\pi \in C^2$ *is increasing,* 1° *homogenous, and concave in* **q**, *and* $\mathbf{h}(\mathbf{p}, \mathbf{q}, y) \equiv \tilde{\mathbf{h}}[\mathbf{p}/\pi(\mathbf{q}), y/\pi(\mathbf{q})]$;

then there is an expenditure function, e: $R^{m+n}_{++} \times R \rightarrow R_{++}$, *such that* $e \in C^2$ *is increasing,* 1° *homogenous, and concave in* (**p**, **q**), *and satisfies Hotelling's lemma* (3) *for all* $(\mathbf{p}, \mathbf{q}, y) \in \boldsymbol{\Omega}$.

Theorem 1 gives one set of sufficient conditions for a C^2 system of incomplete demands to be globally rationalized by the utility maximization hypothesis. These conditions are that the demand functions are positive valued, 0° homogenous, do not exhaust total expenditure, possess a symmetric negative definite submatrix of Slutsky substitution terms, and depend upon the prices of other goods only through a price index that deflates the prices of the goods of interest and total expenditure.

Assumptions (1.1)–(1.3) are standard; there is little difference between complete and incomplete demand systems in this respect. Assumption (1.4) is part of the definition of an incomplete demand model; expenditures on **x** must be less than income, or the demand system would not be incomplete. Negative definiteness of **S** is a stronger condition than that required for integrability of complete demand systems, but Epstein (1982) provides a counterexample when assumption (1.5) is relaxed to negative semidefiniteness of **S**.

The implication of the use of a general price deflator $\pi(\mathbf{q})$ in assumption (1.6) is not trivial. The expenditure function constructed in the proof of theorem 1 is characterized by separability of **q** from (**p**, u),

$$(5) \qquad e(\mathbf{p}, \mathbf{q}, u) \equiv \pi(\mathbf{q})\epsilon[\mathbf{p}/\pi(\mathbf{q}), u].$$

Equivalently, **q** is separable from (**p**, y) in the indirect utility function,

$$(6) \qquad v(\mathbf{p}, \mathbf{q}, y) \equiv \varphi[\mathbf{p}/\pi(\mathbf{q}), y/\pi(\mathbf{q})].$$

Blackorby, Primont, and Russell show that the dual structures (5) and (6) are equivalent to a homothetically separable utility function,

$$(7) \qquad u(\mathbf{x}, \mathbf{z}) \equiv \bar{u}[\mathbf{x}, \mathrm{f}(\mathbf{z})],$$

where f is 1° homogenous (theorem 3.8, pp. 94–97).

The proof of theorem 1 is constructive, however, and relies on the arbitrariness of the constant of integration permitting the definition $\theta(\mathbf{q}, u) \equiv u$. More generally, the hypotheses of theorem 1 imply that the constant of integration is 0° homogenous in, but may depend upon, **q**. The expenditure function then has the form $e(\mathbf{p}, \mathbf{q}, u) \equiv \pi(\mathbf{q})\epsilon[\mathbf{p}/\pi(\mathbf{q}), \theta(\mathbf{q}, u)]$, the indirect utility function has the form $v(\mathbf{p}, \mathbf{q}, y) \equiv \psi[\mathbf{q}, \mathbf{p}/\pi(\mathbf{q}), y/\pi(\mathbf{q})]$, and we cannot prove that the utility function is always homothetically separable. Nevertheless, (1.6) is a very restrictive property, and it is desirable to characterize the conditional preferences underlying incomplete demand models without having to resort to such a strong assumption.

As a result of these considerations, we depart from Epstein's (1982) approach to the integrability of incomplete demand models, which requires the recovered expenditure function to be well behaved in all prices. One reason for this departure is that the Epstein conditions are not implied by utility maximization. Recall that the set of all demand models satisfying (*a*)–(*d*) is given by $\mathcal{A}$, and denote the set of all demand models in $\mathcal{A}$ such that the subset of n demands in (1) also satisfy the added conditions for global integrability by $\mathcal{F}$. It follows that $\mathcal{F} \subset \mathcal{A}$, and not $(\mathcal{A}) \subset$ not $(\mathcal{F})$. A rejection of $\mathcal{F}$ does not imply a rejection of $\mathcal{A}$. Global integrability does not arise from a theoretical construct and, consequently, does not generate any additional testable restrictions on demand functions.

We focus on the properties of the expenditure and indirect utility function with respect to **p** for given **q**, and the properties of the utility function with respect to **x** for given **z**. Focusing on these attributes of the underlying dual preference functions offers several advantages. First, in practical situations the expenditure, indirect utility or utility function is usually not required to be well behaved in the noncentral prices or goods. Second, (*a*′)–(*d*′) can be verified in practice; this is not the case for global integrability. Finally, under (*a*′)–(*d*′) the conditional preference structure for the commodities of interest is recoverable from the demand equations (1), and this conditional preference map is well behaved.

Denote expenditures on all other goods by $s \equiv \mathbf{q}'\mathbf{z} \equiv y - \mathbf{p}'\mathbf{x} > 0$. By its definition s is 1° homogenous in (**p**, **q**, y). We focus on cases where $s > 0$ to avoid a technical continuity issue and define weak integrability as follows.

DEFINITION: *The incomplete demand sys-*

tem (1) *is weakly integrable on* $\mathbf{\Omega}$ *if for all* $(\mathbf{p}, \mathbf{q}, y) \in \mathbf{\Omega}$, *there is a function* $\omega: R_{++}^{n+m+1} \rightarrow R$ *that is continuous, increasing and quasiconcave in* $(\mathbf{x}, s)$, *and* $\mathbf{x} = \mathbf{h}(\mathbf{p}, \mathbf{q}, y)$, $s = \sigma(\mathbf{p}, \mathbf{q}, y) \equiv y - \mathbf{p}'\mathbf{h}(\mathbf{p}, \mathbf{q}, y)$ *are the solutions to* $\max_{\mathbf{x},s}\{\omega(\mathbf{x}, s, \mathbf{q})\text{: } \mathbf{p}'\mathbf{x} + s \leq y, \mathbf{x} \geq \mathbf{0}, s > 0\}$.

A utility function for $(\mathbf{x}, s)$ is $\omega(\mathbf{x}, s, \mathbf{q})$ with $\mathbf{q}$ acting as a vector of shift parameters, while the price of s has been normalized to unity. Thus, we can follow essentially the same line of argument as in the case of a complete demand system; weak integrability is equivalent to the existence of an expenditure function, $e(\mathbf{p}, \mathbf{q}, u)$, that is increasing and concave in $\mathbf{p}$, 1° homogenous in $(\mathbf{p}, \mathbf{q})$, satisfies the adding up condition

$$(8) \quad \mathbf{p}'\mathbf{h}[\mathbf{p}, \mathbf{q}, e(\mathbf{p}, \mathbf{q}, u)] + \sigma[\mathbf{p}, \mathbf{q}, e(\mathbf{p}, \mathbf{q}, u)] \equiv e(\mathbf{p}, \mathbf{q}, u),$$

and Hotelling's lemma (3). Our fundamental result on weak integrability of incomplete demand models is the following.

THEOREM 2. *Given* $\mathbf{h} \in C^2$, *the incomplete system of demand equations* (1) *is weakly integrable throughout* $\mathbf{\Omega}$ *if and only if for all* $(\mathbf{p}, \mathbf{q}, y) \in \mathbf{\Omega}$:

(2.1) $\mathbf{h}$ *is* 0° *homogenous in* $(\mathbf{p}, \mathbf{q}, y)$;

(2.2) $\mathbf{h}(\mathbf{p}, \mathbf{q}, y) \geq 0$;

(2.3) $\mathbf{p}'\mathbf{h}(\mathbf{p}, \mathbf{q}, y) < y$;

(2.4) $\mathbf{S}$ *is symmetric, negative semidefinite.*

Furthermore, $e \in C^3$ *in* $\mathbf{p}$, *and for all* $(\mathbf{p}, \mathbf{q}, y) \in \mathbf{\Omega}$:

$$(9) \quad \partial s_{ij}/\partial y = \partial s_{ji}/\partial y, \qquad i, j = 1, \ldots, n;$$

$$(10) \quad \partial s_{ij}/\partial p_k = \partial s_{ji}/\partial p_k, \qquad i, j, k = 1, \ldots, n.$$

Theorem 2 shows that weak integrability allows us to treat an incomplete demand system in virtually the same manner as a complete system. If we add the composite commodity, $s \equiv y - \mathbf{p}'\mathbf{x} > 0$, to the incomplete system (1) and act as if the augmented system were complete, then the restrictions on the incomplete demand system implied by utility maximization are necessary and sufficient for our incomplete problem to have a well-defined solution.

In addition, theorem 2 provides a way to approach the problem of identifying what the symmetry conditions mean in practice. The expenditure function inherits the smoothness properties of the demand functions plus an additional degree of differentiability due to integration. When the demands are twice continuously differentiable, the symmetry conditions are identities that can be differentiated. This often gives the information needed to identify the parametric restrictions embodied in symmetry.

Suppose that the hypotheses of theorem 2 are satisfied. Then (3) can be integrated with respect to $\mathbf{p}$, and upon integrating we obtain a solution that has the form

$$(11) \quad e(\mathbf{p}, \mathbf{q}, u) \equiv \epsilon[\mathbf{p}, \mathbf{q}, \theta(\mathbf{q}, u)],$$

where $\theta(\mathbf{q}, u)$ is the arbitrary constant of integration for the partial differential equation system and is a function of the prices of the other goods, $\mathbf{q}$, and the level of utility, u, but not $\mathbf{p}$.

Three issues concerning the nature of $\theta(\mathbf{q}, u)$ warrant a brief discussion. First, a constant of integration is unrelated to an empirical constant: $\mathbf{q}$ is not a vector of constants. If it were, the matrix of independent variables would be singular, and no estimate of the demand parameters associated with $\mathbf{q}$ could be obtained. We simply cannot recover the structure of $\theta(\mathbf{q}, u)$ from the incomplete demand model.

Second, one might think $\theta(\mathbf{q}, u)$ is a vector-valued function of $(\mathbf{q}, u)$, but this is impossible. The constant of integration ties down the solution to the partial differential equation system as an initial condition, and only one initial condition exists for a scalar-valued function. The class of functions of the form $\epsilon[\mathbf{p}, \mathbf{q}, \theta(\mathbf{q}, u)]$ completely exhausts the set of solutions to partial differential equation systems of this type.

The third issue is the most important, because it relates the properties of the unobservable function $\theta(\mathbf{q}, u)$ to global integrability of the incomplete demand system. Global integrability requires that an expenditure function, $e(\mathbf{p}, \mathbf{q}, u)$, can be found from the partial solution (11) which is increasing in $(\mathbf{p}, \mathbf{q}, u)$, 1° homogenous and concave in $(\mathbf{p}, \mathbf{q})$. This is equivalent to the existence of an appropriate function $\theta(\mathbf{q}, u)$ such that $\epsilon[\mathbf{p}, \mathbf{q}, \theta(\mathbf{q}, u)]$ is increasing in $(\mathbf{p}, \mathbf{q}, u)$ and 1° homogenous and concave in $(\mathbf{p}, \mathbf{q})$.

With the exception of 1° homogeneity, weak integrability does not address the joint regularity of $e(\mathbf{p}, \mathbf{q}, u)$ with respect to $(\mathbf{p}, \mathbf{q})$. There are good reasons for this. For $e(\mathbf{p}, \mathbf{q}, u)$ to be increasing in $\mathbf{q}$, we must be able to choose $\theta(\mathbf{q}, u)$ such that for all $(\mathbf{p}, \mathbf{q}, y) \in \mathbf{\Omega}$,

$$(12)\quad \frac{\partial e}{\partial \mathbf{q}} \equiv \frac{\partial \epsilon}{\partial \mathbf{q}} + \left(\frac{\partial \epsilon}{\partial \theta}\right)\left(\frac{\partial \theta}{\partial \mathbf{q}}\right) \geq 0.$$

Similarly, for $e(\mathbf{p}, \mathbf{q}, u)$ to be concave in $\mathbf{q}$, the $m \times m$ matrix of second-order partial derivatives,

$$(13)\quad \frac{\partial^2 e}{\partial \mathbf{q} \partial \mathbf{q}'} \equiv \frac{\partial^2 \epsilon}{\partial \mathbf{q} \partial \mathbf{q}'} + 2\left(\frac{\partial \theta}{\partial \mathbf{q}}\right)\left(\frac{\partial^2 \epsilon}{\partial \theta \partial \mathbf{q}'}\right) + \left(\frac{\partial \epsilon}{\partial \theta}\right)\left(\frac{\partial^2 \theta}{\partial \mathbf{q} \partial \mathbf{q}'}\right) + \left(\frac{\partial^2 \epsilon}{\partial \theta^2}\right)\left(\frac{\partial \theta}{\partial \mathbf{q}}\right)\left(\frac{\partial \theta}{\partial \mathbf{q}'}\right),$$

must be negative semidefinite. Finally, for $e(\mathbf{p}, \mathbf{q}, u)$ to be jointly concave in $(\mathbf{p}, \mathbf{q})$, the $m \times n$ matrix of Slutsky substitution terms,

$$(14)\quad \frac{\partial^2 \epsilon}{\partial \mathbf{q} \partial \mathbf{p}'} \equiv \frac{\partial^2 \epsilon}{\partial \mathbf{q} \partial \mathbf{p}'} + \left(\frac{\partial \theta}{\partial \mathbf{q}}\right)\left(\frac{\partial^2 \epsilon}{\partial \theta \partial \mathbf{p}'}\right),$$

must be such that the full $(n + m) \times (n + m)$ matrix of Slutsky substitution terms is also negative semidefinite.

Here is the main difficulty with the notion of global integrability. We simply do not have sufficient information about the structure of $\theta(\mathbf{q}, u)$ to conclude that this complex set of conditions will be satisfied. Even if a specification for $\theta(\mathbf{q}, u)$ satisfying (12)–(14) is found, there is no reason to believe that it is the true specification. Consequently, global integrability does not provide any additional insight into the underlying preference function.

These points are demonstrated in the following example. Let the expenditure function be given by

$$(15)\quad e(\mathbf{p}, \mathbf{q}, u) = \mathbf{a}'\mathbf{p} + \mathbf{b}'\mathbf{q} + .5\left(\frac{\mathbf{p}'\mathbf{C}\mathbf{p} + 2\mathbf{p}'\mathbf{D}\mathbf{q} + \mathbf{q}'\mathbf{F}\mathbf{q}}{\pi(\mathbf{q})}\right) + (\mathbf{r}'\mathbf{p} + \mathbf{s}'\mathbf{q})u,$$

where $\pi(\mathbf{q}) > 0$ is 1° homogenous in $\mathbf{q}$. The demands for $\mathbf{x}$ generated from this expenditure function are

$$(16)\quad \mathbf{x} = \mathbf{a} + \left(\frac{\mathbf{C}\mathbf{p} + \mathbf{D}\mathbf{q}}{\pi(\mathbf{q})}\right) + \mathbf{r}\left(\frac{y - \mathbf{a}'\mathbf{p} - \mathbf{b}'\mathbf{q} - .5(\mathbf{p}'\mathbf{C}\mathbf{p} + 2\mathbf{p}'\mathbf{D}\mathbf{q} + \mathbf{q}'\mathbf{F}\mathbf{q})/\pi(\mathbf{q})}{\mathbf{r}'\mathbf{p} + \mathbf{s}'\mathbf{q}}\right).$$

Equation (16) is invariant to generalizations of the expenditure function that replace u with an arbitrary function $\alpha(\mathbf{q}, u)$ that is 0° homogenous in $\mathbf{q}$. All expenditure functions of the form

$$(17)\quad e(\mathbf{p}, \mathbf{q}, u) = \mathbf{a}'\mathbf{p} + \mathbf{b}'\mathbf{q} + .5\left(\frac{\mathbf{p}'\mathbf{C}\mathbf{p} + 2\mathbf{p}'\mathbf{D}\mathbf{q} + \mathbf{q}'\mathbf{F}\mathbf{q}}{\pi(\mathbf{q})}\right) + (\mathbf{r}'\mathbf{p} + s'\mathbf{q})\alpha(\mathbf{q}, u),$$

and all indirect utility functions of the form

$$(18)\quad v(\mathbf{p}, \mathbf{q}, y) = \psi\left[\mathbf{q}, \left(\frac{y - \mathbf{a}'\mathbf{p} - \mathbf{b}'\mathbf{q} - .5(\mathbf{p}'\mathbf{C}\mathbf{p} + 2\mathbf{p}'\mathbf{D}\mathbf{q} + \mathbf{q}'\mathbf{F}\mathbf{q})/\pi(\mathbf{q})}{\mathbf{r}'\mathbf{p} + \mathbf{s}'\mathbf{q}}\right)\right]$$

generate the same demand functions for $\mathbf{x}$.

This is a fundamental property of incomplete demand models. There is an entire class of expenditure functions that generate identical demand functions for a given subset of the goods consumed. In contrast, there is one and only one expenditure function for a complete demand model. A complete demand system reveals all of the structure of the underlying preferences, while this is impossible for an incomplete demand system.

The demand functions in (16) are 0° homogenous in $(\mathbf{p}, \mathbf{q}, y)$, and for all $\pi(\mathbf{q})$ and $\alpha(\mathbf{q}, u)$ the remaining necessary and sufficient conditions for weak integrability are

$$(\text{E1})\quad \mathbf{a} + \left(\frac{\mathbf{C}\mathbf{p} + \mathbf{D}\mathbf{q}}{\pi(\mathbf{q})}\right) + \mathbf{r}\left(\frac{y - \mathbf{a}'\mathbf{p} - \mathbf{b}'\mathbf{q} - .5(\mathbf{p}'\mathbf{C}\mathbf{p} + 2\mathbf{p}'\mathbf{D}\mathbf{q} + \mathbf{q}'\mathbf{F}\mathbf{q})/\pi(\mathbf{q})}{\mathbf{r}'\mathbf{p} + \mathbf{s}'\mathbf{q}}\right) \geq 0,$$

$$(\text{E2})\quad \mathbf{a}'\mathbf{p} + \mathbf{p}'\left(\frac{\mathbf{C}\mathbf{p} + \mathbf{D}\mathbf{q}}{\pi(\mathbf{q})}\right) + \mathbf{r}'\mathbf{p}\left(\frac{y - \mathbf{a}'\mathbf{p} - \mathbf{b}'\mathbf{q} - .5(\mathbf{p}'\mathbf{C}\mathbf{p} + 2\mathbf{p}'\mathbf{D}\mathbf{q} + \mathbf{q}'\mathbf{F}\mathbf{q})/\pi(\mathbf{q})}{\mathbf{r}'\mathbf{p} + s'\mathbf{q}}\right) < y,$$

(E3) $\mathbf{C}$ symmetric, negative semidefinite.

(E4) $\mathbf{q}'\partial\alpha(\mathbf{q}, u)/\partial\mathbf{q} \equiv \mathbf{0}$.

To solve Hotelling's lemma, we apply the integrating factor

$$(19)\quad \exp\left\{\int\left(\frac{-\mathbf{r}}{\mathbf{r}'\mathbf{p} + \mathbf{s}'\mathbf{q}}\right)' d\mathbf{p}\right\} = \exp\left\{\int\left(\frac{-\partial \ln(\mathbf{r}'\mathbf{p} + \mathbf{s}'\mathbf{q})}{\partial \mathbf{p}}\right)' d\mathbf{p}\right\} = \left(\frac{1}{\mathbf{r}'\mathbf{p} + \mathbf{s}'\mathbf{q}}\right)$$

to (16) and obtain the solution for the expenditure function as

$$(20)\quad \epsilon[\mathbf{p}, \mathbf{q}, \theta(\mathbf{q}, u)] = \mathbf{a}'\mathbf{p} + \mathbf{b}'\mathbf{q} + .5\left(\frac{\mathbf{p}'\mathbf{C}\mathbf{p} + 2\mathbf{p}'\mathbf{D}\mathbf{q} + \mathbf{q}'\mathbf{F}\mathbf{q}}{\pi(\mathbf{q})}\right) + (\mathbf{r}'\mathbf{p} + \mathbf{s}'\mathbf{q})\theta(\mathbf{q}, u),$$

where $\theta(\mathbf{q}, u) \equiv \alpha(\mathbf{q}, u)$ is the unrecoverable part of the expenditure function.

For global integrability, we require a set of parameter values and a functional form for $\theta(\mathbf{q}, u)$ such that $\epsilon[\mathbf{p}, \mathbf{q}, \theta(\mathbf{q}, u)]$ is increasing in $(\mathbf{p}, \mathbf{q}, u)$ and 1° homogenous and concave in $(\mathbf{p}, \mathbf{q})$. In general, this cannot be obtained from (16) because the structure of $\theta(\mathbf{q}, u)$ is not revealed by that subset of demands. The best we can hope for is a set of sufficient conditions for global integrability given some assumptions about the true structure of $\alpha(\mathbf{q}, u)$. For example, if $\alpha(\mathbf{q}, u) \equiv u$, then given (E1)–(E4) the remaining conditions for global integrability are

$$(\text{E5})\quad \mathbf{b} + \left(\frac{\mathbf{D}'\mathbf{p} + \mathbf{F}\mathbf{q}}{\pi(\mathbf{q})}\right) - .5\left(\frac{\mathbf{p}'\mathbf{C}\mathbf{p} + 2\mathbf{p}'\mathbf{D}\mathbf{q} + \mathbf{q}'\mathbf{F}\mathbf{q}}{\pi(\mathbf{q})^2}\right)\left(\frac{\partial\pi(\mathbf{q})}{\partial\mathbf{q}}\right) + \left(\frac{y - \mathbf{a}'\mathbf{p} - \mathbf{b}'\mathbf{q} - .5(\mathbf{p}'\mathbf{C}\mathbf{p} + 2\mathbf{p}'\mathbf{D}\mathbf{q} + \mathbf{q}'\mathbf{F}\mathbf{q})/\pi(\mathbf{q})}{\mathbf{r}'\mathbf{p} + \mathbf{s}'\mathbf{q}}\right)\mathbf{s} \geq 0,$$

(E6)

$$\begin{bmatrix} e_{\mathbf{pp}'} & e_{\mathbf{pq}'} \\ e_{\mathbf{qp}'} & e_{\mathbf{qq}'} \end{bmatrix} \text{ symmetric, negative semidefinite,}$$

where $e_{\mathbf{pp}'} = \mathbf{C}/\pi(\mathbf{q})$,

$$e_{\mathbf{pq}'} = \frac{\mathbf{D}}{\pi(\mathbf{q})} - \left(\frac{\mathbf{C}\mathbf{p} + \mathbf{D}\mathbf{q}}{\pi(\mathbf{q})^2}\right)\left(\frac{\partial\pi(\mathbf{q})}{\partial\mathbf{q}}\right),$$

$$e_{\mathbf{qq}'} = \frac{\mathbf{F}}{\pi(\mathbf{q})} - 2\left(\frac{\mathbf{D}'\mathbf{p} + \mathbf{F}\mathbf{q}}{\pi(\mathbf{q})^2}\right)\left(\frac{\partial\pi(\mathbf{q})}{\partial\mathbf{q}'}\right) + \left(\frac{\mathbf{p}'\mathbf{C}\mathbf{p} + 2\mathbf{p}'\mathbf{D}\mathbf{q} + \mathbf{q}'\mathbf{F}\mathbf{q}}{\pi(\mathbf{q})^3}\right) \times \left[\left(\frac{\partial\pi(\mathbf{q})}{\partial\mathbf{q}}\right)\left(\frac{\partial\pi(\mathbf{q})}{\partial\mathbf{q}'}\right) - .5\pi(\mathbf{q})\left(\frac{\partial^2\pi(\mathbf{q})}{\partial\mathbf{q}\partial\mathbf{q}'}\right)\right],$$

$$(\text{E7})\qquad (\mathbf{r}'\mathbf{p} + \mathbf{s}'\mathbf{q}) > 0.$$

If in fact $\partial\alpha(\mathbf{q}, u)/\partial\mathbf{q} \equiv \mathbf{0}$ and $\partial\alpha(\mathbf{q}, u)/\partial u > 0$, then (E1)–(E7) are necessary and sufficient for the complete demand system to be integrable in the usual sense. Otherwise (E1)–(E4) are necessary, but (E5)–(E7) are neither necessary nor sufficient for the expenditure function to be well behaved. In general, (E5)–(E7) have little to do with utility maximization except as a possible solution to global integrability for this incomplete demand model.

There is no practical reason to employ this or any other set of conditions sufficient for global integrability. Such conditions are not implied by utility maximization, do not generate additional insights into the true preference structure, and result in considerably less generality than arises from conditions (E1)–(E4) and recognition of the fact that $\theta(\mathbf{q}, u)$ cannot be recovered.

Returning now to the main argument, theorem 2 shows that conditions (a')–(d') are equivalent to the existence of the function $\epsilon[\mathbf{p}, \mathbf{q}, \theta(\mathbf{q}, u)]$ that is 1° homogenous in $(\mathbf{p}, \mathbf{q})$, and increasing and concave in $\mathbf{p}$. We now demonstrate that the existence of $\epsilon[\mathbf{p}, \mathbf{q}, \theta(\mathbf{q}, u)]$ is equivalent to the existence of the function $\omega(\mathbf{x}, s, \mathbf{q})$ with the properties stated in the definition of weak integrability. To facilitate the discussion, assume the complete demand system $(\mathbf{h}, \hat{\mathbf{h}})$ is integrable, and C^2 expenditure, indirect utility, and utility functions exist, but $\mathbf{z}$ is not observed and the complete structure of the dual functions with respect to $\mathbf{q}$ or $\mathbf{z}$ cannot be recovered.

Following Hausman, $\epsilon(\mathbf{p}, \mathbf{q}, \theta)$ is the quasi-expenditure function. It is related to the expenditure function by the identity (11). By the envelope theorem and the theorem of the maximum, $\partial e(\mathbf{p}, \mathbf{q}, u)/\partial u$ is continuous and positive valued. Partially differentiating both sides of (11) with respect to u and equating the results gives

$$(21)\quad \frac{\partial e(\mathbf{p}, \mathbf{q}, u)}{\partial u} \equiv \left(\frac{\partial\epsilon[\mathbf{p}, \mathbf{q}, \theta(\mathbf{q}, u)]}{\partial\theta}\right)\left(\frac{\partial\theta(\mathbf{q}, u)}{\partial u}\right) > 0.$$

Since the sign of $\theta(\mathbf{q}, u)$ can be freely chosen, we adopt the convention that $\partial\epsilon/\partial\theta > 0$, so that $\theta(\mathbf{q}, u)$ is a monotonically increasing transformation of u.

Setting $\epsilon(\mathbf{p}, \mathbf{q}, \theta) = y$ and inverting with respect to θ, we obtain the quasi-indirect utility function,

$$\theta \equiv \varphi(\mathbf{p}, \mathbf{q}, y). \tag{22}$$

The indirect utility function, $v(\mathbf{p}, \mathbf{q}, y)$, is related to $\varphi(\mathbf{p}, \mathbf{q}, y)$ by

$$v(\mathbf{p}, \mathbf{q}, y) \equiv \psi[\mathbf{q}, \varphi(\mathbf{p}, \mathbf{q}, y)], \tag{23}$$

where $u = \psi(\mathbf{q}, \theta)$ is the inverse of $\theta(\mathbf{q}, u)$ with respect to u. $\varphi(\mathbf{p}, \mathbf{q}, y)$ exists and can be obtained from $\epsilon(\mathbf{p}, \mathbf{q}, \theta)$ even when $v(\mathbf{p}, \mathbf{q}, y)$ does not exist. It is straightforward to show that $\varphi(\mathbf{p}, \mathbf{q}, y)$ is decreasing and quasi-convex in $\mathbf{p}$ and increasing in y, but $\varphi(\mathbf{p}, \mathbf{q}, y)$ is not necessarily 0° homogenous in $(\mathbf{p}, \mathbf{q}, y)$. The composite commodity, s, is a *numèraire* good, however, and this issue is not a problem. Also, since $v(\mathbf{p}, \mathbf{q}, y)$ is monotonic in y, partially differentiating both sides of (23) with respect to y and equating the results implies

$$\frac{\partial v(\mathbf{p}, \mathbf{q}, y)}{\partial y} \equiv \left(\frac{\partial \psi[\mathbf{q}, \varphi(\mathbf{p}, \mathbf{q}, y)]}{\partial \varphi}\right)\left(\frac{\partial \varphi(\mathbf{p}, \mathbf{q}, y)}{\partial y}\right) > 0, \tag{24}$$

from which it follows that $\partial\psi/\partial\varphi > 0$ since $\partial\varphi/\partial y = 1/(\partial\epsilon/\partial\theta) > 0$.

Finally, consider recovering the utility function, $u(\mathbf{x}, \mathbf{z})$, from the indirect utility function as the solution to the minimization problem

$$u(\mathbf{x}, \mathbf{z}) \equiv \min_{\mathbf{p},\mathbf{q},y} \{v(\mathbf{p}, \mathbf{q}, y):\ \mathbf{p} \geq 0,\ \mathbf{q} \geq 0,\ \mathbf{p}'\mathbf{x} + \mathbf{q}'\mathbf{z} \leq y\}, \tag{25}$$

with $\mathbf{x} \gg \mathbf{0}$ and $\mathbf{z} \gg \mathbf{0}$ for a compact budget set. The solution can be obtained in a two-stage procedure. The first stage minimizes $v(\mathbf{p}, \mathbf{q}, y)$ with respect to $(\mathbf{p}, y)$ for given $\mathbf{q}$. This is equivalent to minimizing $\varphi(\mathbf{p}, \mathbf{q}, y)$ with respect to $(\mathbf{p}, y)$ subject to $\mathbf{p} \geq 0$, $y > 0$, and $\mathbf{p}'\mathbf{x} + s \leq y$, with $\mathbf{x} \gg \mathbf{0}$, $s > 0$. The necessary conditions for an interior solution are

$$\partial\varphi/\partial\mathbf{p} + \mathbf{x}\partial\varphi/\partial y = 0. \tag{26}$$

Solving (26) for $\mathbf{p}(\mathbf{x}, s, \mathbf{q})$ and substituting this into $\varphi(\mathbf{p}, \mathbf{q}, \mathbf{p}'\mathbf{x} + s)$ gives the quasi-utility function,

$$\omega(\mathbf{x}, s, \mathbf{q}) \equiv \varphi[\mathbf{p}(\mathbf{x}, s, \mathbf{q}), \mathbf{q}, \mathbf{p}(\mathbf{x}, s, \mathbf{q})'\mathbf{x} + s]. \tag{27}$$

The utility function is related to $\omega(\mathbf{x}, s, \mathbf{q})$ by

$$u(\mathbf{x}, \mathbf{z}) \equiv \min_{\mathbf{q},s} \{\psi[\mathbf{q}, \omega(\mathbf{x}, s, \mathbf{q})]:\ \mathbf{q} \geq 0,\ \mathbf{q}'\mathbf{z} \leq s\}, \tag{28}$$

with $\mathbf{x} \gg 0$, $\mathbf{z} \gg 0$, which is the second-stage problem.

Accordingly, although we do not know ψ, we can still obtain the quasi-utility function $\omega(\mathbf{x}, s, \mathbf{q})$ from φ. This reveals the individual's conditional preference ordering over $\mathbf{x}$ given $\mathbf{z}$, which follows from applying the envelope theorem to (28),

$$\frac{\partial\omega/\partial x_j}{\partial\omega/\partial x_k} = \frac{\partial u/\partial x_j}{\partial u/\partial x_k}, \quad j, k = 1, \ldots, n. \tag{29}$$

Thus, the structure of the conditional preferences for $\mathbf{x}$ given $\mathbf{z}$ is embodied in the properties of $\omega(\mathbf{x}, s, \mathbf{q})$. Since $\omega(\mathbf{x}, s, \mathbf{q})$ is obtained from $\varphi(\mathbf{p}, \mathbf{q}, y)$, it is straightforward to show that $\omega(\mathbf{x}, s, \mathbf{q})$ is increasing and quasi-concave in $(\mathbf{x}, s)$ whether or not $u(\mathbf{x}, \mathbf{z})$ exists or can be recovered from $v(\mathbf{p}, \mathbf{q}, y)$. Consequently, conditions (a')–(d') and weak integrability are equivalent.

The quasi-utility function is closely related to the concept of a variable indirect utility function (Diewert, Epstein 1975), defined by

$$\nu(\mathbf{x}, s, \mathbf{q}) \equiv \max_{\mathbf{z}} \{u(\mathbf{x}, \mathbf{z}):\ \mathbf{z} \geq \mathbf{0},\ \mathbf{q}'\mathbf{z} \leq s\}, \tag{30}$$

where $\mathbf{x} \geq \mathbf{0}$, $s > 0$, and $\mathbf{q} \gg \mathbf{0}$. Under very weak conditions on $u(\mathbf{x}, \mathbf{z})$, Deiwert shows that $\nu(\mathbf{x}, s, \mathbf{q})$ is (a) continuous in $(\mathbf{x}, \mathbf{q}, s)$; (b) decreasing and quasi-convex in $\mathbf{q}$; (c) 0° homogenous in $(\mathbf{q}, s)$; and (d) increasing and quasi-concave in $(\mathbf{x}, s)$. Epstein (1975) demonstrates the duality of $\nu(\mathbf{x}, s, \mathbf{q})$ and $u(\mathbf{x}, \mathbf{z})$,

$$u(\mathbf{x}, \mathbf{z}) \equiv \min_{\mathbf{q},s} \{\nu(\mathbf{x}, s, \mathbf{q}):\ \mathbf{q} \geq 0,\ s > 0,\ \mathbf{q}'\mathbf{z} \leq s\}, \tag{31}$$

with $\mathbf{x} \gg 0$, $\mathbf{z} \gg 0$. From the analysis above, we have the identity

$$\psi[\mathbf{q}, \omega(\mathbf{x}, s, \mathbf{q})] \equiv \nu(\mathbf{x}, \mathrm{s}, \mathbf{q}). \tag{32}$$

Equation (32) provides another view of the information lost with incomplete demand models. Since ψ is the inverse of θ, we do not recover its structure with respect to $\mathbf{q}$. In principle ψ is quasi-concave in $\mathbf{q}$, but we recover only ω, which does not transmit all of the information about $\mathbf{q}$. Ultimately, we cannot demonstrate that $u(\mathbf{x}, \mathbf{z})$ is increasing and quasi-concave in $(\mathbf{x}, \mathbf{z})$, only that ω is increasing and quasi-concave in $(\mathbf{x}, s)$.

The foregoing analysis is general; nothing has been assumed about the functional structure of $u(\mathbf{x}, \mathbf{z})$, $e(\mathbf{p}, \mathbf{q}, u)$, or $v(\mathbf{p}, \mathbf{q}, y)$. Theorem 2 can be applied to any incomplete demand model to discover the structure of the conditional preference map for the goods of primary interest. We augment the incomplete system with a composite commodity representing total expenditures on all other goods and act as if this augmented system is complete. We cannot recover the structure of the consumer's preferences with respect to the individual elements of this composite commodity, though in practice this should usually be a relatively minor cost.

Incomplete Demand Systems and Welfare Analysis

A common use of incomplete demand models is to estimate welfare effects from a change in the prices of the goods under study. This section presents a set of new results relating incomplete demand models to welfare analysis. We show that (*a*) the exact welfare measures of a change in **p** can be obtained directly from any weakly integrable set of incomplete demand functions, (*b*) the consumer's surplus line integral is a uniquely defined welfare measure if and only if **p** is separable from u in the expenditure function and from y in the indirect utility function, (*c*) consumer's surplus is an exact welfare measure if and only if **p** is additively separable from u in the expenditure function and from y in the indirect utility function, and (*d*) the necessary and sufficient condition for measuring the welfare effects of a change in nonmarket parameters with observable market demand functions is not a testable hypothesis.

The first three results make it clear that considerable additional structure on preferences is required for consumer's surplus to be a useful estimate of welfare effects when several prices change. Since the approximating arguments for the use of consumer's surplus (Willig) are based upon the existence of an underlying preference function, it is not necessary to impose this structure to obtain the theoretically correct measures. The last result shows that it is not generally possible to measure welfare changes due to nonmarket effects using demand functions.

Consider a change in prices from $\mathbf{p}^0$ to $\mathbf{p}^1$, with $(\mathbf{q}, y)$ held constant. Welfare analysis of such a change is often measured by consumer's surplus, cs, defined by the line integral,

$$(33) \qquad cs = \int_{\mathbf{p}^1}^{\mathbf{p}^0} \mathbf{h}(\mathbf{p}, \mathbf{q}, y)' d\mathbf{p}.$$

The exact money measures of the welfare effects of the change in **p** are equivalent variation, ev, defined by

$$(34) \qquad v(\mathbf{p}^0, \mathbf{q}, y + ev) \equiv v(\mathbf{p}^1, \mathbf{q}, y),$$

and compensating variation, cv, defined by

$$(35) \qquad v(\mathbf{p}^0, \mathbf{q}, y) \equiv v(\mathbf{p}^1, \mathbf{q}, y - cv).$$

Equations (23) and (24) in the previous section imply that ev is equivalently defined by

$$(36) \qquad \varphi(\mathbf{p}^0, \mathbf{q}, y + ev) \equiv \varphi(\mathbf{p}^1, \mathbf{q}, y),$$

while cv is equivalently defined by

$$(37) \qquad \varphi(\mathbf{p}^0, \mathbf{q}, y) \equiv \varphi(\mathbf{p}^1, \mathbf{q}, y - cv).$$

Therefore, in general, the exact money measures of the welfare effects of a change in the prices of the goods of interest can be calculated directly from any weakly integrable set of incomplete demand functions.

Two issues are involved with the use of (33) to estimate the welfare effects of a change in **p**: (*a*) Does (33) give a unique measure of the money equivalent for the change in utility due to the change in prices? And (*b*) what is the relationship between consumer's surplus and the exact welfare measures?

When several prices change, the consumer's surplus line integral is path independent and uniquely defined if and only if the ordinary cross-price derivatives of the demands in (1) are symmetric (Chipman and Moore, Dixit and Weller, McKenzie and Pearce),

$$(38) \qquad \partial h^i(\mathbf{p}, \mathbf{q}, y)/\partial p_j \equiv \partial h^j(\mathbf{p}, \mathbf{q}, y)/\partial p_i, \quad i, j = 1, \ldots, n.$$

This condition is equivalent to homotheticity in the demands for **x**, i.e., all income elasticities are equal,

$$(39) \qquad \left(\frac{\partial h^j(\mathbf{p}, \mathbf{q}, y)}{\partial y}\right) \left(\frac{y}{h^j(\mathbf{p}, \mathbf{q}, y)}\right) \equiv \gamma(\mathbf{p}, \mathbf{q}, y), \quad j = 1, \ldots, n.$$

By applying the integrating factor $\exp\{-\int(\gamma(\mathbf{p}, \mathbf{q}, y)/y)dy\}$ to (39) and integrating with respect to y, we find that the demands for **x** will be homothetic if and only if

$$(40) \qquad \mathbf{h}(\mathbf{p}, \mathbf{q}, y) \equiv \boldsymbol{\alpha}(\mathbf{p}, \mathbf{q})\beta(\mathbf{p}, \mathbf{q}, y),$$

where $\beta(\mathbf{p}, \mathbf{q}, y) = \exp\{\int(\gamma(\mathbf{p}, \mathbf{q}, y)/y)dy\}$ and $\boldsymbol{\alpha}(\mathbf{p}, \mathbf{q})$ is the n-vector of constants of integration (one for each element of $\mathbf{x}$), which depend upon $(\mathbf{p}, \mathbf{q})$ but not y. The following result shows that path independence of consumer's surplus is equivalent to separability of $\mathbf{p}$ from u in the expenditure function and from y in the indirect utility function.

THEOREM 3. *Consumer's surplus (33) is path independent if and only if* $\mathbf{p}$ *is separable from* u *in the expenditure function,*

$$(41)\qquad e(\mathbf{p}, \mathbf{q}, u) \equiv \xi[f(\mathbf{p}, \mathbf{q}), \mathbf{q}, u];$$

equivalently, $\mathbf{p}$ *is separable from* y *in the indirect utility function,*

$$(42)\qquad v(\mathbf{p}, \mathbf{q}, y) \equiv \nu[f(\mathbf{p}, \mathbf{q}), \mathbf{q}, y].$$

Theorem 3 shows the dual structure leading to a unique consumer's surplus measure of the change in prices from $\mathbf{p}^0$ to $\mathbf{p}^1$. Also, if (39) holds, compensating and equivalent variation bound consumer's surplus (Dixit and Weller), suggesting that consumer's surplus is a reasonable welfare measure for changes in $\mathbf{p}$ under homotheticity. But because homotheticity is equivalent to (41), it is straightforward to obtain the exact measures of welfare change from the demand functions directly. To see this, use (41) to write Hotelling's lemma in the form

$$(43)\qquad \frac{\partial e(\mathbf{p}, \mathbf{q}, u)}{\partial p_j} \equiv \left(\frac{\partial f(\mathbf{p}, \mathbf{q})}{\partial p_j}\right)\left(\frac{\partial \xi[f(\mathbf{p}, \mathbf{q}), \mathbf{q}, u]}{\partial f}\right).$$

A natural integrating factor for (43) is $(\partial\xi/\partial f)^{-1} = 1/\beta$, since solving (41) for $f(\mathbf{p}, \mathbf{q})$ gives (at least implicitly)

$$(44)\qquad f(\mathbf{p}, \mathbf{q}) = \zeta[e(\mathbf{p}, \mathbf{q}, u), \mathbf{q}, u],$$

where ζ is the inverse of ξ with respect to f, and

$$(45)\qquad \frac{\partial f(\mathbf{p}, \mathbf{q})}{\partial p_j} \equiv \left(\frac{\partial \zeta[e(\mathbf{p}, \mathbf{q}, u), \mathbf{q}, u]}{\partial y}\right)\left(\frac{\partial e(\mathbf{p}, \mathbf{q}, u)}{\partial p_j}\right) \equiv \left(\frac{\partial \xi[f(\mathbf{p}, \mathbf{q}), \mathbf{q}, u]}{\partial f}\right)^{-1}\left(\frac{\partial e(\mathbf{p}, \mathbf{q}, u)}{\partial p_j}\right)$$

by the inverse function theorem. Integrating (45), we obtain

$$(46)\qquad \eta(\mathbf{p}, \mathbf{q}) + \delta(\mathbf{q}) = \zeta[e(\mathbf{p}, \mathbf{q}, u), \mathbf{q}, u],$$

where $f(\mathbf{p}, \mathbf{q}) \equiv \eta(\mathbf{p}, \mathbf{q}) + \delta(\mathbf{q})$. Setting $e(\mathbf{p}, \mathbf{q}, u) = y$, it follows that the equivalent variation for a price change from $\mathbf{p}^0$ to $\mathbf{p}^1$ satisfies

$$(47)\qquad v(\mathbf{p}^0, \mathbf{q}, y + ev) \equiv \nu[\mathbf{f}(\mathbf{p}^0, \mathbf{q}), \mathbf{q}, y + ev] \equiv \nu[f(\mathbf{p}^1, \mathbf{q}), \mathbf{q}, y] \equiv v(\mathbf{p}^1, \mathbf{q}, y)$$

if and only if

$$(48)\qquad \eta(\mathbf{p}^0, \mathbf{q}) - \eta(\mathbf{p}^1, \mathbf{q}) \equiv \zeta(y + ev, \mathbf{q}, u) - \zeta(y, \mathbf{q}, u).$$

The relevant structures of ζ with respect to y and of f with respect to $\mathbf{p}$ are captured by solving (43), and it is no more difficult to obtain the exact welfare measures *cv* or *ev* than to obtain the consumer's surplus approximation for homothetic demands.

Consumer's surplus equals compensating and equivalent variation if and only if the income effects are zero for the demands in (1). Then Slutsky symmetry and symmetry of the cross-price effects are equivalent. Both sets of conditions hold if and only if there is a function $\kappa(\mathbf{p}, \mathbf{q})$ that is 1° homogenous in $(\mathbf{p}, \mathbf{q})$, increasing and concave in $\mathbf{p}$, and satisfies

$$(49)\qquad \mathbf{h}(\mathbf{p}, \mathbf{q}, y) \equiv \partial\kappa(\mathbf{p}, \mathbf{q})/\partial\mathbf{p}.$$

The following result, previously known only for $m = 1$ (Chipman and Moore), shows that (49) holds if and only if $\mathbf{p}$ is additively separable from u in the expenditure function and from y in the indirect utility function.

THEOREM 4. *The demands in (1) are independent of income and cs = cv = ev for changes in* $\mathbf{p}$, *if and only if* $\mathbf{p}$ *is additively separable from* u *in* $e(\mathbf{p}, \mathbf{q}, u)$,

$$(50)\qquad e(\mathbf{p}, \mathbf{q}, u) \equiv \kappa(\mathbf{p}, \mathbf{q}) + \theta(\mathbf{q}, u);$$

equivalently, $\mathbf{p}$ *is additively separable from* y *in* $v(\mathbf{p}, \mathbf{q}, y)$,

$$(51)\qquad v(\mathbf{p}, \mathbf{q}, y) \equiv \psi[\mathbf{q}, y - \kappa(\mathbf{p}, \mathbf{q})].$$

We conclude this section with a caveat concerning the use of demand models to elicit welfare effects from changes in a nonmarket parameter τ. Suppose that the demands for $\mathbf{x}$ depend upon τ in addition to $(\mathbf{p}, \mathbf{q}, y)$. Then Hotelling's lemma has the form

$$(52)\qquad \partial e(\mathbf{p}, \mathbf{q}, u, \tau)/\partial\mathbf{p} \equiv \mathbf{h}[\mathbf{p}, \mathbf{q}, e(\mathbf{p}, \mathbf{q}, u, \tau), \tau].$$

If (52) is integrable, then the solution has the form

(53) $e(\mathbf{p}, \mathbf{q}, u, \tau) \equiv \epsilon[\mathbf{p}, \mathbf{q}, \tau, \theta(\mathbf{q}, u, \tau)]$.

It follows that a necessary and sufficient condition for $\mathbf{h}(\mathbf{p}, \mathbf{q}, y, \tau)$ to reveal the welfare effects of changes in τ is

(54) $$\partial\theta(\mathbf{q}, u, \tau)/\partial\tau \equiv 0.$$

Recall that $\theta(\mathbf{q}, u, \tau)$ is an arbitrary function whose structure is not recovered from the partial differential equation system (52). Consequently, if we wish to obtain the exact welfare effects of a change in τ from market demand functions, then condition (54) must be assumed. But this assumption does not have any behavioral consequences and therefore does not lead to a testable hypothesis, and it is generally impossible to measure unequivocally welfare changes from nonmarket effects using incomplete systems of market demand functions.

Summary and Conclusions

We have outlined a procedure for discovering the set of parameter restrictions and the underlying conditional preference structure implied by weak integrability for an incomplete system of demand equations. By artificially augmenting the incomplete demand system with a composite *numèraire* commodity representing total expenditure on all other goods, we can proceed as if the augmented system is complete. This procedure can be applied to any system of demand equations.

We have shown that the use of incomplete demand systems and the associated quasi functions for the conditional preferences does not result in a loss in generality in several respects. Weak integrability exhausts the implications of the utility maximization hypothesis on incomplete demand models. Also, the dual structure of the conditional preferences for the commodities of interest is revealed by the demand functions. Finally, recovering the structure of the quasi functions permits the calculation of exact welfare measures for changes in the prices of the commodities of interest. Therefore, weak integrability of incomplete demand models can be used in nearly all practical applications.

We also obtained several new results in the theory of welfare measurement. Consumer's surplus is unique if and only if $\mathbf{p}$ is separable from u in the expenditure function and from y in the indirect utility function. Further, consumer's surplus is exact if and only if $\mathbf{p}$ is additively separable from u in the expenditure function and from y in the indirect utility function. Finally, the necessary and sufficient condition for measuring welfare effects due to a change in nonmarket parameters with demand functions is not a testable hypothesis.

[*Received October 1987; final revision received August 1988.*]

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Appendix

Theorem Proofs

In this appendix we state and prove all theorems in the main paper.

THEOREM 1. (Epstein) *If the demand functions (1) satisfy the following conditions for all* $(\mathbf{p}, \mathbf{q}, y) \in \mathbf{\Omega}$:

(1.1) $\mathbf{h} \in C^2$;

(1.2) $\mathbf{h}$ *is* 0° *homogenous in* $(\mathbf{p}, \mathbf{q}, y)$;

(1.3) $\mathbf{h}(\mathbf{p}, \mathbf{q}, y) \geq 0$;

(1.4) $\mathbf{p}'\mathbf{h}(\mathbf{p}, \mathbf{q}, y) < y$;

(1.5) $\mathbf{S}$ *is symmetric, negative definite;*

(1.6) *there is a price index* $\pi\colon R^m_{++} \to R_+$ *such that* $\pi \in C^2$ *is increasing,* 1° *homogenous, and concave in* $\mathbf{q}$, *and* $\mathbf{h}(\mathbf{p}, \mathbf{q}, y) \equiv \tilde{\mathbf{h}}[\mathbf{p}/\pi(\mathbf{q}), y/\pi(\mathbf{q})]$; *then there is an expenditure function* $e\colon R^{m+n}_{++} \times R \to R_{++}$ *such that* $e \in C^2$ *is increasing,* 1° *homogenous, and concave in* $(\mathbf{p}, \mathbf{q})$, *and satisfies Hotelling's lemma* (3) *for all* $(\mathbf{p}, \mathbf{q}, y) \in \mathbf{\Omega}$.

Proof. Substituting $e(\mathbf{p}, \mathbf{q}, u) = y$ into $\mathbf{h}(\mathbf{p}, \mathbf{q}, y) \equiv \tilde{\mathbf{h}}[\mathbf{p}/\pi(\mathbf{q}), y/\pi(\mathbf{q})]$, and partially differentiating with respect to p_k, the Slutsky terms have the form

$$(\text{A.1}) \quad s_{jk} = \frac{\partial h^j[\mathbf{p}, \mathbf{q}, e(\mathbf{p}, \mathbf{q}, u)]}{\partial p_k} + h^k[\mathbf{p}, \mathbf{q}, e(\mathbf{p}, \mathbf{q}, u)] \times \frac{\partial h^j[\mathbf{p}, \mathbf{q}, e(\mathbf{p}, \mathbf{q}, u)]}{\partial_y}$$

$$= \left(\frac{\partial \tilde{h}^j[\mathbf{p}/\pi, e(\mathbf{p}, \mathbf{q}, u)/\pi]}{\partial(p_k/\pi)} + \tilde{h}^k[\mathbf{p}/\pi, e(\mathbf{p}, \mathbf{q}, u)/\pi] \times \frac{\partial \tilde{h}^j[\mathbf{p}/\pi, e(\mathbf{p}, \mathbf{q}, u)/\pi]}{\partial(y/\pi)}\right)\left(\frac{1}{\pi}\right).$$

Symmetry of $\tilde{\mathbf{S}} \equiv [\partial\tilde{h}^i/\partial(p_j/\pi) + \tilde{h}^j\partial\tilde{h}^i/\partial(y/\pi)]$ follows from (1.5), which implies that there is a function $\epsilon\colon R^{n+1}_{++} \to R_+$, such that $\epsilon \in C^2$ and

$$(\text{A.2}) \quad \partial\epsilon[\mathbf{p}/\pi(\mathbf{q}), \theta(\mathbf{q}, u)]/\partial(\mathbf{p}/\pi(\mathbf{q})) \equiv \tilde{h}\{p/\pi(\mathbf{q}), \epsilon[\mathbf{p}/\pi(\mathbf{q}), \theta(\mathbf{q}, u)]/\pi(\mathbf{q})\}.$$

$\epsilon[\mathbf{p}/\pi(\mathbf{q}), \theta(\mathbf{q}, u)]$ is increasing in $\mathbf{p}$ by (1.3) and strictly concave in $\mathbf{p}$ by (1.5). The method of proof is by construction. Define $e(\mathbf{p}, \mathbf{q}, u)$ by

$$(\text{A.3}) \qquad e(\mathbf{p}, \mathbf{q}, u) \equiv \pi(\mathbf{q})\epsilon[\mathbf{p}/\pi(\mathbf{q}), u],$$

where the arbitrary function $\theta(\mathbf{q}, u)$ is chosen such that $\partial\theta(\mathbf{q}, u)/\partial\mathbf{q} \equiv 0$, and without loss of generality $\theta(\mathbf{q}, u) \equiv u$. Taken together, (A.3), (1.1), and (1.6) imply that $\mathrm{e} \in C^2$. Then,

$$(\text{A.4}) \quad \partial e(\mathbf{p}, \mathbf{q}, u)/\partial\mathbf{p} = \partial\epsilon[\mathbf{p}/\pi, u]/\partial(\mathbf{p}/\pi) \geq 0,$$

$$(\text{A.5}) \quad \frac{\partial e(\mathbf{p}, \mathbf{q}, u)}{\partial\mathbf{q}} = \left[\epsilon(\mathbf{p}/\pi, u) - (\mathbf{p}/\pi)'\left(\frac{\partial\epsilon(\mathbf{p}/\pi, u)}{\partial(\mathbf{p}/\pi)}\right)\right]\left(\frac{\partial\pi(\mathbf{q})}{\partial\mathbf{q}}\right) \geq 0,$$

$$(\text{A.6}) \quad \mathbf{p}'\left(\frac{\partial e(\mathbf{p}, \mathbf{q}, u)}{\partial\mathbf{p}}\right) + \mathbf{q}'\left(\frac{\partial e(\mathbf{p}, \mathbf{q}, u)}{\partial\mathbf{q}}\right)$$

$$\equiv \mathbf{p}'\left(\frac{\partial\epsilon(\mathbf{p}/\pi, u)}{\partial(\mathbf{p}/\pi)}\right) + \left[\epsilon(\mathbf{p}/\pi, u) - (\mathbf{p}/\pi)'\left(\frac{\partial\epsilon(\mathbf{p}/\pi, u)}{\partial(\mathbf{p}/\pi)}\right)\right] \mathbf{q}'\left(\frac{\partial\pi(\mathbf{q})}{\partial\mathbf{q}}\right) \equiv e(\mathbf{p}, \mathbf{q}, u)$$

by (1.3), (1.4), and (1.6). Therefore, $e(\mathbf{p}, \mathbf{q}, u)$ is increasing and 1° homogenous in $(\mathbf{p}, \mathbf{q})$. Since $\pi, \mathbf{h} \in C^2$, the hessian matrix exists and is defined by

$$(\text{A.7}) \quad \mathbf{H} = \begin{bmatrix} \left(\dfrac{\partial^2\epsilon}{\partial(\mathbf{p}/\pi)\partial(\mathbf{p}/\pi)'}\right)\left(\dfrac{1}{\pi}\right) & -\left(\dfrac{\partial^2\epsilon}{\partial(\mathbf{p}/\pi)\partial(\mathbf{p}/\pi)'}\right)\left(\dfrac{\mathbf{p}}{\pi}\right)\left(\dfrac{\partial\pi}{\partial\mathbf{q}'}\right) \\ & \left(\dfrac{\mathbf{p}}{\pi}\right)'\left(\dfrac{\partial^2\epsilon}{\partial(\mathbf{p}/\pi)\partial(\mathbf{p}/\pi)'}\right)\left(\dfrac{\mathbf{p}}{\pi}\right)\left(\dfrac{\partial\pi}{\partial\mathbf{q}}\right)\left(\dfrac{\partial\pi}{\partial\mathbf{q}'}\right) \\ -\left(\dfrac{\partial\pi}{\partial\mathbf{q}}\right)\left(\dfrac{\mathbf{p}}{\pi}\right)'\left(\dfrac{\partial^2\epsilon}{\partial(\mathbf{p}/\pi)\partial(\mathbf{p}/\pi)'}\right) & +\left[\epsilon - \left(\dfrac{p}{\pi}\right)'\left(\dfrac{\partial\epsilon}{\partial(\mathbf{q}/\pi)}\right)\right] . \left(\dfrac{\partial^2\pi}{\partial\mathbf{q}\partial\mathbf{q}'}\right) \end{bmatrix}$$

The upper left block of $\mathbf{H}$ is negative definite by (1.5). Maximizing the quadratic form $Q = [\mathbf{r}'\mathbf{s}']\mathbf{H}\begin{bmatrix}\mathbf{r}\\ \mathbf{s}\end{bmatrix}$ with respect to $\mathbf{r}$ for arbitrary fixed $\mathbf{s}$, where $\mathbf{r}$ is $n \times 1$ and $\mathbf{s}$ is $m \times 1$, gives a unique maximizing vector, $\mathbf{r}^* = \mathbf{s}'(\partial\pi/\partial\mathbf{q})\mathbf{p}$. Substituting $\mathbf{r}^*$ into Q gives

$$(\text{A.8}) \quad Q^* = \pi\left[\epsilon - \left(\frac{\mathbf{p}}{\pi}\right)'\left(\frac{\partial\epsilon}{\partial(\mathbf{p}/\pi)}\right)\right]\mathbf{s}'\left(\frac{\partial^2\pi}{\partial\mathbf{q}\partial\mathbf{q}'}\right)\mathbf{s} \leq 0,$$

by (1.4) and (1.6). **Q.E.D.**

THEOREM 2. *Given* $\mathbf{h} \in C^2$, *the incomplete system of demand equations (1) is weakly integrable throughout* $\mathbf{\Omega}$ *if and only if for all* $(\mathbf{p}, \mathbf{q}, y) \in \mathbf{\Omega}$:

(2.1) $\mathbf{h}$ *is* 0° *homogenous in* $(\mathbf{p}, \mathbf{q}, y)$;

(2.2) $\mathbf{h}(\mathbf{p}, \mathbf{q}, y) \geq \mathbf{0}$;

(2.3) $\mathbf{p}'\mathbf{h}(\mathbf{p}, \mathbf{q}, y) < y$;

(2.4) $\mathbf{S}$ *is symmetric, negative semidefinite.*

Furthermore, $e \in C^3$ *in* $\mathbf{p}$, *and for all* $(\mathbf{p}, \mathbf{q}, y) \in \mathbf{\Omega}$:

$$\partial s_{ij}/\partial y = \partial s_{ji}/\partial y, \quad i, j = 1, \ldots, n; \tag{9}$$

$$\partial s_{ij}/\partial p_k = \partial s_{ji}/\partial p_k, \quad i, j, k = 1, \ldots, n. \tag{10}$$

Proof. Set $e(\mathbf{p}, \mathbf{q}, u) = y$. Then, by the arguments in Katzner, chapter 4, symmetry of S is necessary and sufficient for the existence of a solution to the partial differential equation system (3), and S negative semidefinite is necessary and sufficient for the solution to (3) to be concave in $\mathbf{p}$. $e(\mathbf{p}, \mathbf{q}, u)$ increasing in $\mathbf{p}$ is equivalent to (2.2). (2.3) is necessary and sufficient for the existence of $\sigma(\mathbf{p}, \mathbf{q}, y) \equiv y - \mathbf{p}'\mathbf{h}(\mathbf{p}, \mathbf{q}, y) > 0$, and by (2.1) $\sigma(\mathbf{p}, \mathbf{q}, y)$ is 1° homogenous in $(\mathbf{p}, \mathbf{q}, y)$. It follows that

$$\begin{aligned} e(\mathbf{p}, \mathbf{q}, u) \equiv \mathbf{p}'\mathbf{h}[\mathbf{p}, \mathbf{q}, e(\mathbf{p}, \mathbf{q}, u)] \\ + \sigma[\mathbf{p}, \mathbf{q}, e(\mathbf{p}, \mathbf{q}, u)] \end{aligned} \tag{A.9}$$

is 1° homogenous in $(\mathbf{p}, \mathbf{q})$. To demonstrate that $e \in C^3$ in $\mathbf{p}$, partially differentiate (3) with respect to $\mathbf{p}$ to obtain

$$\begin{aligned} \frac{\partial^2 e(\mathbf{p}, \mathbf{q}, u)}{\partial \mathbf{p} \partial \mathbf{p}'} &\equiv \frac{\partial \mathbf{h}[\mathbf{p}, \mathbf{q}, e(\mathbf{p}, \mathbf{q}, u)]}{\partial \mathbf{p}'} \\ &+ \left(\frac{\partial \mathbf{h}[\mathbf{p}, \mathbf{q}, e(\mathbf{p}, \mathbf{q}, u)]}{\partial y}\right) \left(\frac{\partial e(\mathbf{p}, \mathbf{q}, u)}{\partial \mathbf{p}'}\right). \end{aligned} \tag{A.10}$$

Partially differentiating an element of (A.10) with respect to u,

$$\begin{aligned} \frac{\partial^3 e(\mathbf{p}, \mathbf{q}, u)}{\partial p_i \partial p_j \partial u} &\equiv \left[\frac{\partial^2 h^i[\mathbf{p}, \mathbf{q}, e(\mathbf{p}, \mathbf{q}, u)]}{\partial p_j \partial y}\right. \\ &+ \left(\frac{\partial h^i[\mathbf{p}, \mathbf{q}, e(\mathbf{p}, \mathbf{q}, u)]}{\partial y}\right) \left(\frac{\partial h^j[\mathbf{p}, \mathbf{q}, e(\mathbf{p}, \mathbf{q}, u)]}{\partial y}\right) \\ &+ \left.\left(\frac{\partial^2 h^i[\mathbf{p}, \mathbf{q}, e(\mathbf{p}, \mathbf{q}, u)]}{\partial y^2}\right) h^j[\mathbf{p}, \mathbf{q}, e(\mathbf{p}, \mathbf{q}, u)]\right] \\ &\times \left(\frac{\partial e(\mathbf{p}, \mathbf{q}, u)}{\partial u}\right). \end{aligned} \tag{A.11}$$

Each term on the right-hand side is continuous, and $\partial^3 e(\mathbf{p}, \mathbf{q}, u)/\partial p_i \partial p_j \partial u$ exists and is continuous. It follows that

$$\begin{aligned} \frac{\partial s_{ij}}{\partial y} &= \frac{\partial^2 h^i}{\partial p_j \partial y} + \frac{\partial h^i}{\partial y} \cdot \frac{\partial h^j}{\partial y} + \frac{\partial^2 h^i}{\partial y^2} h^j \\ &= \frac{\partial^2 h^j}{\partial p_i \partial y} + \frac{\partial h^j}{\partial y} \cdot \frac{\partial h^i}{\partial y} + \frac{\partial^2 h^j}{\partial y^2} h^i = \frac{\partial s_{ji}}{\partial y}. \end{aligned} \tag{A.12}$$

Partially differentiating an element of (A.10) with respect to p_k,

(A.13)

$$\begin{aligned} \frac{\partial^3 e}{\partial p_i \partial p_j \partial p_k} &= \frac{\partial^2 h^i}{\partial p_j \partial p_k} + \frac{\partial h^i}{\partial y} \cdot \frac{\partial h^j}{\partial p_k} + \frac{\partial^2 h^i}{\partial y \partial p_k} h^j \\ &+ \left(\frac{\partial^2 h^i}{\partial p_j \partial y} + \frac{\partial h^i}{\partial y} \cdot \frac{\partial h^j}{\partial y} + \frac{\partial^2 h^i}{\partial y^2} h^j\right) \frac{\partial e}{\partial p_k}. \end{aligned}$$

Each term on the right-hand side is continuous, and $\partial^3 e(\mathbf{p}, \mathbf{q}, \theta)/\partial p_i \partial p_j \partial p_k$ exists and is continuous. The terms in parentheses are symmetric. It follows that

(A.14)

$$\begin{aligned} \frac{\partial s_{ij}}{\partial p_k} &= \frac{\partial^2 h^i}{\partial p_j \partial p_k} + \frac{\partial h^i}{\partial y} \cdot \frac{\partial h^j}{\partial p_k} + \frac{\partial^2 h^i}{\partial y \partial p_k} h^j \\ &= \frac{\partial^2 h^j}{\partial p_i \partial p_k} + \frac{\partial h^j}{\partial y} \cdot \frac{\partial h^i}{\partial p_k} + \frac{\partial^2 h^j}{\partial y \partial p_k} h^i = \frac{\partial s_{ji}}{\partial p_k}. \end{aligned}$$

Q.E.D.

THEOREM 3. *Consumer's surplus (33) is path independent if and only if* $\mathbf{p}$ *is separable from u in the expenditure function,*

$$e(\mathbf{p}, \mathbf{q}, u) \equiv \xi[f(\mathbf{p}, \mathbf{q}), \mathbf{q}, u]; \tag{41}$$

equivalently, $\mathbf{p}$ *is separable from y in the indirect utility function,*

$$v(\mathbf{p}, \mathbf{q}, y) \equiv \nu[f(\mathbf{p}, \mathbf{q}), \mathbf{q}, y]. \tag{42}$$

Proof. Sufficiency is obvious from Hotelling's lemma applied to (41). Equations (41) and (42) are dual functional structures, hence equivalent. Necessity is proven as follows. If (40) is integrable, then from Roy's identity we have

$$\begin{aligned} -\frac{\partial v(\mathbf{p}, \mathbf{q}, y)/\partial p_j}{\partial v(\mathbf{p}, \mathbf{q}, y)/\partial y} &\equiv h^j(\mathbf{p}, \mathbf{q}, y) \\ &\equiv \alpha_j(\mathbf{p}, \mathbf{q})\beta(\mathbf{p}, \mathbf{q}, y), \quad j = 1, \ldots, n. \end{aligned} \tag{A.15}$$

Because the cross-price effects are symmetric,

$$\begin{aligned} \frac{\partial h^j}{\partial p_k} &\equiv \left(\frac{\partial \alpha_j}{\partial p_k}\right)\beta + \left(\frac{\partial \beta}{\partial p_k}\right)\alpha_j \\ &\equiv \left(\frac{\partial \alpha_k}{\partial p_j}\right)\beta + \left(\frac{\partial \beta}{\partial p_j}\right)\alpha_k \equiv \frac{\partial h^k}{\partial p_j}, \\ &\qquad j, k = 1, \ldots, n. \end{aligned} \tag{A.16}$$

Partially differentiating (A.15) with respect to p_k gives, by Young's theorem,

$$\begin{aligned} \frac{\partial^2 v}{\partial p_j \partial p_k} &\equiv -\left[\left(\frac{\partial \alpha_j}{\partial p_k}\right)\beta + \left(\frac{\partial \beta}{\partial p_k}\right)\alpha_j\right] \left(\frac{\partial v}{\partial y}\right) \\ &- \left(\frac{\partial^2 v}{\partial y \partial p_k}\right)\alpha_j \beta \\ &\equiv -\left[\left(\frac{\partial \alpha_k}{\partial p_j}\right)\beta + \left(\frac{\partial \beta}{\partial p_j}\right)\alpha_k\right] \left(\frac{\partial v}{\partial y}\right) \\ &- \left(\frac{\partial^2 v}{\partial y \partial p_j}\right)\alpha_k \beta \equiv \frac{\partial^2 v}{\partial p_k \partial p_j}. \end{aligned} \tag{A.17}$$

Taken together, (A.15), (A.16), and (A.17) imply that

$$\frac{\partial}{\partial y}\left(\frac{\partial v/\partial p_j}{\partial v/\partial p_k}\right) \equiv 0, \qquad j, k = 1, \ldots, n, \tag{A.18}$$

and (42) follows immediately. **Q.E.D.**

THEOREM 4. *The demands in (1) are independent of income and cs = cv = ev for changes in* $\mathbf{p}$, *if and only if* $\mathbf{p}$ *is additively separable from u in* $e(\mathbf{p}, \mathbf{q}, u)$,

$$e(\mathbf{p}, \mathbf{q}, u) \equiv \kappa(\mathbf{p}, \mathbf{q}) + \theta(\mathbf{q}, u); \tag{50}$$

equivalently, $\mathbf{p}$ *is additively separable from y in* $v(\mathbf{p}, \mathbf{q}, y)$,

$$v(\mathbf{p}, \mathbf{q}, y) \equiv \psi[\mathbf{q}, y - \kappa(\mathbf{p}, \mathbf{q})]. \tag{51}$$

Proof. Again, sufficiency is obvious. Necessity is proven by integrating (49) directly.

Q.E.D.