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The envelope theorem in dynamic optimization*

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The dynamic envelope theorem is presented for optimal control problems with nondifferential constraints. Some of these constraints may switch from binding to nonbinding, or vice versa, along the optimal path. Twice continuous differentiability of the optimal performance function and intertemporal symmetry and reciprocity conditions are shown to follow from the envelope theorem and twice continuous differentiability of the integrand, state equations, and constraints. Conditions implying convexity or concavity of the optimal performance function in the parameters are derived. Dynamic versions of Hotelling's Lemma, Roy's Identity, and the Slutsky equation are presented for an intertemporal consumption problem

1. Introduction

The envelope theorem is a powerful tool in static economic analysis [Samuelson (1947, 1960a, 1960b), Silberberg (1971, 1974, 1978)]. Perhaps the single most important implication of the envelope theorem is the straightforward elucidation of the symmetry relationships which result from maximization subject to constraint [Silberberg (1974)]. This information is forthcoming because the static envelope theorem gives three equivalent expressions for the rate of change of a value function with respect to a parameter. These

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expressions are obtained by differentiating the objective function with respect to the parameter after maximization and differentiating the Lagrangian both before and after maximization. In all three cases the choice functions are evaluated at their optimal solutions for the initial value of the parameter.

There is extensive literature on the existence, stability, and comparative statics properties of steady state equilibria in optimal control problems [e.g., Araujo and Scheinkman (1977, 1979), Brock (1977, 1983), Brock and Scheinkman (1976, 1977), Cass and Shell (1976), Magill and Scheinkman (1979), Rockafellar (1976), and Scheinkman (1976, 1978)]. But the nature, application, and practical implications of the dynamic analogue to the envelope theorem are not nearly so well understood.

Oniki (1973) alluded to the dynamic envelope theorem. He suggested that a simple expression for the rate of change of the optimal performance function with respect to an exogenous parameter can be obtained by substituting the optimal controls and state variables into the integral functional and differentiating with respect to the parameter. He also suggested that an equivalent expression could be obtained in terms of the shadow prices and costate variables for the optimal control problem. Hochman, LaFrance, and Zilberman (1984) correctly applied the dynamic envelope theorem, but made no attempt to demonstrate the validity of their heuristic argument.

Hadley and Kemp (1971) derived expressions for the partial derivatives of the optimal performance function with respect to the limits of integration and the fixed end points of the state variables in calculus of variations problems. For optimal control problems, expressions for the partial derivatives of the optimal performance function with respect to fixed end points for the state variables and the limits of integration were obtained by Benveniste and Scheinkman (1979, 1982), Seierstad (1982), and Seierstad and Sydsaeter (1987). Seierstad and Sydsaeter also developed an expression for the partial derivative of the optimal performance function with respect to an exogenous parameter that is essentially the dynamic envelope theorem for optimal control problems whose only constraints are the equations of motion. Another result along this line is the expression for the rate of change of the optimal performance function with respect to a parameter obtained by Epstein (1978).

Utilizing the Hamilton–Jacobi equation for autonomous current value problems with static expectations, intertemporal duality for the firm has been analyzed by McLaren and Cooper (1980) and Epstein (1981), and for the consumer by Cooper and McLaren (1980). A dynamic generalization of Hotelling/Shephard's Lemma gives the optimal controls and the equations of motion from the open loop with feedback solution for the first instant in the planning horizon. Epstein and Denny (1983) generalized these intertemporal duality results to expectations generated by linear first-order differential equations.

This paper considers the dynamic envelope theorem in optimal control problems with nondifferential constraints. Some of the constraints may switch from a binding to a nonbinding condition along the optimal path, leading to corners in the optimal path. Under conditions guaranteeing that the problem is sufficiently well-behaved, it is shown that there are two, though *not* three, equivalent expressions for the rate of change of the optimal performance function with respect to an exogenous parameter. The two equivalent expressions are obtained by differentiating the integrand *after* maximization and the Lagrangian *before* maximization, evaluating the choice functions of both along their optimal paths, and integrating over the planning horizon. The third expression, obtained by differentiating the Lagrangian *after* maximization and integrating over the planning horizon, does *not* give the same value as the other two, in contrast to the static case.

Twice continuous differentiability of the optimal performance function and dynamic symmetry and reciprocity conditions follow from the dynamic envelope theorem when the integrand, the state equations, and the constraint functions are twice continuously differentiable. We obtain two results that generalize the recent results of Caputo (1989, 1990) utilizing the primal–dual methodology adapted for dynamic optimization. For parameters that only affect the integrand, convexity of the optimal performance function in the parameters follows from convexity of the integrand in the parameters. Conversely, the optimal performance function is concave in the parameters if the equality constraints and the equations of motion are jointly linear and the integrand and the inequality constraints are jointly concave in the state variables, control variables, and parameters. Also, the added structure of infinite horizon, autonomous current value problems is considered, and the special case of time-variant parameters which can be represented by first-order differential equations is discussed.

Finally, an application to an intertemporal consumer problem produces dynamic versions of Hotelling's Lemma, Roy's Identity, and the Slutsky equation. These dynamic duality results are contrasted to their static counterparts. Properties such as symmetry and negative semidefiniteness for the price effects in a system of dynamic compensated demand functions hold over the entire planning horizon, but not necessarily at a given point in time. A compensated own-price effect may be positive at a given point in time, but it must be negative in integral form over the entire planning horizon. Similarly, compensated cross-price effects may not be symmetric at a point in time, but an integral version of Slutsky symmetry must hold.

The paper is organized as follows. In section 2 a dynamic optimization problem containing exogenous parameters, nondifferential equality constraints, and nondifferential inequality constraints is developed, and the dynamic envelope theorem is stated and proven for this class of optimal control problems. A simple example illustrates the main arguments, and the special cases of time-variant parameters and autonomous current value problems are discussed. An application to an intertemporal consumer choice problem is presented in section 3. Section 4 contains a summary and conclusion.

2. The dynamic envelope theorem

The problem of interest is to

$$\text{maximize } J=\int_0^T f[x(t),u(t),\alpha,t]\,\mathrm{d}t, \tag{P1}$$

subject to

$$\dot{x}(t)=g[x(t),u(t),\alpha,t], \qquad \forall t\in[0,T],$$

$$x(0)=x_0, \quad \text{fixed},$$

$$h[x(t),u(t),\alpha,t]=0, \qquad \forall t\in[0,T],$$

$$\omega[x(t),u(t),\alpha,t]\geq 0, \qquad \forall t\in[0,T],$$

where $x\in\mathbb{R}^n$ is a vector of state variables, $u\in\mathbb{R}^m$ is a vector of control variables, $\alpha\in\mathscr{A}\subset\mathbb{R}^r$ is a vector of parameters with values in an open, bounded, and convex set, h: $\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}^r\times\mathbb{R}\rightarrow\mathbb{R}^\kappa$ and ω: $\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}^r\times\mathbb{R}\rightarrow\mathbb{R}^l$. Let the Hamiltonian be defined by

$$H=f[x(t),u(t),\alpha,t]+\lambda(t)'g[x(t),u(t),\alpha,t], \tag{1}$$

where $\lambda\in\mathbb{R}^n$ is an n-vector of costate variables for the state equations and a prime (′) denotes vector transposition. The optimal control solution maximizes H with respect to u subject to $h=0$ and $\omega\geq 0$ for each $t\in[0,T]$. The Lagrangian for the constrained maximization problem is

$$\begin{aligned}L=&f[x(t),u(t),\alpha,t]+\lambda(t)'g[x(t),u(t),\alpha,t]\\&+\mu(t)'h[x(t),u(t),\alpha,t]+\varphi(t)'\omega[x(t),u(t),\alpha,t],\end{aligned} \tag{2}$$

where $\mu\in\mathbb{R}^\kappa$ is a κ-vector of Lagrange multipliers for the nondifferential equality constraints and $\varphi\in\mathbb{R}^l_+$ is an l-vector of Lagrange multipliers for the nondifferential inequality constraints.

In problems of this type, the possibility exists that some of the inequality constraints will be nonbinding for part of the planning horizon and that the set of binding constraints will change along the optimal path. Such changes produce corners in the optimal path, and the solution may not be differentiable in the parameters $\boldsymbol{\alpha}$ at those corners. Nevertheless, many economic problems include inequality constraints. Examples include nonnegativity constraints on variable input use, capacity constraints in production due to a finite stock of physical capital, and budget constraints in consumption. It is therefore essential to incorporate inequality constraints in the analysis and to account for changes in the set of binding constraints along the optimal path.

Thus, $\forall(\boldsymbol{\alpha}, t) \in \mathscr{A} \times [0,T]$, let the optimal controls be $\boldsymbol{u}^*(\boldsymbol{\alpha}, t)$, the optimal levels of the state variables be $\boldsymbol{x}^*(\boldsymbol{\alpha}, t)$, and the set of binding inequality constraints be $E(\boldsymbol{\alpha}, t) \equiv \{j\colon \omega_j[\boldsymbol{x}^*(\boldsymbol{\alpha}, t), \boldsymbol{u}^*(\boldsymbol{\alpha}, t), \boldsymbol{\alpha}, t] = 0\}$. Define a switch point in the optimal path as a time s, $0 < s < T$, when the set $E(\boldsymbol{\alpha}, t)$ changes. That is, as the optimal path passes through a switch point s, at least one inequality constraint becomes just binding or just nonbinding. Let the set of all switch points be $S(\boldsymbol{\alpha})$ and the set of all other times in the planning horizon be $T(\boldsymbol{\alpha}) \equiv [0,T] \sim S(\boldsymbol{\alpha})$. The following structure is imposed on the problem:

A1. $f, g, \boldsymbol{h}, \boldsymbol{\omega}$ are twice continuously differentiable.

A2. H is strictly quasiconcave in $\boldsymbol{u}$, and $\forall(\boldsymbol{x}, \boldsymbol{\alpha}, t) \in \mathbb{R}^n \times \mathscr{A} \times [0,T]$,

$$U(\boldsymbol{x}, \boldsymbol{\alpha}, t) \equiv \{\boldsymbol{u} \in \mathbb{R}^m\colon \boldsymbol{h}(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{\alpha}, t) = \boldsymbol{0},\ \boldsymbol{\omega}(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{\alpha}, t) \geq \boldsymbol{0}\},$$

is compact, convex, and $\exists \boldsymbol{u} \in U(\boldsymbol{x}, \boldsymbol{\alpha}, t)$ such that $\boldsymbol{\omega}(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{\alpha}, t) \gg \boldsymbol{0}$.

A3. The optimal path, $\{\boldsymbol{x}^*(\boldsymbol{\alpha}, t), \boldsymbol{\lambda}^*(\boldsymbol{\alpha}, t)\}$, is unique $\forall \boldsymbol{\alpha} \in \mathscr{A}$.

A4. For each $\boldsymbol{\alpha} \in \mathscr{A}$ there is a finite number of distinct switch points $s_j(\boldsymbol{\alpha}) \in S(\boldsymbol{\alpha})$, $j = 0, 1, \ldots, q$, and without loss in generality,

$$0 \equiv s_0(\boldsymbol{\alpha}) < s_1(\boldsymbol{\alpha}) < s_2(\boldsymbol{\alpha}) < \cdots < s_q(\boldsymbol{\alpha}) \equiv T.$$

Before proceeding, assumptions A1–A4 and their relationship to previous research warrant some discussion.

There is a rapidly growing literature in mathematical programming focusing on nonsmooth analysis [Clarke (1975, 1983), Clarke and Loewen (1986), Clarke and Vinter (1986), Gauvin (1979), Rockafellar (1981, 1985), and Vinter (1988)]. Most of these studies replace hypotheses like A1–A4 with Lipschitz continuity conditions, nonempty compact convex-valued differential inclusions to describe the dynamics, and the assumption that a feasible solution exists. Generalized gradients are then utilized to characterize conditions under which the optimal performance function is Lipschitz-continuous, admits directional derivatives, or is actually differentiable [Clarke and Loewen (1986)].

Although the solution to problem (P1) will often be nonsmooth at the switch points, there are several reasons why we adopt A1–A4 and follow the standard approach of optimal control theory rather than approach the problem with these newer methods. First, and perhaps foremost, the calculus of generalized gradients takes the form of set inclusions. The main thrust of this paper is to demonstrate the *equivalence* of different algebraic expressions for the rate of change of the optimal performance function with respect to parameters affecting the optimal control problem. In terms of generalized gradients, such equivalences cannot be obtained for such elementary operations as finite sums, chain rules, or product rules [Clarke (1983, pp. 38–50)]. Since the equivalence of the different algebraic expressions for the rate of change of the optimal performance function generates all of the interesting qualitative results, the methods of nonsmooth analysis and generalized gradients are neither applicable nor relevant to the envelope theorem in dynamic optimization problems.

Second, the recent results that have been obtained with nonsmooth analysis, including the generalized maximum principle [Clarke (1983, theorem 5.2.1)] and the results on the generalized gradient of the optimal performance function [Clarke and Loewen (1986)], do not carry over to problems where the set of feasible controls depends explicitly on the state variables [Clarke (1983, p. 212)]. This is clearly the case in the current problem, and many if not all economic problems are characterized by precisely this sort of dependence. For example, outputs are produced with variable inputs and capital assets. The feasible combinations of the control variables, i.e., the rates of output that can be produced with a given set of variable input use rates, depend on the state variables, i.e., the stock of productive capital assets, through the firm's technology.

Third, although Lipschitz continuity is less restrictive than twice continuous differentiability, it is well-known that Lipschitz-continuous functions are continuously differentiable almost everywhere, that convex functions are twice differentiable almost everywhere, and that the twice differentiable convex functions form a dense subset of all proper convex functions [Alexandroff (1939), Clarke (1975), Rockafellar (1970)]. Furthermore, most economic problems are characterized by convexity due to such considerations as decreasing returns to scale and diminishing marginal rates of substitution. Thus, assumption A1 merely amounts to a strengthening of twice differentiability almost everywhere to twice continuous differentiability everywhere. Moreover, the full strength of this assumption is really only needed to obtain the twice continuous differentiability of the optimal performance function.

Fourth, rather than assert that a solution exists, for obvious reasons we prefer to identify properties which ensure that such a solution exists. It is clear that a feasible solution to (P1) exists if and only if a feasible control, $\boldsymbol{u}$, exists given (x, α, t). And this is precisely what the conditions on the set

$U(x, \alpha, t)$ guarantee. Combining the conditions on $U(x, \alpha, t)$ with the property that H is strictly quasiconcave in u then implies that the Arrow, Hurwicz, and Uzawa (1961) constraint qualification is satisfied and that the first-order Kuhn–Tucker conditions for a constrained maximum are necessary and sufficient for identifying the unique closed-loop optimal control.

Fifth, it is well-known that uniqueness of the optimal path is a sufficient condition for the differentiability of the optimal performance function, and usually a necessary condition, including cases where nonsmooth analysis and generalized gradients are used [Clarke and Loewen (1986), Seierstad and Sydsaeter (1987)]. Oniki (1973) pointed out that twice continuous differentiability of f, g, h, and ω and uniqueness of the optimal path are sufficient for the state and costate variables and their time derivatives to be continuous on $[0, T]$ and continuously differentiable on $T(\alpha)$, and for the switch points to be uniquely defined and continuously differentiable in α. If the maximized Hamiltonian,

$$H^*(x, \lambda, \alpha, t) \equiv \max_{u} \{H: u \in \mathbb{R}^m,\ h = 0,\ \omega \geq 0\}, \tag{3}$$

is strictly concave in x, then it is well-known that the optimal path is unique [Arrow and Kurz (1970), Epstein (1978)]. Moreover, existing sufficiency theorems require concavity of H^* in x [Arrow and Kurz (1970), Clarke (1983), Seierstad and Sydsaeter (1987)], and most economic problems possess this property and therefore also satisfy condition A3.

Finally, at most a countable number of distinct switch points is a condition that will be satisfied by any problem of practical interest. This property is essential when we interchange the role of differentiation and integration in evaluating the partial derivative of the optimal performance function with respect to a parameter. Strengthening this to a finite number is a very mild simplification that will be satisfied by most problems, since the number of switch points, although finite, can be very large. Therefore, properties A1–A4, or very similar properties, appear to be crucial to the existence, proof, and qualitative implications of the dynamic envelope theorem presented in the sequel.

We define the optimal performance function by[1]

$$J^*(\alpha) = \int_0^T f[x^*(\alpha, t), u^*(\alpha, t), \alpha, t]\, \mathrm{d}t. \tag{4}$$

With these preliminaries, we now state and prove our main result, which

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[1] J^* also depends on x_0 and T. Our focus is on the relationship between J^* and parameters other than the initial values of the state variables or the limits of integration, and these arguments have been omitted to reduce the notational burden.

gives two equivalent ways of expressing the marginal effect of a change in the parameters on the optimal performance function.

Theorem 1. If problem (P1) satisfies conditions A1–A4, then $J^*(\boldsymbol{\alpha})$ is twice continuously differentiable in $\boldsymbol{\alpha}$, and

$$\begin{aligned}\frac{\partial J^*}{\partial \alpha} &= \int_0^T \left[\left(\frac{\partial x^{*\prime}}{\partial \alpha}\right)\left(\frac{\partial f}{\partial x}\right) + \left(\frac{\partial u^{*\prime}}{\partial \alpha}\right)\left(\frac{\partial f}{\partial u}\right) + \left(\frac{\partial f}{\partial \alpha}\right)\right] \mathrm{d}t \\ &= \int_0^T \left[\left(\frac{\partial f}{\partial \alpha}\right) + \left(\frac{\partial g'}{\partial \alpha}\right)\lambda^* + \left(\frac{\partial h'}{\partial \alpha}\right)\mu^* + \left(\frac{\partial \omega'}{\partial \alpha}\right)\varphi^*\right] \mathrm{d}t \\ &= \int_0^T \frac{\partial L(x^*, u^*, \lambda^*, \mu^*, \varphi^*, \alpha, t)}{\partial \alpha} \mathrm{d}t. \end{aligned} \tag{5}$$

Proof. The first-order conditions for a constrained maximum are

$$\frac{\partial L}{\partial u} = \frac{\partial f}{\partial u} + \left(\frac{\partial g'}{\partial u}\right)\lambda + \left(\frac{\partial h'}{\partial u}\right)\mu + \left(\frac{\partial \omega'}{\partial u}\right)\varphi = 0, \tag{6}$$

$$\frac{\partial L}{\partial \mu} = h(x, u, \alpha, t) = 0, \tag{7}$$

$$\frac{\partial L}{\partial \varphi} = \omega(x, u, \alpha, t) \geq 0, \quad \varphi \geq 0, \quad \varphi'\omega(x, u, \alpha, t) = 0. \tag{8}$$

By A1 and A2, the closed-loop solutions for the control variables and the Lagrange multipliers, $\hat{u}(x, \lambda, \alpha, t)$, $\hat{\mu}(x, \lambda, \alpha, t)$, and $\hat{\varphi}(x, \lambda, \alpha, t)$, are unique and continuous in (x, λ, α, t) on $[0, T]$ by the maximum theorem and continuously differentiable on $T(\alpha)$ by the implicit function theorem. Furthermore, the first-order conditions for a constrained maximum with respect to u are necessary and sufficient. In addition to (6)–(8), the optimal control solution requires that along the optimal path, λ and x must satisfy the following conjugate differential equations on $[0, T]$, [Hestenes (1965, 1966), Pontryagin et al. (1962)]:

$$\dot{\lambda} = -\left[\frac{\partial f}{\partial x} + \left(\frac{\partial g'}{\partial x}\right)\lambda + \left(\frac{\partial h'}{\partial x}\right)\mu + \left(\frac{\partial \omega'}{\partial x}\right)\varphi\right], \quad \lambda(T) = 0, \quad \forall \alpha \in \mathscr{A}, \tag{9}$$

$$\dot{x} = g(x, u, \alpha, t), \quad x(0) = x_0, \quad \text{fixed} \quad \forall \alpha \in \mathscr{A}. \tag{10}$$

It follows from A1, A3, and Oniki (1973) that $\boldsymbol{x}^*(\boldsymbol{\alpha},t)$ and $\boldsymbol{\lambda}^*(\boldsymbol{\alpha},t)$ are continuous functions on $[0,T]$ and continuously differentiable on $\boldsymbol{T}(\boldsymbol{\alpha})$ and that the switch points $s_j(\boldsymbol{\alpha})$ are uniquely defined and continuously differentiable. Substitute $\boldsymbol{x}^*(\boldsymbol{\alpha},t)$, $\boldsymbol{\lambda}^*(\boldsymbol{\alpha},t)$ for $\boldsymbol{x}, \boldsymbol{\lambda}$ in $\hat{\boldsymbol{u}}(\boldsymbol{x},\boldsymbol{\lambda},\boldsymbol{\alpha},t)$, $\hat{\boldsymbol{\mu}}(\boldsymbol{x},\boldsymbol{\lambda},\boldsymbol{\alpha},t)$, and $\hat{\boldsymbol{\varphi}}(\boldsymbol{x},\boldsymbol{\lambda},\boldsymbol{\alpha},t)$ to obtain the open-loop controls and Lagrange multipliers as continuous functions on $[0,T]$ and continuously differentiable on $\boldsymbol{T}(\boldsymbol{\alpha})$,

$$\boldsymbol{u}^*(\boldsymbol{\alpha},t) \equiv \hat{\boldsymbol{u}}[\boldsymbol{x}^*(\boldsymbol{\alpha},t),\boldsymbol{\lambda}^*(\boldsymbol{\alpha},t),\boldsymbol{\alpha},t], \tag{11}$$

$$\boldsymbol{\mu}^*(\boldsymbol{\alpha},t) \equiv \hat{\boldsymbol{\mu}}[\boldsymbol{x}^*(\boldsymbol{\alpha},t),\boldsymbol{\lambda}^*(\boldsymbol{\alpha},t),\boldsymbol{\alpha},t], \tag{12}$$

$$\boldsymbol{\varphi}^*(\boldsymbol{\alpha},t) \equiv \hat{\boldsymbol{\varphi}}[\boldsymbol{x}^*(\boldsymbol{\alpha},t),\boldsymbol{\lambda}^*(\boldsymbol{\alpha},t),\boldsymbol{\alpha},t]. \tag{13}$$

Along the optimal path the state equations can be written as

$$\frac{\partial \boldsymbol{x}^*(\boldsymbol{\alpha},t)}{\partial t} \equiv \boldsymbol{g}[\boldsymbol{x}^*(\boldsymbol{\alpha},t),\boldsymbol{u}^*(\boldsymbol{\alpha},t),\boldsymbol{\alpha},t], \quad \boldsymbol{x}^*(\boldsymbol{\alpha},0) \equiv \boldsymbol{x}_0. \tag{14}$$

Since $\boldsymbol{g}$, $\boldsymbol{x}^*$, and $\boldsymbol{u}^*$ are continuously differentiable on $\boldsymbol{T}(\boldsymbol{\alpha})$,

$$\frac{\partial^2 \boldsymbol{x}^*}{\partial t\,\partial \boldsymbol{\alpha}} \equiv \left(\frac{\partial \boldsymbol{g}}{\partial \boldsymbol{x}'}\right)\left(\frac{\partial \boldsymbol{x}^*}{\partial \boldsymbol{\alpha}}\right) + \left(\frac{\partial \boldsymbol{g}}{\partial \boldsymbol{u}'}\right)\left(\frac{\partial \boldsymbol{u}^*}{\partial \boldsymbol{\alpha}}\right) + \left(\frac{\partial \boldsymbol{g}}{\partial \boldsymbol{\alpha}}\right) \tag{15}$$

exists and is continuous on $\boldsymbol{T}(\boldsymbol{\alpha})$. Now, rewrite (4) as

$$J^*(\boldsymbol{\alpha}) = \sum_{j=1}^{q}\left[\int_{s_{j-1}(\boldsymbol{\alpha})}^{s_j(\boldsymbol{\alpha})} f[\boldsymbol{x}^*(\boldsymbol{\alpha},t),\boldsymbol{u}^*(\boldsymbol{\alpha},t),\boldsymbol{\alpha},t]\,\mathrm{d}t\right]. \tag{16}$$

Apply Leibnitz' rule to the right-hand side of (16),

$$\begin{aligned}\frac{\partial J^*}{\partial \boldsymbol{\alpha}} &= \sum_{j=1}^{q}\left\{\int_{s_{j-1}(\boldsymbol{\alpha})}^{s_j(\boldsymbol{\alpha})}\left[\left(\frac{\partial \boldsymbol{x}^{*\prime}}{\partial \boldsymbol{\alpha}}\right)\left(\frac{\partial f}{\partial \boldsymbol{x}}\right) + \left(\frac{\partial \boldsymbol{u}^{*\prime}}{\partial \boldsymbol{\alpha}}\right)\left(\frac{\partial f}{\partial \boldsymbol{u}}\right) + \left(\frac{\partial f}{\partial \boldsymbol{\alpha}}\right)\right]\mathrm{d}t\right\}\\ &\quad + \sum_{j=1}^{q} f\big(\boldsymbol{x}^*(\boldsymbol{\alpha},s_j(\boldsymbol{\alpha})),\boldsymbol{u}^*(\boldsymbol{\alpha},s_j(\boldsymbol{\alpha})),\boldsymbol{\alpha},s_j(\boldsymbol{\alpha})\big)\frac{\partial s_j(\boldsymbol{\alpha})}{\partial \boldsymbol{\alpha}}\\ &\quad - \sum_{j=1}^{q} f\big(\boldsymbol{x}^*(\boldsymbol{\alpha},s_{j-1}(\boldsymbol{\alpha})),\boldsymbol{u}^*(\boldsymbol{\alpha},s_{j-1}(\boldsymbol{\alpha})),\boldsymbol{\alpha},s_{j-1}(\boldsymbol{\alpha})\big)\\ &\qquad \times \frac{\partial s_{j-1}(\boldsymbol{\alpha})}{\partial \boldsymbol{\alpha}},\\ &= \int_0^T\left[\left(\frac{\partial \boldsymbol{x}^{*\prime}}{\partial \boldsymbol{\alpha}}\right)\left(\frac{\partial f}{\partial \boldsymbol{x}}\right) + \left(\frac{\partial \boldsymbol{u}^{*\prime}}{\partial \boldsymbol{\alpha}}\right)\left(\frac{\partial f}{\partial \boldsymbol{u}}\right) + \left(\frac{\partial f}{\partial \boldsymbol{\alpha}}\right)\right]\mathrm{d}t,\end{aligned} \tag{17}$$

by the continuity of f, $\boldsymbol{x}^*$, $\boldsymbol{u}^*$, and $\partial s_j(\boldsymbol{\alpha})/\partial \boldsymbol{\alpha}$, $\forall j = 0,\dots,q$, and since $s_0(\boldsymbol{\alpha}) \equiv 0$ and $s_q(\boldsymbol{\alpha}) \equiv T$. The integrand in the last expression for the right-hand side of (17) is piecewise continuous, with at most finite jumps at the switch points. Therefore $\partial J^*/\partial \boldsymbol{\alpha}$ exists and is continuous. From (7) and (8) the following conditions hold on $[0, T]$:

$$\boldsymbol{h}[\boldsymbol{x}^*(\boldsymbol{\alpha},t),\boldsymbol{u}^*(\boldsymbol{\alpha},t),\boldsymbol{\alpha},t] \equiv \mathbf{0}, \tag{18}$$

$$\boldsymbol{\varphi}^*(\boldsymbol{\alpha},t)'\boldsymbol{\omega}[\boldsymbol{x}^*(\boldsymbol{\alpha},t),\boldsymbol{u}^*(\boldsymbol{\alpha},t),\boldsymbol{\alpha},t] \equiv 0. \tag{19}$$

Partially differentiate (18) with respect to $\boldsymbol{\alpha}'$, transpose and take the vector product with $\boldsymbol{\mu}^*(\boldsymbol{\alpha}, t)$, $\forall t \in T(\boldsymbol{\alpha})$,

$$\left[\left(\frac{\partial \boldsymbol{x}^{*\prime}}{\partial \boldsymbol{\alpha}}\right)\left(\frac{\partial \boldsymbol{h}'}{\partial \boldsymbol{x}}\right) + \left(\frac{\partial \boldsymbol{u}^{*\prime}}{\partial \boldsymbol{\alpha}}\right)\left(\frac{\partial \boldsymbol{h}'}{\partial \boldsymbol{u}}\right) + \left(\frac{\partial \boldsymbol{h}'}{\partial \boldsymbol{\alpha}}\right)\right]\boldsymbol{\mu}^* \equiv \mathbf{0}. \tag{20}$$

Now partially differentiate (19) with respect to $\boldsymbol{\alpha}$, $\forall t \in T(\boldsymbol{\alpha})$,

$$\left(\frac{\partial \boldsymbol{\varphi}^{*\prime}}{\partial \boldsymbol{\alpha}}\right)\boldsymbol{\omega} + \left[\left(\frac{\partial \boldsymbol{x}^{*\prime}}{\partial \boldsymbol{\alpha}}\right)\left(\frac{\partial \boldsymbol{\omega}'}{\partial \boldsymbol{x}}\right) + \left(\frac{\partial \boldsymbol{u}^{*\prime}}{\partial \boldsymbol{\alpha}}\right)\left(\frac{\partial \boldsymbol{\omega}'}{\partial \boldsymbol{u}}\right) + \left(\frac{\partial \boldsymbol{\omega}'}{\partial \boldsymbol{\alpha}}\right)\right]\boldsymbol{\varphi}^* \equiv \mathbf{0}. \tag{21}$$

Now, $\forall j = 1,\dots,l$, $\forall t \in T(\boldsymbol{\alpha})$, if $\omega_j > 0$, then $\varphi_j^* = 0$ and $\partial \varphi_j^*/\partial \boldsymbol{\alpha} = \mathbf{0}$; otherwise $\omega_j = 0$. The control variables, state and costate variables, and Lagrange multipliers pass through the boundaries of the inequality constraints continuously, and their derivatives with respect to $\boldsymbol{\alpha}$ are piecewise continuous, with at most finite jumps at the switch points. Therefore, $(\partial \boldsymbol{\varphi}^{*\prime}/\partial \boldsymbol{\alpha})\boldsymbol{\omega} = \mathbf{0}$, $\forall t \in [0\,T]$. Integrate (20) and (21) and add the results (which equal zero $\forall \boldsymbol{\alpha} \in \mathscr{A}$) to (17),

$$\begin{aligned}\frac{\partial J^*}{\partial \boldsymbol{\alpha}} = \int_0^T \Bigg\{ &\left(\frac{\partial \boldsymbol{x}^{*\prime}}{\partial \boldsymbol{\alpha}}\right)\left[\left(\frac{\partial f}{\partial \boldsymbol{x}}\right) + \left(\frac{\partial \boldsymbol{h}'}{\partial \boldsymbol{x}}\right)\boldsymbol{\mu}^* + \left(\frac{\partial \boldsymbol{\omega}'}{\partial \boldsymbol{x}}\right)\boldsymbol{\varphi}^*\right] \\ &+ \left(\frac{\partial \boldsymbol{u}^{*\prime}}{\partial \boldsymbol{\alpha}}\right)\left[\left(\frac{\partial f}{\partial \boldsymbol{u}}\right) + \left(\frac{\partial \boldsymbol{h}'}{\partial \boldsymbol{u}}\right)\boldsymbol{\mu}^* + \left(\frac{\partial \boldsymbol{\omega}'}{\partial \boldsymbol{u}}\right)\boldsymbol{\varphi}^*\right] \\ &+ \left(\frac{\partial f}{\partial \boldsymbol{\alpha}}\right)\left(\frac{\partial \boldsymbol{h}'}{\partial \boldsymbol{\alpha}}\right)\boldsymbol{\mu}^* + \left(\frac{\partial \boldsymbol{\omega}'}{\partial \boldsymbol{\alpha}}\right)\boldsymbol{\varphi}^*\Bigg\}\,\mathrm{d}t.\end{aligned} \tag{22}$$

Substitute for $\partial f/\partial u$ from (6) and cancel common terms,

$$\frac{\partial J^*}{\partial \alpha} = \int_0^T \left\{ \left(\frac{\partial x^{*\prime}}{\partial \alpha} \right) \left[\left(\frac{\partial f}{\partial x} \right) + \left(\frac{\partial h'}{\partial x} \right) \mu^* + \left(\frac{\partial \omega'}{\partial x} \right) \varphi^* \right] \right.$$
$$\left. - \left(\frac{\partial u^{*\prime}}{\partial \alpha} \right) \left(\frac{\partial g'}{\partial u} \right) \lambda^* + \left(\frac{\partial f}{\partial \alpha} \right) + \left(\frac{\partial h'}{\partial \alpha} \right) \mu^* + \left(\frac{\partial \omega'}{\partial \alpha} \right) \varphi^* \right\} \mathrm{d}t. \tag{23}$$

Solve (15) for $(\partial g/\partial u')(\partial u^*/\partial \alpha)$, substitute the result into (23), and regroup terms,

$$\frac{\partial J^*}{\partial \alpha} = \int_0^T \left\{ \left(\frac{\partial x^{*\prime}}{\partial \alpha} \right) \left[\left(\frac{\partial f}{\partial x} \right) + \left(\frac{\partial g'}{\partial x} \right) \lambda^* + \left(\frac{\partial h'}{\partial x} \right) \mu^* + \left(\frac{\partial \omega'}{\partial x} \right) \varphi^* \right] \right.$$
$$- \left(\frac{\partial^2 x^{*\prime}}{\partial t\, \partial \alpha} \right) \lambda^* + \left(\frac{\partial f}{\partial \alpha} \right) + \left(\frac{\partial g'}{\partial \alpha} \right) \lambda^*$$
$$\left. + \left(\frac{\partial h'}{\partial \alpha} \right) \mu^* + \left(\frac{\partial \omega'}{\partial \alpha} \right) \varphi^* \right\} \mathrm{d}t. \tag{24}$$

Since $\partial^2 x^{*\prime}/\partial t\, \partial \alpha = \partial^2 x^*/\partial \alpha\, \partial t$, $\forall t \in T(\alpha)$, by Young's theorem, the second expression in (24) can be integrated by parts, which gives

$$\int_0^T \left(\frac{\partial^2 x^*(\alpha, t)'}{\partial \alpha\, \partial t} \right) \lambda^*(\alpha, t)\, \mathrm{d}t$$
$$= \left(\frac{\partial x^*(\alpha, t)'}{\partial \alpha} \right) \lambda^*(\alpha, t) \bigg|_0^T - \int_0^T \frac{\partial x^*(\alpha, t)'}{\partial \alpha} \frac{\partial \lambda^*(\alpha, t)}{\partial t} \mathrm{d}t$$
$$= - \int_0^T \frac{\partial x^*(\alpha, t)'}{\partial \alpha} \frac{\partial \lambda^*(\alpha, t)}{\partial t} \mathrm{d}t, \tag{25}$$

since $\lambda^*(\alpha, T) \equiv 0$ and $x(\alpha, 0) \equiv x_0$, $\forall \alpha \in \mathscr{A}$. Substitute the open-loop solution $\{x^*(\alpha, t), u^*(\alpha, t), \lambda^*(\alpha, t), \mu^*(\alpha, t), \varphi^*(\alpha, t)\}$ for $\{x, u, \lambda, \mu, \varphi\}$ in (9),

$$\frac{\partial \lambda^*(\alpha, t)}{\partial t} = - \left[\frac{\partial f}{\partial x} + \left(\frac{\partial g'}{\partial x} \right) \lambda^* + \left(\frac{\partial h'}{\partial x} \right) \mu^* + \left(\frac{\partial \omega'}{\partial x} \right) \varphi^* \right]. \tag{26}$$

Combine (24)–(26) and rearrange terms,

$$\frac{\partial J^*}{\partial \alpha} = \int_0^T \left[\left(\frac{\partial f}{\partial \alpha} \right) + \left(\frac{\partial g'}{\partial \alpha} \right) \lambda^* + \left(\frac{\partial h'}{\partial \alpha} \right) \mu^* + \left(\frac{\partial \omega'}{\partial \alpha} \right) \varphi^* \right] \mathrm{d}t$$

$$= \int_0^T \frac{\partial L(x^*, u^*, \lambda^*, \mu^*, \varphi^*, \alpha, t)}{\partial \alpha} \mathrm{d}t.$$

Finally, by nearly identical arguments to those above, the integrand on the right-hand side is continuously differentiable in α on $T(\alpha)$ with at most finite jumps in the derivatives on $S(\alpha)$. Hence $J^*(\alpha)$ is twice continuously differentiable. Q.E.D.

In one respect, Theorem 1 is precisely the result we would expect for the dynamic envelope theorem. The static envelope theorem states that the rate of change of an indirect objective function with respect to a parameter in constrained optimization is equal to the partial derivative of the Lagrangian with respect to the parameter, evaluating the choice variables at their optimal levels for the initial value of the parameter and taking the derivative prior to maximization. Similarly, Theorem 1 states that the rate of change of the optimal performance function with respect to a parameter is the integral of the partial derivative of the Lagrangian function with respect to the parameter, evaluated along the optimal path for the initial value of α. This derivative is obtained by partially differentiating the Lagrangian with respect to the parameters *prior to* finding the constrained maximum with respect to the control variables, substituting the open-loop solutions for the control, state and costate variables, and the Lagrange multipliers into the resulting expression for $\partial L/\partial \alpha$, and integrating over the planning horizon. An equivalent expression is obtained by substituting the optimal solutions for the state and control variables directly into the objective function, partially differentiating with respect to the parameters, and interchanging the operations of differentiation and integration. This is also analogous to the static case.

But there is an important difference between the dynamic and static envelope theorems. In static problems, substituting the optimal solutions for the choice variables into the Lagrangian and then differentiating with respect to the parameters gives the same result as reversing these steps. In dynamic problems, substituting the optimal solutions into the Lagrangian, differentiating with respect to the parameters, and then integrating gives a different result than when the first two steps are reversed. This is because the optimal performance function for the interval $(t, T]$, $J^*(\alpha, t)$, satisfies the partial differential equation

$$\begin{aligned}
&-\partial J^*(\alpha,t)/\partial t \\
&= H[x^*(\alpha,t), u^*(\alpha,t), \lambda^*(\alpha,t), \alpha, t] \\
&= f[x^*(\alpha,t), u^*(\alpha,t), \alpha, t] + \lambda^*(\alpha,t)' g[x^*(\alpha,t), u^*(\alpha,t), \alpha, t] \\
&= f[x^*(\alpha,t), u^*(\alpha,t), \alpha, t] + \lambda^*(\alpha,t)' g[x^*(\alpha,t), u^*(\alpha,t), \alpha, t] \\
&\quad + \mu^*(\alpha,t)' h[x^*(\alpha,t), u^*(\alpha,t), \alpha, t] \\
&\quad + \varphi^*(\alpha,t)' \omega[x^*(\alpha,t), u^*(\alpha,t), \alpha, t] \\
&= L[x^*(\alpha,t), u^*(\alpha,t), \lambda^*(\alpha,t), \mu^*(\alpha,t), \varphi^*(\alpha,t), \alpha, t] \\
&\equiv L^*(\alpha,t), \qquad (27)
\end{aligned}$$

as well as the ordinary differential equation

$$\begin{aligned}
-\mathrm{d}J^*(\alpha,t)/\mathrm{d}t &\equiv f[x^*(\alpha,t), u^*(\alpha,t), \alpha, t] \\
&\equiv H[x^*(\alpha,t), u^*(\alpha,t), \lambda^*(\alpha,t), \alpha, t] \\
&\quad - \lambda^*(\alpha,t)' \dot{x}^*(\alpha,t) \\
&\equiv L^*(\alpha,t) - \lambda^*(\alpha,t)' \dot{x}^*(\alpha,t). \qquad (28)
\end{aligned}$$

Differentiating $f[x^*(\alpha,t), u^*(\alpha,t), \alpha, t] \equiv L^*(\alpha,t) - \lambda^*(\alpha,t)'\dot{x}^*(\alpha,t)$ with respect to α, integrating over the planning horizon $[0,T]$, and applying Theorem 1 gives

$$\begin{aligned}
\frac{\partial J^*(\alpha)}{\partial \alpha} - \int_0^T \left(\frac{\partial L^*(\alpha,t)}{\partial \alpha} \right) \mathrm{d}t &= -\int_0^T \left(\frac{\partial \lambda^{*\prime}}{\partial \alpha} \frac{\partial x^*}{\partial t} + \frac{\partial^2 x^{*\prime}}{\partial t\, \partial \alpha} \lambda^* \right) \mathrm{d}t \\
&= \int_0^T \left(\frac{\partial x^{*\prime}}{\partial \alpha} \frac{\partial \lambda^*}{\partial t} - \frac{\partial \lambda^{*\prime}}{\partial \alpha} \frac{\partial x^*}{\partial t} \right) \mathrm{d}t. \qquad (29)
\end{aligned}$$

Another important difference between the envelope theorem for static and dynamic optimization problems is the nature of the symmetry conditions that result. As a simple consequence of Theorem 1 and Young's theorem, the matrix of second-order derivatives of the optimal performance function, $\partial^2 J^*(\alpha)/\partial\alpha\,\partial\alpha'$, is symmetric. As a result, the following complex set of terms are symmetric $\forall i, j = 1, \ldots, r$:

$$\begin{aligned}\frac{\partial^2 J^*(\boldsymbol{\alpha})}{\partial\alpha_i\partial\alpha_j} = \int_0^T \Bigg[& \frac{\partial^2 f}{\partial\alpha_i\partial\alpha_j} + \boldsymbol{\lambda}^{*\prime}\frac{\partial^2 \boldsymbol{g}}{\partial\alpha_i\partial\alpha_j} + \boldsymbol{\mu}^{*\prime}\frac{\partial^2 \boldsymbol{h}}{\partial\alpha_i\partial\alpha_j} + \boldsymbol{\varphi}^{*\prime}\frac{\partial^2 \boldsymbol{\omega}}{\partial\alpha_i\partial\alpha_j} \\ & + \left(\frac{\partial^2 f}{\partial\alpha_i\partial\boldsymbol{x}'} + \boldsymbol{\lambda}^{*\prime}\frac{\partial^2 \boldsymbol{g}}{\partial\alpha_i\partial\boldsymbol{x}'} + \boldsymbol{\mu}^{*\prime}\frac{\partial^2 \boldsymbol{h}}{\partial\alpha_i\partial\boldsymbol{x}'} + \boldsymbol{\varphi}^{*\prime}\frac{\partial^2 \boldsymbol{\omega}}{\partial\alpha_i\partial\boldsymbol{x}'}\right)\frac{\partial\boldsymbol{x}^*}{\partial\alpha_j} \\ & + \left(\frac{\partial^2 f}{\partial\alpha_i\partial\boldsymbol{u}'} + \boldsymbol{\lambda}^{*\prime}\frac{\partial^2 \boldsymbol{g}}{\partial\alpha_i\partial\boldsymbol{u}'} + \boldsymbol{\mu}^{*\prime}\frac{\partial^2 \boldsymbol{h}}{\partial\alpha_i\partial\boldsymbol{u}'} + \boldsymbol{\varphi}^{*\prime}\frac{\partial^2 \boldsymbol{\omega}}{\partial\alpha_i\partial\boldsymbol{u}'}\right)\frac{\partial\boldsymbol{u}^*}{\partial\alpha_j} \\ & + \frac{\partial\boldsymbol{g}'}{\partial\alpha_i}\frac{\partial\boldsymbol{\lambda}^*}{\partial\alpha_j} + \frac{\partial\boldsymbol{h}'}{\partial\alpha_i}\frac{\partial\boldsymbol{\mu}^*}{\partial\alpha_j} + \frac{\partial\boldsymbol{\omega}'}{\partial\alpha_i}\frac{\partial\boldsymbol{\varphi}^*}{\partial\alpha_j}\Bigg]\mathrm{d}t. \qquad (30)\end{aligned}$$

Note, in particular, that symmetry in dynamic optimization problems holds only in integral form, not at a given point in time. Indeed, since the integrand will often not be continuous in the parameters at the switch points, at those points there will certainly not be any sort of temporal symmetry. It is only because of the smoothing process of integration that we obtain continuity of the second-order derivatives of the optimal performance function. This result generalizes the symmetry result obtained by Caputo (1990) applying primal–dual methods to optimal control problems with no inequality constraints.

Caputo (1990) also obtained an important curvature result for optimal control problems with nondifferential equality constraints. This result is that if the integrand, $f(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{\alpha}, t)$, is locally convex in the parameters, $\boldsymbol{\alpha}$, and the equations of motion and equality constraints are independent of $\boldsymbol{\alpha}$, then the optimal performance function is locally convex in $\boldsymbol{\alpha}$.

Curvature properties such as convexity of indirect profit functions in prices or the concavity of cost functions in input prices, and their qualitative implications, e.g., negatively sloped input demands or positively sloped output supplies, are well understood in static economic analysis. However, this is not the case in dynamic optimization. Moreover, some results regarding the qualitative properties of optimal control problems are only forthcoming when the optimal performance function is convex.

For example, the impact on the initial rate of soil erosion of an increase in the price of the cultivated crop is very difficult to determine unless the optimal performance function is convex in the price of the crop. But given this convexity and a comparative statics analysis of the long-run equilibrium showing that the steady state level of crop production is a decreasing function of the price of the crop (which it will be under reasonable conditions), it follows immediately that an increase in output prices accelerates the initial rate of soil erosion.

Thus, it is important to have an understanding of conditions which imply that the optimal performance function is convex in the parameters. The following result generalizes those of Caputo (1989, 1990) to optimal control problems with inequality constraints and an integrand that is globally convex in the parameters. This implies that the optimal performance function is globally convex in the parameters.

Theorem 2. Given A1–A4, if f *is convex in* $\boldsymbol{\alpha}$, $\partial g(\boldsymbol{x},\boldsymbol{u},\boldsymbol{\alpha},t)/\partial\boldsymbol{\alpha} \equiv \mathbf{0}$, $\partial h(\boldsymbol{x},\boldsymbol{u},\boldsymbol{\alpha},t)/\partial\boldsymbol{\alpha} \equiv \mathbf{0}$, *and* $\partial \omega(\boldsymbol{x},\boldsymbol{u},\boldsymbol{\alpha},t)/\partial\boldsymbol{\alpha} \equiv \mathbf{0}$, *then* $J^*(\boldsymbol{\alpha})$ *is convex in* $\boldsymbol{\alpha}$.

Proof. Let $\boldsymbol{\alpha}_0, \boldsymbol{\alpha}_1 \in \mathscr{A}$, and $\boldsymbol{\alpha}_2 = \theta\boldsymbol{\alpha}_0 + (1-\theta)\boldsymbol{\alpha}_1$ for arbitrary $\theta \in [0,1]$. Let $\{\boldsymbol{x}_0, \boldsymbol{u}_0\}$, $\{\boldsymbol{x}_1, \boldsymbol{u}_1\}$, and $\{\boldsymbol{x}_2, \boldsymbol{u}_2\}$ denote the optimal paths for the parameter vectors $\boldsymbol{\alpha}_0$, $\boldsymbol{\alpha}_1$, and $\boldsymbol{\alpha}_2$, respectively. Then $\boldsymbol{\alpha}_2 \in \mathscr{A}$ by convexity of $\mathscr{A}$, and since none of the constraints depend on $\boldsymbol{\alpha}$, $\{\boldsymbol{x}_2, \boldsymbol{u}_2\}$ is a feasible path for $\boldsymbol{\alpha}_0$ and $\boldsymbol{\alpha}_1$. Therefore,

$$\begin{aligned}
J^*(\boldsymbol{\alpha}_2) &\equiv \int_0^T f(\boldsymbol{x}_2, \boldsymbol{u}_2, \boldsymbol{\alpha}_2, t)\,\mathrm{d}t \\
&\leq \int_0^T \left[\theta f(\boldsymbol{x}_2, \boldsymbol{u}_2, \boldsymbol{\alpha}_0, t) + (1-\theta) f(\boldsymbol{x}_2, \boldsymbol{u}_2, \boldsymbol{\alpha}_1, t)\right]\mathrm{d}t \\
&\leq \int_0^T \left[\theta f(\boldsymbol{x}_0, \boldsymbol{u}_0, \boldsymbol{\alpha}_0, t) + (1-\theta) f(\boldsymbol{x}_1, \boldsymbol{u}_1, \boldsymbol{\alpha}_1, t)\right]\mathrm{d}t \\
&\equiv \theta J^*(\boldsymbol{\alpha}_0) + (1-\theta) J^*(\boldsymbol{\alpha}_1).
\end{aligned}$$

The first inequality follows from the convexity of f in $\boldsymbol{\alpha}$ and the second inequality follows from the fact that $\{\boldsymbol{x}_2, \boldsymbol{u}_2\}$ is feasible but may not be optimal for either $\boldsymbol{\alpha}_0$ or $\boldsymbol{\alpha}_1$. Q.E.D.

This result covers a large number of cases, but it is unlikely that a general statement concerning convexity can be obtained for parameters that enter the functions $\boldsymbol{g}$, $\boldsymbol{h}$, or $\boldsymbol{\omega}$ in a nontrivial way. The reason is that the optimal path for the convex combination of the parameter vectors $\boldsymbol{\alpha}_0$ and $\boldsymbol{\alpha}_1$, $\boldsymbol{\alpha}_2 = \theta\boldsymbol{\alpha}_0 + (1-\theta)\boldsymbol{\alpha}_1$, must be a feasible path for $\boldsymbol{\alpha}_0$ and $\boldsymbol{\alpha}_1$. When changes in $\boldsymbol{\alpha}$ either shift or rotate the boundaries of the feasible set, this condition may not be satisfied.

An analogy to static problems is the fact that, without a cardinal choice of the utility function, a consumer's expenditure function may be concave, convex, or neither in the utility index. The analogy also illustrates an exception to this concern. If f and ω are jointly concave in (x, u, α), and g and h are jointly linear in (x, u, α), then $J^*(\alpha)$ is concave in α, as shown by the next result.

Theorem 3. Given A1–A4, if f and ω are jointly concave in (x, u, α) and g and h are jointly linear in (x, u, α), then $J^*(\alpha)$ is concave in α.

Proof. Define α_0, α_1, α_2, $\{x_0, u_0\}$, $\{x_1, u_1\}$, and $\{x_2, u_2\}$ as in the proof of Theorem 2. By the concavity of ω in (x, u, α);

$$\omega[\theta x_0 + (1-\theta)x_1, \theta u_0 + (1-\theta)u_1, \alpha_2, t] \geq 0, \qquad \forall t \in [0, T].$$

Therefore, $\{\theta x_0 + (1-\theta)x_1, \theta u_0 + (1-\theta)u_1\}$ is a feasible path for α_2, and

$$\begin{aligned} J^*(\alpha_2) &\equiv \int_0^T f(x_2, u_2, \alpha_2, t)\,\mathrm{d}t \\ &\geq \int_0^T f[\theta x_0 + (1-\theta)x_1, \theta u_0 + (1-\theta)u_1, \alpha_2, t]\,\mathrm{d}t \\ &\geq \int_0^T [\theta f(x_0, u_0, \alpha_0, t) + (1-\theta) f(x_1, u_1, \alpha_1, t)]\,\mathrm{d}t \\ &\equiv \theta J^*(\alpha_0) + (1-\theta) J^*(\alpha_1). \end{aligned}$$

The first inequality follows since $\{\theta x_0 + (1-\theta)x_1, \theta u_0 + (1-\theta)u_1\}$ is feasible but not necessarily optimal for α_2 and the second inequality follows from the joint concavity of f in (x, u, α). Q.E.D.

An example

We present a simple example to illustrate the effect of corners in the optimal path and the differentiability and convexity of J^* in α. We seek to

$$\max J = \int_0^2 (\alpha x + ut - 0.5u^2)\,\mathrm{d}t,$$

subject to

$$\mathrm{d}x/\mathrm{d}t = u, \quad x(0) = 0, \quad u \leq 1, \tag{31}$$

where $-\infty < \alpha < \frac{1}{2}$. The Lagrangian is

$$L = \alpha x + ut - 0.5u^2 + \lambda u + \varphi(1-u), \tag{32}$$

and the necessary and sufficient conditions for an optimal control are

$$\partial L/\partial u = t - u + \lambda - \varphi = 0, \tag{33}$$

$$\partial L/\partial \varphi = 1 - u \geq 0, \quad \varphi \geq 0, \quad \varphi(1-u) = 0, \tag{34}$$

$$\partial L/\partial x = \alpha = -\mathrm{d}\lambda/\mathrm{d}t, \quad \lambda(2) = 0, \tag{35}$$

$$\partial L/\partial \lambda = u = \mathrm{d}x/\mathrm{d}t, \quad x(0) = 0. \tag{36}$$

Solving (35) for $\lambda^*(\alpha, t)$ gives

$$\lambda^*(\alpha, t) = \alpha(2-t). \tag{37}$$

There is a unique switch point, s, given by

$$s(\alpha) = (1-2\alpha)/(1-\alpha). \tag{38}$$

Note that $\lim_{\alpha \to 1/2} s(\alpha) = 0$, while $\lim_{\alpha \to -\infty} s(\alpha) = 2$, and that $s(\alpha)$ is continuously differentiable $\forall \alpha \in (-\infty, \frac{1}{2})$. For arbitrary $\alpha \in (-\infty, \frac{1}{2})$, the optimal solutions for $u^*(\alpha, t)$, $x^*(\alpha, t)$, and $\varphi^*(\alpha, t)$ are

$$u^*(\alpha, t) = \begin{cases} 2\alpha + (1-\alpha)t, & 0 \leq t \leq s(\alpha), \\ 1, & s(\alpha) \leq t \leq 2, \end{cases}$$

$$x^*(\alpha, t) = \begin{cases} 2\alpha t + (1-\alpha)t^2/2, & 0 \leq t \leq s(\alpha), \\ t - (1-2\alpha)^2/2(1-\alpha), & s(\alpha) \leq t \leq 2, \end{cases}$$

$$\varphi^*(\alpha, t) = \begin{cases} 0, & 0 \leq t \leq s(\alpha), \\ (1-\alpha)t - (1-2\alpha), & s(\alpha) \leq t \leq 2. \end{cases}$$

As expected, u^*, φ^*, and $\partial x^*/\partial t$ are not differentiable in α at the switch point $s(\alpha)$, but all three are continuously differentiable in α for $t \in [0,2] \sim s(\alpha)$. A simple calculation shows that x^* and λ^* are continuously differentiable in α on $[0,2]$. The optimal paths for x, u, λ, and φ are depicted in fig. 1.

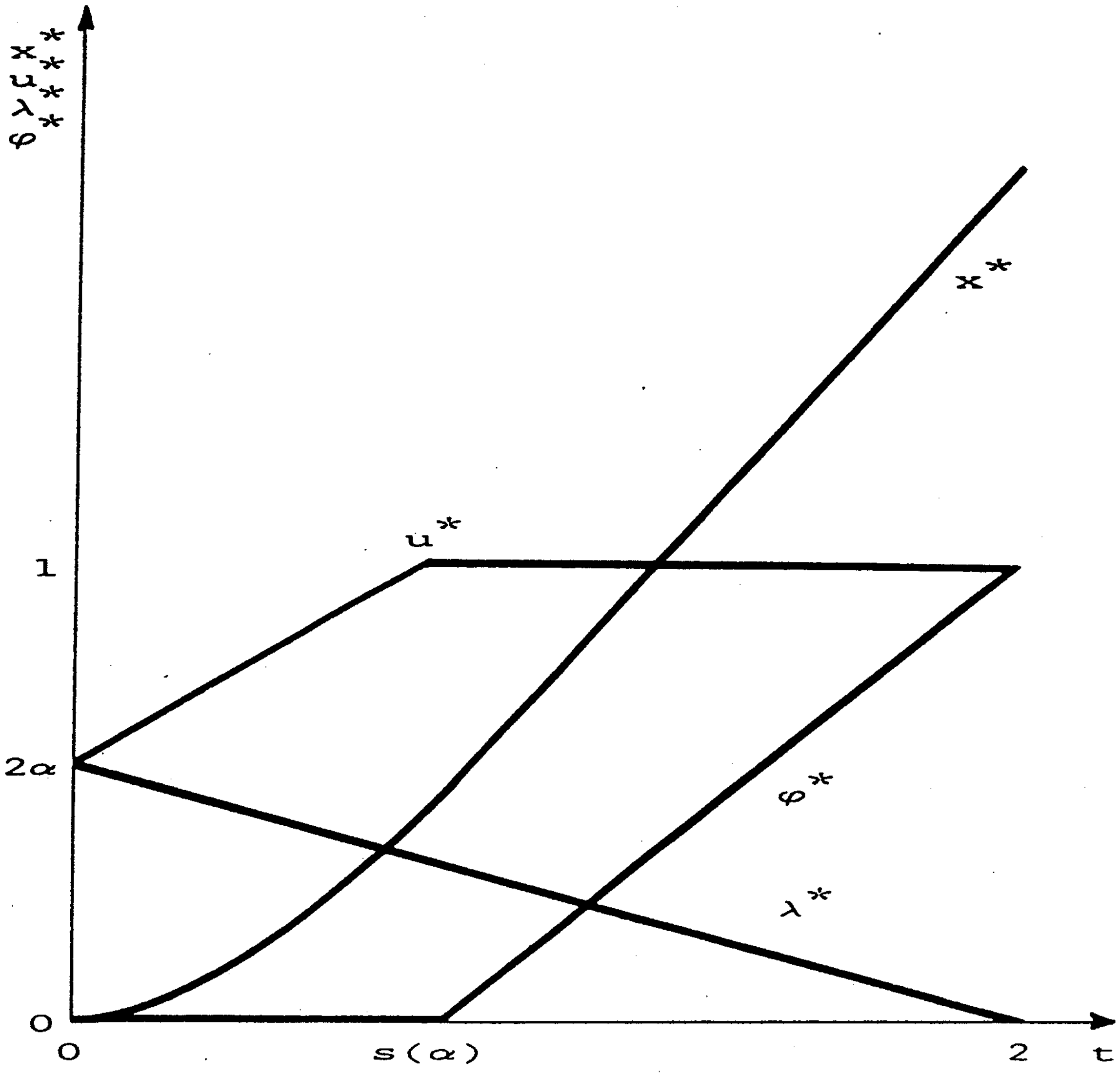

Fig. 1. An optimal path with a corner.

Substituting the optimal solutions into (31), the optimal performance function is given by

$$J^*(\alpha) = 2\alpha + 1 + \frac{(1-2\alpha)^3}{6(1-\alpha)}. \tag{39}$$

Clearly, $J^*(\alpha)$ is twice continuously differentiable in α on $(-\infty, \frac{1}{2})$, and the derivative of (39) can be obtained directly as

$$\frac{\mathrm{d}J^*(\alpha)}{\mathrm{d}\alpha} = 2 - \frac{(1-2\alpha)^2}{(1-\alpha)} + \frac{(1-2\alpha)^3}{6(1-\alpha)^2}. \tag{40}$$

Since $\partial L/\partial\alpha = x$, we can also obtain $\mathrm{d}J^*(\alpha)/\mathrm{d}\alpha$ using Theorem 1 simply by integrating $x^*(\alpha, t)$ over $[0, 2]$,

$$\frac{\mathrm{d}J^*(\alpha)}{\mathrm{d}\alpha} = \int_0^{s(\alpha)} \left[2\alpha t + (1-\alpha)t^2/2\right] \mathrm{d}t$$

$$+ \int_{s(\alpha)}^{2} \left[t - (1-2\alpha)^2/2(1-\alpha)\right] \mathrm{d}t$$

$$= 2 - \frac{(1-2\alpha)^2}{(1-\alpha)} + \frac{(1-2\alpha)^3}{6(1-\alpha)^2}. \tag{41}$$

Also, for $\alpha = 0$, a simple calculation shows that $\mathrm{d}J^*(0)/\mathrm{d}\alpha = \frac{7}{6}$, while $\int_0^T \partial L(x^*, u^*, \lambda^*, \varphi^*, 0, t)/\partial\alpha\, \mathrm{d}t = \frac{10}{3}$, illustrating the fact that, in general, these two expressions are not equal.

We conclude our discussion of this example by noting that the hypotheses of Theorem 2 are satisfied, while the hypotheses of Theorem 3 are violated by the appearance of the interaction term αx in f. Therefore, $J^*(\alpha)$ must be convex in α. Indeed, differentiating $J^*(\alpha)$ a second time,

$$\frac{\mathrm{d}^2 J^*(\alpha)}{\mathrm{d}\alpha^2} = 4\left(\frac{1-2\alpha}{1-\alpha}\right) + \tfrac{1}{3}\left(\frac{1-2\alpha}{1-\alpha}\right)^3, \tag{42}$$

which clearly is positive $\forall \alpha \in (-\infty, \frac{1}{2})$.

Time-varying parameters

Dynamic economic problems frequently contain state variables which are exogenous to the decision maker, i.e., are not subject to control, but are not constant over the planning horizon. Examples include time-dependent prices of inputs and outputs in a model of a competitive firm, a time-dependent discount rate, the size of the labor force in a model of economic growth, or prices faced by a consumer. If some of the state variables vary over time but are not subject to control, it seems appropriate to assume that decision makers form expectations about the movement of these variables such that the initial values are not necessarily expected to remain constant throughout the planning period.

To introduce uncontrolled time-dependent state variables into the analysis, let $\boldsymbol{\alpha}(t)$ denote the vector of time-varying parameters, and augment problem

(P1) with

$$\mathrm{d}\boldsymbol{\alpha}(t)/\mathrm{d}t = \boldsymbol{\xi}[\boldsymbol{\alpha}(t), t], \qquad \boldsymbol{\alpha}(0) = \boldsymbol{\alpha}_0, \tag{43}$$

and the following additional hypothesis:

A5. $|\partial\boldsymbol{\xi}/\partial\boldsymbol{\alpha}| \neq 0$ and $\boldsymbol{\xi}$ is twice continuously differentiable.

Given A5, a unique solution to (43), $\boldsymbol{\psi}$: $\mathscr{A} \times [0, T] \rightarrow \mathscr{A}$, exists such that $\boldsymbol{\psi}(\boldsymbol{\alpha}_0, t) = \boldsymbol{\alpha}(t)$, $\forall t \in [0, T]$, $\boldsymbol{\psi}(\boldsymbol{\alpha}_0, 0) = \boldsymbol{\alpha}_0$, and $\boldsymbol{\psi}$ is twice continuously differentiable on $\mathscr{A} \times [0, T]$. In such a situation, it is natural to substitute $\boldsymbol{\psi}(\boldsymbol{\alpha}_0, t)$ for $\boldsymbol{\alpha}$ and consider the effect on the optimal performance function due to a change in $\boldsymbol{\alpha}_0$. It follows that A1–A5 are sufficient to guarantee the existence of a twice continuously differentiable optimal performance function. Indeed, this is only one of many ways in which f, $\boldsymbol{g}$, $\boldsymbol{h}$, and $\boldsymbol{\omega}$ can depend explicitly on t. By replacing $\boldsymbol{\alpha}$ with $\boldsymbol{\psi}(\boldsymbol{\alpha}_0, t)$ in f, $\boldsymbol{g}$, $\boldsymbol{h}$, and $\boldsymbol{\omega}$ and replacing $\boldsymbol{\alpha}$ with $\boldsymbol{\alpha}_0$ in $\boldsymbol{x}^*$, $\boldsymbol{u}^*$, $\boldsymbol{\lambda}^*$, $\boldsymbol{\mu}^*$, and $\boldsymbol{\varphi}^*$, the proof of Theorem 1 is unaltered for the next result.

Theorem 4. If (P1) is augmented by (43) and satisfies A1–A5, then

$$\begin{aligned}
\frac{\partial J^*}{\partial \boldsymbol{\alpha}_0} &= \int_0^T \left[\left(\frac{\partial \boldsymbol{x}^{*\prime}}{\partial \boldsymbol{\alpha}_0} \right) \left(\frac{\partial f}{\partial \boldsymbol{x}} \right) + \left(\frac{\partial \boldsymbol{u}^{*\prime}}{\partial \boldsymbol{\alpha}_0} \right) \left(\frac{\partial f}{\partial \boldsymbol{u}} \right) + \left(\frac{\partial \boldsymbol{\psi}'}{\partial \boldsymbol{\alpha}_0} \right) \left(\frac{\partial f}{\partial \boldsymbol{\alpha}} \right) \right] \mathrm{d}t \\
&= \int_0^T \left(\frac{\partial \boldsymbol{\psi}'}{\partial \boldsymbol{\alpha}_0} \right) \left[\left(\frac{\partial f}{\partial \boldsymbol{\alpha}} \right) + \left(\frac{\partial \boldsymbol{g}'}{\partial \boldsymbol{\alpha}} \right) \boldsymbol{\lambda}^* + \left(\frac{\partial \boldsymbol{h}'}{\partial \boldsymbol{\alpha}} \right) \boldsymbol{\mu}^* + \left(\frac{\partial \boldsymbol{\omega}'}{\partial \boldsymbol{\alpha}} \right) \boldsymbol{\varphi}^* \right] \mathrm{d}t \\
&= \int_0^T \left(\frac{\partial \boldsymbol{\psi}'}{\partial \boldsymbol{\alpha}_0} \right) \left(\frac{\partial L(\boldsymbol{x}^*, \boldsymbol{u}^*, \boldsymbol{\lambda}^*, \boldsymbol{\mu}^*, \boldsymbol{\varphi}^*, \boldsymbol{\psi}, t)}{\partial \boldsymbol{\alpha}} \right) \mathrm{d}t..
\end{aligned} \tag{44}$$

Autonomous current value problems

The following autonomous current value problem has been utilized to study intertemporal duality by Cooper and McLaren (1980), McLaren and Cooper (1980), Epstein (1981, 1982), and Epstein and Denny (1983). Let $T = \infty$, and

$$f(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{\alpha}, t) \equiv v(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{\alpha})\, \mathrm{e}^{-rt}, \tag{45}$$

$$\partial \boldsymbol{g}(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{\alpha}, t)/\partial t \equiv \boldsymbol{0}. \tag{46}$$

There are no nondifferential constraints, and the following assumption is satisfied [see Arrow and Kurz (1970, pp. 46–51), Halkin (1974), and Brock (1977)]:

A6. A finite, unique, and globally asymptotically stable steady-state solution $\{x^{\infty}(\alpha), u^{\infty}(\alpha)\}$ exists such that the transversality condition $\lim_{t\to\infty} \lambda^*(\alpha,t)'x^*(\alpha,t) \equiv 0$ is satisfied $\forall \alpha \in \mathscr{A}$.

It is straightforward to show that Theorem 1 remains valid in this infinite horizon problem. Indeed, since there are no corners in the optimal path, the optimal choice functions are continuously differentiable throughout the planning horizon, and the proof is greatly simplified.

As pointed out by Arrow and Kurz, the current value optimal performance function for the interval $[t,\infty)$ does not depend explicitly on t,

$$\begin{aligned}
\tilde{V}(x^*(\alpha,t),\alpha,t) &\equiv \mathrm{e}^{rt} V^*(x^*(\alpha,t),\alpha,t)\\
&\equiv \mathrm{e}^{rt}\left\{\int_t^{\infty} v[x^*(\alpha,\tau),u'(\alpha,\tau),\alpha]\,\mathrm{e}^{-r\tau}\,\mathrm{d}\tau\right\}\\
&\equiv \int_t^{\infty} v[x^*(\alpha,\tau),u^*(\alpha,\tau),\alpha]\,\mathrm{e}^{-r(\tau-t)}\,\mathrm{d}\tau\\
&\equiv \int_0^{\infty} v[x^*(\alpha,s+t),u^*(\alpha,s+t),\alpha]\,\mathrm{e}^{-rs}\,\mathrm{d}s\\
&\equiv \tilde{V}(x^*(\alpha,t),\alpha,0),
\end{aligned} \tag{47}$$

where $s=\tau-t$ and $V^*(x^*(\alpha,t),\alpha,t)$ is the present value optimal performance function for the interval $[t,\infty)$ starting with initial state vector $x^*(\alpha,t)$. It follows from (47) that

$$\partial\tilde{V}/\partial t \equiv 0, \qquad \forall t\in[0,\infty). \tag{48}$$

The Hamilton–Jacobi–Bellman equation at time t is

$$\begin{aligned}
&-\left(\frac{\partial V^*(x^*(\alpha,t),\alpha,t)}{\partial t}\right)\\
&\equiv \max_u \left\{ v[x^*(\alpha,t),u,\alpha]\,\mathrm{e}^{-rt}\right.\\
&\qquad \left. +\left(\frac{\partial V^*(x^*(\alpha,t),\alpha,t)}{\partial x}\right)' g[x^*(\alpha,t),u,\alpha]\right\}.
\end{aligned} \tag{49}$$

Because $\tilde{V}(x^*(\alpha,t),\alpha,t) \equiv \mathrm{e}^{rt}V^*(x^*(\alpha,t),\alpha,t)$, it is a simple consequence of (48) that $rV^*(x^*(\alpha,t),\alpha,t) = -\partial V^*(x^*(\alpha,t),\alpha,t)/\partial t$. Partially differentiating (49) with respect to α then implies, by the static envelope theorem, $\forall t \in [0,\infty)$,

$$
\begin{aligned}
&r\left[\left(\frac{\partial V^*}{\partial \alpha}\right) + \left(\frac{\partial x^{*\prime}}{\partial \alpha}\right)\left(\frac{\partial V^*}{\partial x}\right)\right] \\
&\quad = \left(\frac{\partial x^{*\prime}}{\partial \alpha}\right)\left[\left(\frac{\partial v}{\partial x}\right)\mathrm{e}^{-rt} + \left(\frac{\partial g'}{\partial x}\right)\left(\frac{\partial V^*}{\partial x}\right) + \left(\frac{\partial^2 V^*}{\partial x\,\partial x'}\right)g\right] \\
&\qquad + \left(\frac{\partial v}{\partial \alpha}\right)\mathrm{e}^{-rt} + \left(\frac{\partial g'}{\partial \alpha}\right)\left(\frac{\partial V^*}{\partial x}\right) + \left(\frac{\partial^2 V^*}{\partial \alpha\,\partial x'}\right)g. \qquad (50)
\end{aligned}
$$

At $t=0$, $x^*(\alpha,0) \equiv x_0$ and $\mathrm{e}^{-r0}=1$, so that at the first instant in the planning horizon, (50) reduces to

$$
\begin{aligned}
\frac{\partial V^*(x_0,\alpha,0)}{\partial \alpha} = \left(\frac{1}{r}\right)&\left[\left(\frac{\partial v[x_0,u^*(\alpha,0),\alpha]}{\partial \alpha}\right)\right. \\
&+ \left(\frac{\partial g[x_0,u^*(\alpha,0),\alpha]}{\partial \alpha}\right)'\left(\frac{\partial V^*(x_0,\alpha,0)}{\partial x}\right) \\
&\left. + \left(\frac{\partial^2 V^*(x_0,\alpha,0)}{\partial \alpha\,\partial x'}\right)g[x_0,u^*(\alpha,0),\alpha]\right]. \qquad (51)
\end{aligned}
$$

Cooper and McLaren (1980), McLaren and Cooper (1980), Epstein (1981, 1982), and Epstein and Denny (1983) employed (51) to obtain an intertemporal analogue to Hotelling's/Shepard's lemma for producer and consumer behavior. If V^* is quadratic in the variables, then (51) provides an estimable relationship for the open-loop controls at the first instant in the planning horizon.

As pointed out by Epstein and Denny (1983, pp. 649–650): 'Current prices... are expected to persist indefinitely. As the base period changes and new prices... are observed, the firm revises its expectations and its previous plans. Thus only the $t=0$ portion of the plan... is carried out in general.' The implications are that the decision maker looks infinitely forward into the future when designing an optimal plan, but is totally myopic when forming expectations.

To compare (51) with the dynamic envelope theorem, note that (45) and (46) imply (5) has the form

$$\frac{\partial J^*(\boldsymbol{\alpha})}{\partial \boldsymbol{\alpha}} = \int_0^\infty \left[\left(\frac{\partial \boldsymbol{x}^{*\prime}}{\partial \boldsymbol{\alpha}}\right)\left(\frac{\partial v}{\partial \boldsymbol{x}}\right) + \left(\frac{\partial \boldsymbol{u}^{*\prime}}{\partial \boldsymbol{\alpha}}\right)\left(\frac{\partial v}{\partial \boldsymbol{u}}\right) + \left(\frac{\partial v}{\partial \boldsymbol{\alpha}}\right) \right] \mathrm{e}^{-rt}\,\mathrm{d}t$$

$$= \int_0^\infty \left[\left(\frac{\partial v}{\partial \boldsymbol{\alpha}}\right)\mathrm{e}^{-rt} + \left(\frac{\partial \boldsymbol{g}'}{\partial \boldsymbol{\alpha}}\right)\boldsymbol{\lambda}^* \right] \mathrm{d}t. \tag{52}$$

The next result, which reveals the additional structure of infinite horizon, autonomous, current value control problems, follows from (51), (52), and the identities $J^*(\boldsymbol{\alpha}) \equiv V^*(\boldsymbol{x}_0, \boldsymbol{\alpha}, 0)$ and $\boldsymbol{\lambda}^*(\boldsymbol{\alpha}, 0) \equiv \partial V^*(\boldsymbol{x}_0, \boldsymbol{\alpha}, 0)/\partial \boldsymbol{x}_0$:

Theorem 5. If (P1) satisfies A1–A4, A6, (45), and (46), then

$$\frac{\partial J^*(\boldsymbol{\alpha})}{\partial \boldsymbol{\alpha}} = \int_0^\infty \left[\left(\frac{\partial \boldsymbol{x}^{*\prime}}{\partial \boldsymbol{\alpha}}\right)\left(\frac{\partial v}{\partial \boldsymbol{x}}\right) + \left(\frac{\partial \boldsymbol{u}^{*\prime}}{\partial \boldsymbol{\alpha}}\right)\left(\frac{\partial v}{\partial \boldsymbol{u}}\right) + \left(\frac{\partial v}{\partial \boldsymbol{\alpha}}\right) \right] \mathrm{e}^{-rt}\,\mathrm{d}t$$

$$= \int_0^\infty \left[\left(\frac{\partial v}{\partial \boldsymbol{\alpha}}\right)\mathrm{e}^{-rt} + \left(\frac{\partial \boldsymbol{g}'}{\partial \boldsymbol{\alpha}}\right)\boldsymbol{\lambda}^* \right] \mathrm{d}t$$

$$= \frac{1}{r}\left[\frac{\partial v[\boldsymbol{x}_0, \boldsymbol{u}^*(\boldsymbol{\alpha},0), \boldsymbol{\alpha}]}{\partial \boldsymbol{\alpha}} + \frac{\partial \boldsymbol{g}[\boldsymbol{x}_0, \boldsymbol{u}^*(\boldsymbol{\alpha},0), \boldsymbol{\alpha}]'}{\partial \boldsymbol{\alpha}}\boldsymbol{\lambda}^*(\boldsymbol{\alpha},0) \right.$$

$$\left. + \frac{\partial \boldsymbol{\lambda}^*(\boldsymbol{\alpha},0)'}{\partial \boldsymbol{\alpha}}\boldsymbol{g}[\boldsymbol{x}_0, \boldsymbol{u}^*(\boldsymbol{\alpha},0), \boldsymbol{\alpha}] \right].$$

3. A dynamic consumption problem

Like the static envelope theorem, the dynamic envelope theorem can be applied to a variety of problems. In this section the dynamic envelope theorem is applied to a problem of a consumer maximizing discounted utility from consumption subject to a lifetime budget constraint. When applied to this problem, Theorem 4 gives rise to intertemporal analogues of Hotelling's lemma, Roy's identity, and the Slutsky equation from the static theory of consumer choice.

Let $p(t)$ be an n-vector of expected prices which depends upon the initial set of prices and time,

$$p(t)=\eta(p_0,t), \qquad p(0)=p_0. \tag{53}$$

The price path (53) might be generated by any form of smooth, deterministic expectations process, such as static or adaptive expectations, or perfect foresight. For example, static expectations results from (53) taking the form $\eta(p_0,t)=p_0$, $\forall t \in [0,\infty)$.

Consider a consumer who wishes to choose the time path of a consumption vector $c(t)$ to

$$\text{maximize } W=\int_0^T w[c(t)]\,\mathrm{e}^{-\rho t}\,\mathrm{d}t, \tag{54}$$

subject to the intertemporal budget constraint

$$\mathrm{d}M(t)/\mathrm{d}t=\eta(p_0,t)'c(t)\,\mathrm{e}^{-rt}, \qquad M(0)=0, \quad M(T)=M^0, \tag{55}$$

where ρ is the consumer's personal discount rate, w is the instantaneous flow of utility, r is the market discount rate at which the consumer can freely borrow or lend, and M^0 is the present value of lifetime earnings.

The optimal consumption path can be written as $c^m(p_0,M^0,t)$, and the optimal level of discounted utility flows is given by[2]

$$W^*(p_0,M^0)=\int_0^T w\left[c^m(p_0,M^0,t)\right]\mathrm{e}^{-\rho t}\,\mathrm{d}t. \tag{56}$$

Now consider the dual problem of choosing a consumption path $c(t)$ to minimize the present value of lifetime expenditures on consumption, subject to the constraint that discounted lifetime utility is constant at some level, W^0,

$$\text{minimize } E=\int_0^T \eta(p_0,t)'c(t)\,\mathrm{e}^{-rt}\,\mathrm{d}t, \tag{57}$$

subject to

$$\mathrm{d}W(t)/\mathrm{d}t=w[c(t)]\,\mathrm{e}^{-\rho t}, \qquad W(0)=0, \quad W(T)=W^0. \tag{58}$$

---

[2] The optimal discounted utility flow and the minimum net present value of consumption expenditures also depend on r, ρ, and T. Because our focus is on the initial values for the commodity prices and the duality between W and E, the arguments r, ρ, and T have been suppressed.

Denote this problem's solution by $c^w(p_0, W^0, t)$, and the minimal net present value of consumption expenditures by

$$E^*(p_0, W^0) = \int_0^T \eta(p_0, t)' c^w(p_0, W^0, t)\, \mathrm{e}^{-rt}\, \mathrm{d}t. \tag{59}$$

Both of these problems can be represented as isoperimetric control problems by rewriting (55) and (58) in integral form,

$$\int_0^T \eta(p_0, t)' c(t)\, \mathrm{e}^{-rt}\, \mathrm{d}t = M^0, \tag{60}$$

$$\int_0^T w[c(t)]\, \mathrm{e}^{-\rho t}\, \mathrm{d}t = W^0. \tag{61}$$

A well-known result from the calculus of variations is that, for isoperimetric control problems, the solution to (54) is equivalent to the solution to (57) if $M^0 = E^*$, or equivalently $W^0 = W^*$ [Clegg (1968, pp. 117–121)]. Therefore, we have the following identities:

$$c^w(p_0, W^0, t) \equiv c^m[p_0, E^*(p_0, W^0), t], \quad \forall t \in [0, T], \tag{62}$$

$$c^m(p_0, M^0, t) \equiv c^w[p_0, W^*(p_0, M^0), t], \quad \forall t \in [0, T], \tag{63}$$

$$E^*[p_0, W^*(p_0, M^0)] \equiv M^0, \tag{64}$$

$$W^*[p_0, E^*(p_0, W^0)] \equiv W^0. \tag{65}$$

These relations generate a complete set of intertemporal analogues to the static duality theorems of consumer choice theory. To see this, first note Theorems 1 and 4 remain valid for problems with both end points of the state variable fixed. The only modification that is required in the proof is to show that the first term on the right-hand side of (25) now vanishes because $x^*(\alpha, T) \equiv x_T$, $\forall \alpha \in \mathscr{A}$. Therefore, applying Theorem 4 to (59) gives

$$\frac{\partial E^*(p_0, W^0)}{\partial p_0} = \int_0^T \left(\frac{\partial \eta(p_0, t)'}{\partial p_0} \right) c^w(p_0, W^0, t)\, \mathrm{e}^{-rt}\, \mathrm{d}t. \tag{66}$$

This is a dynamic analogue to Hotelling's lemma. Next, we partially differentiate (62) with respect to p_0',

$$\frac{\partial c^w(p_0, W^0, t)}{\partial p_0'} \equiv \frac{\partial c^m[p_0, E^*(p_0, W^0), t]}{\partial p_0'} + \left(\frac{\partial c^m[p_0, E^*(p_0, W^0), t]}{\partial M}\right)\left(\frac{\partial E^*(p_0, W^0)}{\partial p_0'}\right), \tag{67}$$

to obtain a dynamic version of the Slutsky equation. Combining (62), (66), and (67) we see that the impact of a change in an initial price on the consumer's optimal consumption path at each point in time can be decomposed into two effects; a 'utility-held-constant' substitution effect and a 'wealth' effect. The wealth effect in this case includes the integral (66). Note, however, that the cross substitution terms in (67) generally are not symmetric, nor is the matrix of instantaneous substitution terms necessarily negative semidefinite.

If η is concave in p_0, then Theorem 2 implies that E^* is also concave in p_0 (since $-E^*$ is a maximum and therefore convex). Combining the dynamic envelope theorem with the intertemporal Hotelling's lemma, it therefore follows that

$$\frac{\partial^2 E^*}{\partial p_0 \partial p_0'} = \int_0^T \left[\left(\frac{\partial^2 \eta'}{\partial p_0 \partial p_0'}\right) c^w + \left(\frac{\partial \eta'}{\partial p_0}\right)\left(\frac{\partial c^w}{\partial p_0'}\right)\right] \mathrm{d}t \tag{68}$$

is symmetric and negative semidefinite. But this does not imply that the instantaneous Slutsky matrix (67) is symmetric or negative semidefinite. In particular, a rejection of either in (67) does not imply a rejection of these properties for (68).

These conclusions can be understood quite clearly by observing that there is a static expenditure minimization problem embedded in the dynamic problem (57). Indeed, the Hamiltonian for the maximum principle is

$$H = \mathrm{e}^{-rt} p(t)' c(t) + \lambda(t)\, \mathrm{e}^{-\rho t} w[c(t)], \tag{69}$$

where $p(t)$ rather than $\eta(p_0, t)$ is being used for the moment to derive the closed-loop optimal controls and develop the relationship between static and dynamic symmetry conditions. The first-order conditions for an interior minimum of H with respect to $\boldsymbol{c}$ are

$$-\boldsymbol{p}(t) = \mathrm{e}^{(r-\rho)t}\lambda(t)\frac{\partial w[\boldsymbol{c}(t)]}{\partial \boldsymbol{c}}. \tag{70}$$

These are the same necessary conditions as for minimizing $\boldsymbol{p}(t)'\boldsymbol{c}(t)$ subject to a constraint that $w[\boldsymbol{c}(t)] \geq w(t)$, say. Note that the closed-loop optimal controls, $\hat{\boldsymbol{c}}[\boldsymbol{p}(t), \lambda(t), t]$, do not depend on the state variable $W(t)$. Also note that $\lambda(t) < 0$ follows from eq. (70) so long as $\partial w[\boldsymbol{c}(t)]/\partial \boldsymbol{c} \gg \mathbf{0}$. Since there are no nondifferential constraints for this problem, the necessary and sufficient second-order condition for the existence of a unique optimal control is that w is strictly concave in $\boldsymbol{c}$. Therefore, assuming that w is twice continuously differentiable and strongly concave in $\boldsymbol{c}$, the static symmetry and negativity conditions for the expenditure minimization problem are obtained by substituting $\hat{\boldsymbol{c}}[\boldsymbol{p}(t), \lambda(t), t]$ for $\boldsymbol{c}(t)$ in (70) and differentiating with respect to $\boldsymbol{p}'$,

$$\frac{\partial \hat{\boldsymbol{c}}}{\partial \boldsymbol{p}'} = -\lambda^{-1}\mathrm{e}^{(\rho-r)t}\left(\frac{\partial^2 w}{\partial \boldsymbol{c}\,\partial \boldsymbol{c}'}\right)^{-1}. \tag{71}$$

Clearly, the closed-loop short-run compensated price effects are symmetric and negative definite.

We now construct the open-loop solutions for $\boldsymbol{c}$. The remaining conditions for the maximum principle are

$$\dot{\lambda} = \frac{-\partial H}{\partial W} = 0, \tag{72}$$

$$\dot{W} = \frac{\partial H}{\partial \lambda} = \mathrm{e}^{-rt}w(\boldsymbol{c}), \qquad W(0) = 0, \quad W(T) = W^0. \tag{73}$$

Eqs. (72) and (73) imply that $\lambda(t) \equiv \lambda$, a constant $\forall t \in [0, T]$, and the optimal path for the discounted utility flow satisfies (61). Therefore, the open-loop solution for the costate variable, $\lambda^w(\boldsymbol{p}_0, W^0)$, is the solution to the integral equation

$$\int_0^T \mathrm{e}^{-\rho t} w[\hat{\boldsymbol{c}}(\boldsymbol{\eta}(\boldsymbol{p}_0, t), \lambda, t)]\,\mathrm{d}t = W^0. \tag{74}$$

Thus, the open-loop solutions for the controls are defined by

$$\boldsymbol{c}^w(\boldsymbol{p}_0, W^0, t) \equiv \hat{\boldsymbol{c}}\left[\boldsymbol{\eta}(\boldsymbol{p}_0, t), \lambda^w(\boldsymbol{p}_0, W^0), t\right]. \tag{75}$$

Differentiating (75) with respect to p_0' gives

$$\frac{\partial c^w}{\partial p_0'} = \left(\frac{\partial \hat{c}}{\partial p'}\right)\left(\frac{\partial \eta}{\partial p_0'}\right) + \left(\frac{\partial \hat{c}}{\partial \lambda}\right)\left(\frac{\partial \lambda^w}{\partial p_0'}\right), \tag{76}$$

and only the term $\partial \hat{c}/\partial p'$ is symmetric. This can be seen by using (70) to evaluate $\partial \hat{c}/\partial \lambda$ and (74) to evaluate $\partial \lambda^w/\partial p_0'$,

$$\frac{\partial \hat{c}}{\partial \lambda} = -\lambda^{-1}\left(\frac{\partial^2 w}{\partial c\, \partial c'}\right)^{-1}\left(\frac{\partial w}{\partial c}\right), \tag{77}$$

$$\frac{\partial \lambda^w}{\partial p_0'} = \frac{-\int_0^T e^{-\rho t}\left(\frac{\partial w}{\partial c'}\right)\left(\frac{\partial \hat{c}}{\partial p'}\right)\left(\frac{\partial \eta}{\partial p_0'}\right)\mathrm{d}t}{\int_0^T e^{-\rho t}\left(\frac{\partial w}{\partial c'}\right)\left(\frac{\partial \hat{c}}{\partial \lambda}\right)\mathrm{d}t}. \tag{78}$$

Thus, even when expectations are static, it is clear from (76)–(78) that the instantaneous substitution terms for the open-loop controls will not be symmetric.

We conclude this section with a dynamic version of Roy's Identity. This is accomplished by differentiating (64) with respect to p_0,

$$\left(\frac{\partial E^*[p_0, W^*(p_0, M^0)]}{\partial p_0}\right) + \left(\frac{\partial E^*[p_0, W^*(p_0, M^0)]}{\partial W}\right)\left(\frac{\partial W^*(p_0, M^0)}{\partial p_0}\right) \equiv 0, \tag{79}$$

and then with respect to M^0,

$$\left(\frac{\partial E^*[p_0, W^*(p_0, M^0)]}{\partial W}\right)\left(\frac{\partial W^*(p_0, M^0)}{\partial M}\right) \equiv 1. \tag{80}$$

Combining (63), (66), (79), and (80) then gives a dynamic Roy's Identity,

$$-\frac{\partial W^*(p_0, M^0)/\partial p_0}{\partial W^*(p_0, M^0)/\partial M} \equiv \int_0^T \left(\frac{\partial \eta(p_0, t)'}{\partial p_0}\right) c^m(p_0, M^0, t)\, e^{-rt}\,\mathrm{d}t. \tag{81}$$

4. Conclusions

The envelope theorem has numerous applications in economics. In static models, it can be employed to obtain factor demand and output supply equations as partial derivatives of indirect cost and profit functions, greatly simplifying estimation problems. It is also quite useful in conducting comparative statics, in welfare analysis, and in providing the economic interpretation of Lagrange multipliers.

We have shown that the solutions to dynamic optimization problems are characterized by an exact intertemporal analogue to the static envelope theorem. We have proven that, for a large class of optimal control problems, the rate of change of the optimal performance function with respect to the underlying parameters can be obtained by partially differentiating the Lagrangian with respect to the parameters *prior to* finding the constrained maximum with respect to the control variables, substituting the optimal solutions for the control, state and costate variables, and the Lagrange multipliers into the resulting expression for $\partial L/\partial \alpha$ and integrating over the planning horizon. An equivalent expression is obtained by substituting the optimal solutions for the state and control variables into the objective function, partially differentiating with respect to the parameters, and interchanging the operations of differentiation and integration.

The results we have obtained are general enough to account for a variety of nondifferential constraints, corners in the optimal path, dynamic processes on the parameters affecting the intertemporal choice problem, and to encompass the special cases in the recent literature. These results should prove to be useful in many applications of dynamic optimization.

Through the use of the dynamic envelope theorem we have also obtained an intuitively appealing intertemporal analogue to each of the static duality theorems in consumer choice theory. Further, the structure of these intertemporal duality results does not depend substantially on the way in which the expectations process is modeled, and therefore is a significant generalization of results which rely on static expectations. Moreover, application of the methods developed in this paper clearly illustrates the differences between static and dynamic optimization problems.

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