Title: *AJAE* Appendix for "Homogeneity and Supply"

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Unique Representations and Linear Independence

In this section of the Appendix, we discuss the concept of *linear independence* of the input and output price functions used throughout this article. Let the $K\times 1$ vector of input price functions be $\boldsymbol{\alpha}(\boldsymbol{w}) = [\alpha_1(\boldsymbol{w})\cdots\alpha_K(\boldsymbol{w})]^\top$ and the $K\times 1$ vector of output price functions be $\boldsymbol{h}(p)$. For the supply equation to have a unique representation on $\mathbb{R}^n_{++}\times\mathbb{R}_{++}$, we need two conditions.

The first condition is that the output price functions, $\{h_k(p)\}_{k=1}^K$, must be linearly independent with respect to the K–dimensional constants. In other words, there can not exist any $\boldsymbol{c}\in\mathbb{R}^K$, $\boldsymbol{c}\neq\boldsymbol{0}$, such that $\boldsymbol{c}^\top\boldsymbol{h}(p^1) = 0\ \forall\, p^1\in\mathcal{N}(p)\subset\mathbb{R}$, over any open neighborhood $\mathcal{N}(p)$ of any point in the interior of the domain for p. The reason we must have this condition is if it were not satisfied, then $\forall\ \boldsymbol{d}\in\mathbb{R}^K$, if we add $\boldsymbol{\alpha}(\boldsymbol{w})^\top\boldsymbol{d}[\boldsymbol{c}^\top\boldsymbol{h}(p)]\equiv 0$ to the supply equation, we do not change its value,

$$\begin{aligned} q &= \sum_{k=1}^K \alpha_k(\boldsymbol{w})h_k(p) + \left[\sum_{k=1}^K \alpha_k(\boldsymbol{w})d_k\right]\left[\sum_{\ell=1}^K c_\ell h_\ell(p)\right] \\ &= \sum_{k=1}^K \alpha_k(\boldsymbol{w})\left[h_k(p) + d_k\sum_{\ell=1}^K c_\ell h_\ell(p)\right] \qquad \text{(A.1)} \\ &= \boldsymbol{\alpha}(\boldsymbol{w})^\top(\boldsymbol{I}+\boldsymbol{d}\boldsymbol{c}^\top)\boldsymbol{h}(p). \end{aligned}$$

But then, since $\boldsymbol{c}\neq\boldsymbol{0}$, we can choose $\boldsymbol{d}$ to make $\tilde{\boldsymbol{\alpha}}(\boldsymbol{w})\equiv(\boldsymbol{I}+\boldsymbol{c}\boldsymbol{d}^\top)\boldsymbol{\alpha}(\boldsymbol{w})$ anything, and the supply model is observationally meaningless.

The second condition we must have is that the elements of $\boldsymbol{\alpha}(\boldsymbol{w})$ are linearly independent with respect to the K–dimensional constants. For this property to hold, there can be no $\boldsymbol{c}\in\mathbb{R}^K$, $\boldsymbol{c}\neq\boldsymbol{0}$, such that $\boldsymbol{\alpha}(\boldsymbol{w}^1)^\top\boldsymbol{c} = 0\ \forall\,\boldsymbol{w}^1\in\mathcal{N}(\boldsymbol{w})$, for $\mathcal{N}(\boldsymbol{w})$ a neighborhood of an arbitrary point in the interior of the domain for $\boldsymbol{w}$. We need this property because if it is not satisfied, then $\forall\ \boldsymbol{d} \in \mathbb{R}^K$, if we add $[\boldsymbol{\alpha}(\boldsymbol{w})^\top \boldsymbol{c}]\boldsymbol{d}^\top \boldsymbol{h}(p) \equiv 0$ to the supply equation, we do not change its value,

$$q = \boldsymbol{\alpha}(\boldsymbol{w})^\top (\boldsymbol{I} + \boldsymbol{c}\boldsymbol{d}^\top)\boldsymbol{h}(p). \tag{A.2}$$

But then, since $\boldsymbol{c} \neq \boldsymbol{0}$, we could choose $\boldsymbol{d}$ to make $\tilde{\boldsymbol{h}}(p) \equiv \left(\boldsymbol{I} + \boldsymbol{c}\boldsymbol{d}^\top\right)\boldsymbol{h}(p)$ anything, and the supply model again has no empirical content.

These conditions are necessary and sufficient for the present purpose. If both are not satisfied, then we can always reduce the number of both the input and the output price functions by a linear combination of the original functions with no change in the model. To illustrate, without loss of generality (WLOG), assume that $h_K(p) = \sum_{k=1}^{K-1} c_k h_k(p)$, so that the supply equation is

$$\begin{aligned} q &= \sum_{k=1}^{K-1} \alpha_k(\boldsymbol{w}) h_k(p) + \alpha_K(\boldsymbol{w}) \sum_{k=1}^{K-1} c_k h_k(p) \\ &= \sum_{k=1}^{K-1} [\alpha_k(\boldsymbol{w}) + c_k \alpha_K(\boldsymbol{w})] h_k(p) \\ &\equiv \sum_{k=1}^{K-1} \tilde{\alpha}_k(\boldsymbol{w}) h_k(p). \end{aligned} \tag{A.3}$$

Thus, the supply model can always be written with the Gorman structure as a sum of at most K–1 products if the output price functions are linearly dependent. Alternatively, again WLOG, assume that $\alpha_K(\boldsymbol{w}) = \sum_{k=1}^{K-1} c_k \alpha_k(\boldsymbol{w})$, so that now the supply equation is

$$\begin{aligned} q &= \sum_{k=1}^{K-1} \alpha_k(\boldsymbol{w}) h_k(p) + \left[\sum_{k=1}^{K-1} c_k \alpha_k(\boldsymbol{w})\right] h_K(p) \\ &= \sum_{k=1}^{K-1} \alpha_k(\boldsymbol{w}) [h_k(p) + c_k h_K(p)] \\ &\equiv \sum_{k=1}^{K-1} \alpha_k(\boldsymbol{w}) \tilde{h}_k(p). \end{aligned} \tag{A.4}$$

Once again, the supply model can always be written with the Gorman structure as a sum of at most K–1 products if the input price functions are linearly dependent.

A unique representation requires that no linear reductions of this type are possible. Various ways have been developed to check for the linear independence of a K–vector of functions. For example, in the case of the output price functions where there is a single argument, p, if the Wronksian – which is the determinant of the $K\times K$ matrix whose first row is the vector of functions, $[h_1(p)\cdots h_K(p)]$, the second row is the vector of first-order derivatives, $[h_1'(p)\cdots h_K'(p)]$, and so on through K–1 derivatives – does not vanish at any point in an interval, then the K functions are linearly independent on that interval.

For vector-valued functions of several variables, such as the input price functions, $\{\alpha_k(\boldsymbol{w})\}_{k=1}^K$, the matter is significantly more involved. However, for each element of $\boldsymbol{w}$, a sufficient condition for the linear independence across the K–dimensional constants is that each Wronksian made up of the $K\times K$ matrix of levels of the $\{\alpha_k(\boldsymbol{w})\}_{k=1}^K$ functions plus the row vectors of their partial derivatives with respect to w_j through order K–1 does not vanish on any one-dimensional open interval, for each j=1,…,n. Interested readers are referred to Gorman (1981), the appendix in Russell and Farris (1998) written by Robert Bryant, Cohen (1933), or Boyce and diPrima (1977) for additional details on linear independence of a vector of functions of one or several variables.

Proof of Proposition 1

***Proposition* 1**: *Let the supply function take the Gorman form,* $q=\sum_{k=1}^K \alpha_k(\boldsymbol{w})h_k(p)$, *with K smooth, linearly independent, functions of input prices,* **w**, *and K smooth, linearly* *independent, functions of output price, p. If q is 0° homogeneous in* $(\boldsymbol{w}, p)$*, then each output price function is either:* (*i*) p^{ε}, *with* $\varepsilon \in \mathbb{R}$*;* (*ii*) $p^{\varepsilon}(\ln p)^{j}$, *with* $\varepsilon \in \mathbb{R}$, $j \in \{1,...,K\}$*;* (*iii*) $p^{\varepsilon}\sin(\tau \ln p)$, $p^{\varepsilon}\cos(\tau \ln p)$, *with* $\varepsilon \in \mathbb{R}$, $\tau \in \mathbb{R}_{+}$, *appearing in pairs with the same* $\{\varepsilon, \tau\}$ *for each pair; or* (*iv*) $p^{\varepsilon}(\ln p)^{j}\sin(\tau \ln p)$, $p^{\varepsilon}(\ln p)^{j}\cos(\tau \ln p)$, *with* $\varepsilon \in \mathbb{R}$, $j \in \{1,...,[½K]\}$, $\tau \in \mathbb{R}_{+}$, *and* $K \geq 4$, *appearing in pairs with the same* $\{\varepsilon, j, \tau\}$ *for each pair, where* $[½K]$ *is the largest integer no greater than ½K. If* $K \in \{1,2,3\}$*, then the supply of q can be written as:*

(*a*) *K*=1

$$q = [p/\alpha_1(\boldsymbol{w})]^{\varepsilon_1};$$

(*b*) *K*=2

i. $q = [p/\alpha_1(\boldsymbol{w})]^{\varepsilon_1} + [p/\alpha_2(\boldsymbol{w})]^{\varepsilon_2}$;

ii. $q = [p/\alpha_1(\boldsymbol{w})]^{\varepsilon_1} \ln\left(p/\alpha_2(\boldsymbol{w})\right)$; *or*

iii. $q = [p/\alpha_1(\boldsymbol{w})]^{\varepsilon_1}\left[\sin\left(\tau \ln\left(p/\alpha_2(\boldsymbol{w})\right)\right) + \cos\left(\tau \ln\left(p/\alpha_2(\boldsymbol{w})\right)\right)\right]$;

(*c*) *K*=3

i. $q = [p/\alpha_1(\boldsymbol{w})]^{\varepsilon_1} + [p/\alpha_2(\boldsymbol{w})]^{\varepsilon_2} + [p/\alpha_3(\boldsymbol{w})]^{\varepsilon_3}$;

ii. $q = [p/\alpha_1(\boldsymbol{w})]^{\varepsilon_1} + [p/\alpha_2(\boldsymbol{w})]^{\varepsilon_2} \ln\left(p/\alpha_3(\boldsymbol{w})\right)$;

iii. $q = [p/\alpha_1(\boldsymbol{w})]^{\varepsilon_1}\left\{\alpha_2(\boldsymbol{w}) + \left[\ln\left(p/\alpha_3(\boldsymbol{w})\right)\right]^2\right\}$; *or*

iv. $q = [p/\alpha_1(\boldsymbol{w})]^{\varepsilon_1} + [p/\alpha_2(\boldsymbol{w})]^{\varepsilon_2}\left\{\sin\left(\tau \ln[p/\alpha_3(\boldsymbol{w})]\right) + \cos\left(\tau \ln[p/\alpha_3(\boldsymbol{w})]\right)\right\}$.

In each case except (c) iii, where $\alpha_2(\boldsymbol{w})$ is homogeneous of degree zero, each $\alpha_i(\boldsymbol{w})$ is positively linearly homogeneous for $i = 1, 2, 3$.

Proof: The Euler equation for 0° homogeneity is:

$$\sum_{k=1}^{K} \frac{\partial \alpha_k(\boldsymbol{w})}{\partial \boldsymbol{w}^{\mathsf{T}}} \boldsymbol{w} h_k(p) + \sum_{k=1}^{K} \alpha_k(\boldsymbol{w}) h_k'(p) p = 0. \tag{A.5}$$

If K=1 and $h_1'(p) = 0$, this reduces to $\partial \alpha_1(\boldsymbol{w}) / \partial \boldsymbol{w}^{\mathsf{T}} \boldsymbol{w} = 0$, so that $h_1(p) = c$ and $\alpha_1(\boldsymbol{w})$ is homogeneous of degree zero. Absorb the constant c into the price index and set $\varepsilon_1 = 0$ to obtain a special case of (a) i. If either K=1 and $h_1'(p) \neq 0$ or $K \geq 2$, then neither sum in (A.5) can vanish without contradicting the linear independence of the $\{\alpha_k(\boldsymbol{w})\}$ or the $\{h_k(p)\}$.[1] Write the Euler equation as

$$\frac{\sum_{k=1}^{K} \alpha_k(\boldsymbol{w}) h_k'(p) p}{\sum_{k=1}^{K} \left[\partial \alpha_k(\boldsymbol{w}) / \partial \boldsymbol{w}^{\mathsf{T}} \boldsymbol{w} \right] h_k(p)} = -1. \tag{A.6}$$

Since the right-hand side is constant, we must be able to recombine the left-hand side to be independent of both $\boldsymbol{w}$ and p. In other words, the terms in the numerator must recombine in some way so that it is proportional to the denominator, with –1 as the proportionality factor. Clearly, if these two sums are proportional, identically in $(\boldsymbol{w}, p)$, then the functional forms of the two sums must be the same.

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[1] Note, in particular, that the terms $\partial \alpha_k(\boldsymbol{w}) / \partial \boldsymbol{w}^{\mathsf{T}} \boldsymbol{w}$ are constant with respect to p, and that the terms $h_k'(p) p$ are constant with respect to $\boldsymbol{w}$.

To see this, for any $\boldsymbol{w} \in \mathbb{R}^n_{++}$, let $\mathcal{N}(\boldsymbol{w})$ be an open neighborhood of $\boldsymbol{w}$. Fix K unique vectors, $\boldsymbol{w}_\ell \in \mathcal{N}(\boldsymbol{w})$, $\ell = 1, \cdots, K$, and define the $K \times K$ matrices $\boldsymbol{B} = \left[\alpha_k(\boldsymbol{w}_\ell)\right]_{k,\ell=1,\cdots,K}$ and $\boldsymbol{D} = \left[\partial\alpha_k(\boldsymbol{w}_\ell)/\partial\boldsymbol{w}^\top\boldsymbol{w}_\ell\right]_{k,\ell=1,\cdots,K}$. The linear independence of the input price functions implies that we can choose $\{\boldsymbol{w}_\ell\}$ such that $\boldsymbol{B}$ is nonsingular, and therefore write (A.5) in the form

$$\boldsymbol{h}'(p)p = -\boldsymbol{B}^{-1}\boldsymbol{D}\boldsymbol{h}(p) \equiv \boldsymbol{C}\boldsymbol{h}(p), \tag{A.7}$$

Now, since both $\boldsymbol{h}'(p)p$ and $\boldsymbol{h}(p)$ only depend on p and not on $\boldsymbol{w}$, the $K \times K$ matrix $\boldsymbol{C}$ also must be independent of $\boldsymbol{w}$, i.e., each of its elements must be constant.

Thus, the linear independence of $\{h_1(p),\ldots, h_K(p)\}$ and $\{\alpha_1(\boldsymbol{w}), \cdots, \alpha_K(\boldsymbol{w})\}$ implies that each $h_k'(p)p$ is a linear function of $\{h_1(p),\ldots, h_K(p)\}$ with constant coefficients:

$$h_k'(p)p = \sum\nolimits_{\ell=1}^{K} c_{k,\ell} h_\ell(p), \;\; k = 1, \cdots, K. \tag{A.8}$$

This is a complete system of K linear, homogeneous, ordinary differential equations (odes), of the form commonly known as Cauchy's linear differential equation. Our strategy is the following. First, we convert (A.8) through the change of variables from p to $x = \ln p$ to a system of linear odes with constant coefficients (Cohen 1933, pp. 124-125). Second, we identify the set of solutions for the converted system of odes. Third, we return to (A.5) with these solutions in hand and identify the implied restrictions among the input price functions for each K=1,2,3.

Since $p(x) = e^x$ and $p'(x) = p(x)$, defining $\tilde{h}_k(x) \equiv h_k(p(x))$, $k = 1, \ldots, K$, and applying this change of variables yields:

$$\tilde{h}_k'(x) = \sum_{\ell=1}^{K} c_{k,\ell}\tilde{h}_\ell(x), \;\; k = 1,\cdots,K. \tag{A.9}$$

In matrix form, this system of linear, first-order, homogeneous odes is $\tilde{\boldsymbol{h}}'(x) - \boldsymbol{C}\tilde{\boldsymbol{h}}(x) = \boldsymbol{0}$, and the characteristic equation is $|\boldsymbol{C} - \lambda \boldsymbol{I}| = 0$. This is a K^{th} order polynomial in λ, for which the fundamental theorem of algebra implies that there are exactly K roots. Some of these roots may repeat and some may be complex conjugate pairs. Let the characteristic roots be denoted by $\lambda_k, \;\; k = 1,\cdots,K$.

The general solution to a linear, homogeneous, ode of order K is the sum of K linearly independent particular solutions (Cohen 1933, Chapter 6; Boyce and DiPrima 1977, Chapter 5), where linear independence of the K functions, $\{f_1,\cdots,f_K\}$ of the scalar x means that no non-vanishing vector, $[a_1,\cdots,a_K]^{\mathsf{T}}$, satisfies $a_1 f_1 + \cdots + a_K f_K = 0$ for all values of the variables in an open neighborhood of any point $[x, f_1(x),\cdots,f_K(x)]$. Cohen (1933), pp. 303-306 contains a statement of necessary and sufficient conditions.

Let there be $R \geq 0$ roots that repeat and reorder the output price functions as necessary in the following way. Label the first repeating root (if one exists) as λ_1 and let its multiplicity be denoted by $M_1 \geq 1$. Let the second repeating root (if one exists) be the M_1+1^{st} root. Label this root as λ_2 and its multiplicity as $M_2 \geq 1$. Continue in this manner until there are no more repeating roots. Let the total number of repeated roots be $\tilde{M} = \sum_{k=1}^{R} M_k$. Label the remaining $K - \tilde{M} \geq 0$ unique roots as λ_k for each $k = \tilde{M}+1,\cdots,K$. Then WLOG, the general solution to (A.9) can be written as

$$\tilde{h}_k(x) = \sum_{r=1}^{R} \left[\sum_{\ell=1}^{M_r} d_{k\ell} x^{(\ell-1)} e^{\lambda_r x} \right] + \sum_{\ell=\tilde{M}+1}^{K} d_{k\ell} e^{\lambda_\ell x}, \ \ k = 1, \cdots, K. \qquad \text{(A.10)}$$

Now substitute (A.10) into the supply of q to obtain:

$$\begin{aligned} q &= \sum_{k=1}^{K} \alpha_k(\boldsymbol{w}) \left[\sum_{r=1}^{R} \sum_{\ell=1}^{M_r} d_{k\ell} p^{\lambda_r} (\ln p)^{(\ell-1)} + \sum_{\ell=\tilde{M}+1}^{K} d_{k\ell} p^{\lambda_\ell} \right] \\ &= \sum_{r=1}^{R} \sum_{\ell=1}^{M_r} \left[\sum_{k=1}^{K} d_{k\ell} \alpha_k(\boldsymbol{w}) \right] p^{\lambda_r} (\ln p)^{(\ell-1)} + \sum_{\ell=\tilde{M}+1}^{K} \left[\sum_{k=1}^{K} d_{k\ell} \alpha_k(\boldsymbol{w}) \right] p^{\lambda_\ell} \qquad \text{(A.11)} \\ &\equiv \sum_{r=1}^{R} \sum_{k=1}^{M_r} \tilde{\alpha}_{kr}(\boldsymbol{w}) p^{\lambda_r} (\ln p)^{(k-1)} + \sum_{k=\tilde{M}+1}^{K} \tilde{\alpha}_k(\boldsymbol{w}) p^{\lambda_k}. \end{aligned}$$

The terms in the first double sum give cases (i) and (ii), and case (iv) when $K \geq 4$ and at least one pair of complex conjugate roots repeats, while the terms in the sum on the far right give cases (i) for unique real roots and (iii) for unique pairs of complex conjugate roots, completing the proof of the functional form of the output price terms.

Turn now to the representation of the supply function for K=1,2, or 3.

K=1: $$q = \alpha_1(\boldsymbol{w}) h_1(p). \qquad \text{(A.12)}$$

Equation (A.8) simplifies to

$$h_1'(p) p = c_{11} h_1(p). \qquad \text{(A.13)}$$

Direct integration leads to $h_1(p) = d_{11} p^{c_{11}}$. The Euler equation then reduces to

$$\partial \alpha_1(\boldsymbol{w}) / \partial \boldsymbol{w}^{\mathsf{T}} \boldsymbol{w} = -c_{11} \alpha_1(\boldsymbol{w}). \qquad \text{(A.14)}$$

Hence, the input price function must be homogeneous of degree $-c_{11}$. Set $\varepsilon_1 = c_{11}$, absorb the multiplicative constant d_{11} into the homogeneous price function, and omit the subscripts to obtain the expression found in (a) of the proposition.

K=2: $$q = \alpha_1(\boldsymbol{w}) h_1(p) + \alpha_2(\boldsymbol{w}) h_2(p). \qquad \text{(A.15)}$$

The characteristic equation for (A.9) is

$$\lambda^2 - (c_{11} + c_{22})\lambda + (c_{11}c_{22} - c_{12}c_{21}) = 0 . \tag{A.16}$$

The characteristic roots are

$$\lambda_1, \lambda_2 = ½(c_{11} + c_{22}) \pm ½\sqrt{(c_{11} - c_{22})^2 + 4c_{12}c_{21}} . \tag{A.17}$$

Three cases are possible:

(1) unique real roots, $\lambda_1 \neq \lambda_2$, $\lambda_1, \lambda_2 \in \mathbb{R}$, and $(c_{11} - c_{22})^2 + 4c_{12}c_{21} > 0$;

(2) one real root, $\lambda_1 = \lambda_2 = ½(c_{11} + c_{22}) \in \mathbb{R}$, and $(c_{11} - c_{22})^2 + 4c_{12}c_{21} = 0$; or

(3) complex conjugate roots, $\lambda_1 = \kappa + \imath\tau$, $\lambda_2 = \kappa - \imath\tau$, $\kappa = ½(c_{11} + c_{22})$,

$\tau = ½\sqrt{|(c_{11} - c_{22})^2 + 4c_{12}c_{21}|}$, and $(c_{11} - c_{22})^2 + 4c_{12}c_{21} < 0$.

With unique roots (whether real or complex), the general solution is

$$\tilde{h}_k(x) = d_{k1}e^{\lambda_1 x} + d_{k2}e^{\lambda_2 x}, \; k = 1,2. \tag{A.18}$$

Substituting these expressions into the supply of q yields

$$\begin{aligned} q &= \left[d_{11}\alpha_1(\boldsymbol{w}) + d_{12}\alpha_2(\boldsymbol{w})\right] p^{\lambda_1} + \left[d_{21}\alpha_1(\boldsymbol{w}) + d_{22}\alpha_2(\boldsymbol{w})\right] p^{\lambda_2} \\ &\equiv \tilde{\alpha}_1(\boldsymbol{w}) p^{\lambda_1} + \tilde{\alpha}_2(\boldsymbol{w}) p^{\lambda_2}. \end{aligned} \tag{A.19}$$

If the roots are real, then the Euler equation is

$$\left[\partial\tilde{\alpha}_1(\boldsymbol{w})/\partial\boldsymbol{w}^{\mathsf{T}}\boldsymbol{w} + \lambda_1\tilde{\alpha}_1(\boldsymbol{w})\right] p^{\lambda_1} + \left[\partial\tilde{\alpha}_2(\boldsymbol{w})/\partial\boldsymbol{w}^{\mathsf{T}}\boldsymbol{w} + \lambda_2\tilde{\alpha}_2(\boldsymbol{w})\right] p^{\lambda_2} = 0. \tag{A.20}$$

Linear independence of the output price functions implies that the term premultiplying each output price function vanishes. Hence, $\tilde{\alpha}_i(\boldsymbol{w})$ must be homogeneous of degree $-\lambda_i$ for i=1,2. Relabel terms so that $\varepsilon_i = \lambda_i$ and $\alpha_i(\boldsymbol{w}) = \tilde{\alpha}_i(\boldsymbol{w})^{-1/\varepsilon_i}$, $i = 1,2,$ for case (b) i.

If the characteristic root repeats, the general solution is

$$\tilde{h}_k(x) = d_{k1}e^{\lambda x} + d_{k2}xe^{\lambda x}, \;\; k = 1,2. \tag{A.21}$$

Making the same substitutions as before yields:

$$q = \tilde{\alpha}_1(\boldsymbol{w})p^{\lambda} + \tilde{\alpha}_2(\boldsymbol{w})p^{\lambda}\ln p. \tag{A.22}$$

The Euler equation now is

$$\left[\left(\frac{\partial\tilde{\alpha}_1(\boldsymbol{w})}{\partial\boldsymbol{w}^{\mathsf{T}}}\boldsymbol{w} + \lambda\tilde{\alpha}_1(\boldsymbol{w}) + \tilde{\alpha}_2(\boldsymbol{w})\right) + \left(\frac{\partial\tilde{\alpha}_2(\boldsymbol{w})}{\partial\boldsymbol{w}^{\mathsf{T}}}\boldsymbol{w} + \lambda\tilde{\alpha}_2(\boldsymbol{w})\right)\ln p\right]p^{\lambda} = 0. \tag{A.23}$$

Since $p^{\lambda} > 0$ and $\{1,\ln p\}$ is linearly independent, $\tilde{\alpha}_2(\boldsymbol{w})$ must be homogeneous of degree $-\lambda$. Therefore, factor it and p^{λ} out on the right-hand side of (A.22),

$$q = \tilde{\alpha}_2(\boldsymbol{w})p^{\lambda}\left[\hat{\alpha}_1(\boldsymbol{w}) + \ln p\right], \tag{A.24}$$

where $\hat{\alpha}_1(\boldsymbol{w}) \equiv \tilde{\alpha}_1(\boldsymbol{w})/\tilde{\alpha}_2(\boldsymbol{w})$. The Euler equation then simplifies to

$$\partial\hat{\alpha}_1(\boldsymbol{w})/\partial\boldsymbol{w}^{\mathsf{T}}\boldsymbol{w} = -1. \tag{A.25}$$

Let $\beta(\boldsymbol{w}) = \exp\{-\hat{\alpha}(\boldsymbol{w})\}$ and note that $\partial\beta(\boldsymbol{w})/\partial\boldsymbol{w}^{\mathsf{T}}\boldsymbol{w} = -\beta(\boldsymbol{w})\partial\hat{\alpha}_1(\boldsymbol{w})/\partial\boldsymbol{w}^{\mathsf{T}}\boldsymbol{w} = \beta(\boldsymbol{w})$ if and only if $\hat{\alpha}_1(\boldsymbol{w})$ satisfies (A.25). Relabel terms so that $\varepsilon_1 = \lambda$, $\alpha_1(\boldsymbol{w}) = \tilde{\alpha}_2(\boldsymbol{w})$, and $\alpha_2(\boldsymbol{w}) = \beta(\boldsymbol{w})$ to obtain case (b) ii.

When the roots are complex, we first require conditions on the input price functions so that q is real-valued. From (A.19), we have

$$q = p^{\kappa}\left[\tilde{\alpha}_1(\boldsymbol{w})p^{\imath\tau} + \tilde{\alpha}_2(\boldsymbol{w})p^{-\imath\tau}\right], \tag{A.26}$$

while deMoivre's theorem implies (Abramowitz and Stegun 1972)

$$p^{\pm\imath\tau} = \cos(\tau\ln p) \pm \imath\sin(\tau\ln p). \tag{A.27}$$

Thus, complex functions $\tilde{\alpha}_1(\boldsymbol{w}) = \hat{\alpha}_0(\boldsymbol{w}) + \imath\hat{\alpha}_1(\boldsymbol{w})$ and $\tilde{\alpha}_2(\boldsymbol{w}) = \hat{\beta}_0(\boldsymbol{w}) + \imath\hat{\beta}_1(\boldsymbol{w})$ are required if q is real-valued. Substituting these definitions and (A.27) into (A.26) yields:

$$q = p^{\kappa}\left[(\hat{\alpha}_0 + \imath\hat{\alpha}_1 + \hat{\beta}_0 + \imath\hat{\beta}_1)\cos(\tau\ln p) + (\imath\hat{\alpha}_0 - \hat{\alpha}_1 - \imath\hat{\beta}_0 + \hat{\beta}_1)\sin(\tau\ln p)\right]. \quad \text{(A.28)}$$

We must have $\hat{\beta}_1(\boldsymbol{w}) = -\hat{\alpha}_1(\boldsymbol{w})$ for the term in front of $\cos(\tau\ln p)$ to be real-valued and $\hat{\beta}_0(\boldsymbol{w}) = \hat{\alpha}_0(\boldsymbol{w})$ for the term in front of $\sin(\tau\ln p)$ to be real-valued, so that the input price functions are complex conjugates. Omitting the ^'s and the subscripts and absorbing the multiplicative constant 2 into the price functions for conciseness, we then have:

$$q = p^{\kappa}\left[\alpha(\boldsymbol{w})\cos(\tau\ln p) + \beta(\boldsymbol{w})\sin(\tau\ln p)\right]. \quad \text{(A.29)}$$

The Euler equation now has the form:

$$\begin{aligned}&\Big\{\Big[\alpha_{\boldsymbol{w}}(\boldsymbol{w})^{\mathsf{T}}\boldsymbol{w} + \kappa\alpha(\boldsymbol{w}) + \tau\beta(\boldsymbol{w})\Big]\cos(\tau\ln p)\\ &+\Big[\beta_{\boldsymbol{w}}(\boldsymbol{w})^{\mathsf{T}}\boldsymbol{w} - \tau\alpha(\boldsymbol{w}) + \kappa\beta(\boldsymbol{w})\Big]\sin(\tau\ln p)\Big\}p^{\kappa} = 0.\end{aligned} \quad \text{(A.30)}$$

Define the smooth and invertible transformation

$$\begin{aligned}\alpha(\boldsymbol{w}) &= \tilde{\alpha}(\boldsymbol{w})\Big[\cos\big(\tau\ln\tilde{\beta}(\boldsymbol{w})\big) - \sin\big(\tau\ln\tilde{\beta}(\boldsymbol{w})\big)\Big],\\ \beta(\boldsymbol{w}) &= \tilde{\alpha}(\boldsymbol{w})\Big[\sin\big(\tau\ln\tilde{\beta}(\boldsymbol{w})\big) + \cos\big(\tau\ln\tilde{\beta}(\boldsymbol{w})\big)\Big],\end{aligned} \quad \text{(A.31)}$$

for $\tilde{\alpha}(\boldsymbol{w}) \neq 0$ any smooth, homogeneous of degree $-\kappa$ function and $\tilde{\beta}(\boldsymbol{w}) > 0$ any positive linearly homogeneous function. A direct calculation then yields

$$\begin{aligned}\alpha_{\boldsymbol{w}}(\boldsymbol{w})^{\mathsf{T}}\boldsymbol{w} &= -\kappa\alpha(\boldsymbol{w}) - \tau\beta(\boldsymbol{w}),\\ \beta_{\boldsymbol{w}}(\boldsymbol{w})^{\mathsf{T}}\boldsymbol{w} &= \tau\alpha(\boldsymbol{w}) - \kappa\beta(\boldsymbol{w}),\end{aligned} \quad \text{(A.32)}$$

as required.

Relabeling with $\varepsilon_1 = \kappa$, $\alpha_1(\boldsymbol{w}) = \tilde{\alpha}(\boldsymbol{w})^{-1/\kappa}$, and $\alpha_2(\boldsymbol{w}) = \tilde{\beta}(\boldsymbol{w})$ yields:

$$q = \left[p/\alpha_1(\boldsymbol{w})\right]^{\varepsilon_1} \left\{\left[\cos\left(\tau \ln \alpha_2(\boldsymbol{w})\right) - \sin\left(\tau \ln \alpha_2(\boldsymbol{w})\right)\right]\cos(\tau \ln p) + \left[\sin\left(\tau \ln \alpha_2(\boldsymbol{w})\right) + \cos\left(\tau \ln \alpha_2(\boldsymbol{w})\right)\right]\sin(\tau \ln p)\right\}. \tag{A.33}$$

Some tedious but straightforward algebra using the trigonometric identities (Abramowitz and Stegun 1972, pp. 72-74):

$\sin(a+b) = \sin(a)\cos(b) + \cos(a)\sin(b)$;

$\cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b)$;

$\sin(-b) = -\sin(b)$; and

$\cos(-b) = \cos(b)$;

with $a = \tau \ln p$ and $b = -\tau \ln \alpha_2(\boldsymbol{w})$ then gives the form in (b) iii of the proposition.

K=3: $$q = \alpha_1(\boldsymbol{w})h_1(p) + \alpha_2(\boldsymbol{w})h_2(p) + \alpha_3(\boldsymbol{w})h_3(p). \tag{A.34}$$

In this case, the characteristic equation is a third-order polynomial in λ, and by the fundamental theorem of algebra, there are four mutually exclusive and exhaustive cases:

(1) three unique real roots $\lambda_1 \neq \lambda_2 \neq \lambda_3$, $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$;

(2) one repeated real root, $\lambda_1 = \lambda_2 \in \mathbb{R}$ and one unique real root $\lambda_3 \in \mathbb{R}$;

(3) one real root repeated thrice $\lambda_1 = \lambda_2 = \lambda_3 \equiv \lambda \in \mathbb{R}$; and

(4) a real root $\lambda_1 \in \mathbb{R}$ and two complex conjugate roots $\lambda_2 = \kappa + \iota\tau$, $\lambda_3 = \kappa - \iota\tau$.

First, if (1) holds, the argument leading to the representation in (c) i is identical to that of the previous cases *K*=1 or *K*=2 when the roots are real and unique. Second, if (2) holds, then we have the sum of one term of the form given in (a) and a second term of the form given in (b) ii of the proposition, leading to case (c) ii. Third, if (4) holds, then we have the sum of one term of the form given in (a) and a second term of the form given in (b) iii of the proposition, leading to case (c) iv.

Therefore, consider case (3), for which the general solution to (A.9) has the form:

$$\tilde{h}_k(x) = d_{k1}e^{\lambda x} + d_{k2}xe^{\lambda x} + d_{k3}x^2e^{\lambda x},\ \ k = 1,2,3. \tag{A.35}$$

Rewriting this in terms of p and the $\{h_k(p)\}$, substituting the result into (A.34), and re-grouping terms as before yields:

$$q = p^{\lambda}\left[\tilde{\alpha}_1(\boldsymbol{w}) + \tilde{\alpha}_2(\boldsymbol{w})\ln p + \tilde{\alpha}_3(\boldsymbol{w})(\ln p)^2\right]. \tag{A.36}$$

The Euler equation is:

$$\begin{aligned} p^{\lambda}\Big\{&\left[\partial\tilde{\alpha}_1(\boldsymbol{w})/\partial\boldsymbol{w}^{\top}\boldsymbol{w} + \lambda\tilde{\alpha}_1(\boldsymbol{w}) + \tilde{\alpha}_2(\boldsymbol{w})\right] \\ &+\left[\partial\tilde{\alpha}_2(\boldsymbol{w})/\partial\boldsymbol{w}^{\top}\boldsymbol{w} + \lambda\tilde{\alpha}_2(\boldsymbol{w}) + 2\tilde{\alpha}_3(\boldsymbol{w})\right]\ln p \\ &+\left[\partial\tilde{\alpha}_3(\boldsymbol{w})/\partial\boldsymbol{w}^{\top}\boldsymbol{w} + \lambda\tilde{\alpha}_3(\boldsymbol{w})\right](\ln p)^2\Big\} = 0. \end{aligned} \tag{A.37}$$

As before, $p^{\lambda} > 0$ and the linear independence of $\left\{1, \ln p, (\ln p)^2\right\}$ requires each sum in square brackets to vanish. In particular, $\tilde{\alpha}_3(\boldsymbol{w})$ must be homogeneous of degree $-\lambda$, and we can factor it out of the term in square brackets in (A.36), yielding:

$$q = \tilde{\alpha}_3(\boldsymbol{w})p^{\lambda}\left[\hat{\alpha}_1(\boldsymbol{w}) + \hat{\alpha}_2(\boldsymbol{w})\ln p + (\ln p)^2\right], \tag{A.38}$$

with $\hat{\alpha}_1(\boldsymbol{w}) = \tilde{\alpha}_1(\boldsymbol{w})/\tilde{\alpha}_3(\boldsymbol{w})$ and $\hat{\alpha}_2(\boldsymbol{w}) = \tilde{\alpha}_2(\boldsymbol{w})/\tilde{\alpha}_3(\boldsymbol{w})$.

Now the term in brackets on the right-hand side must be homogeneous of degree zero, which implies:

$$\partial \hat{\alpha}_1(\boldsymbol{w}) / \partial \boldsymbol{w}^{\mathsf{T}} \boldsymbol{w} = -\hat{\alpha}_2(\boldsymbol{w});$$
$$\partial \hat{\alpha}_2(\boldsymbol{w}) / \partial \boldsymbol{w}^{\mathsf{T}} \boldsymbol{w} = -2. \tag{A.39}$$

Therefore, define the smooth and invertible transformation

$$\hat{\alpha}_1(\boldsymbol{w}) = \breve{\alpha}_1(\boldsymbol{w}) + [\ln \breve{\alpha}_2(\boldsymbol{w})]^2,$$
$$\hat{\alpha}_2(\boldsymbol{w}) = -2 \ln \breve{\alpha}_2(\boldsymbol{w}), \tag{A.40}$$

where $\breve{\alpha}_1(\boldsymbol{w})$ is an arbitrary homogeneous of degree zero function and $\breve{\alpha}_2(\boldsymbol{w}) > 0$ is an arbitrary positive linearly homogeneous function. A direct calculation shows that $\hat{\alpha}_1(\boldsymbol{w})$ and $\hat{\alpha}_2(\boldsymbol{w})$ satisfy (A.39) if and only if they are related to the two homogeneous functions $\breve{\alpha}_1(\boldsymbol{w})$ and $\breve{\alpha}_2(\boldsymbol{w})$ by (A.40). Substituting (A.40) into (A.38), grouping terms, and relabeling with $\varepsilon = \lambda$, $\alpha_1(\boldsymbol{w}) = \hat{\alpha}_3(\boldsymbol{w})$, $\alpha_2(\boldsymbol{w}) = \breve{\alpha}_1(\boldsymbol{w})$, and $\alpha_3(\boldsymbol{w}) = \breve{\alpha}_2(\boldsymbol{w})$ yields the representation in (c) iii of the proposition. ■

Proof of Proposition 2

Proposition 2: *Let the supply of q take the form in Proposition 1, then homogeneity requires profit functions of the following forms:*

(a) K=1 $\varepsilon_1 > 0$

$$\pi(p, \boldsymbol{w}) = \frac{p}{(1+\varepsilon_1)} \left(\frac{p}{\alpha_1(\boldsymbol{w})} \right)^{\varepsilon_1} - \beta(\boldsymbol{w});$$

(b) K=2

i.a. $\varepsilon_1, \varepsilon_2 \neq -1$

$$\pi(p, \boldsymbol{w}) = \frac{p}{(1+\varepsilon_1)} \left(\frac{p}{\alpha_1(\boldsymbol{w})} \right)^{\varepsilon_1} + \frac{p}{(1+\varepsilon_2)} \left(\frac{p}{\alpha_2(\boldsymbol{w})} \right)^{\varepsilon_2} - \beta(\boldsymbol{w});$$

i.b. $\varepsilon_1 \neq -1, \varepsilon_2 = -1$

$$\pi(p,\boldsymbol{w}) = \frac{p}{(1+\varepsilon_1)}\left(\frac{p}{\alpha_1(\boldsymbol{w})}\right)^{\varepsilon_1} + \alpha_2(\boldsymbol{w})\ln\left(\frac{p}{\beta(\boldsymbol{w})}\right) - \gamma(\boldsymbol{w});$$

ii.a. $\varepsilon_1 \neq -1$

$$\pi(p,\boldsymbol{w}) = \frac{p}{(1+\varepsilon_1)}\left(\frac{p}{\alpha_1(\boldsymbol{w})}\right)^{\varepsilon_1}\left[\ln\left(\frac{p}{\alpha_2(\boldsymbol{w})}\right) - \frac{1}{(1+\varepsilon_1)}\right] - \beta(\boldsymbol{w});$$

ii.b. $\varepsilon_1 = -1$

$$\pi(p,\boldsymbol{w}) = \tfrac{1}{2}\alpha_1(\boldsymbol{w})\left[\ln\left(\frac{p}{\alpha_2(\boldsymbol{w})}\right)\right]^2 - \beta(\boldsymbol{w});$$

iii.

$$\pi(p,\boldsymbol{w}) = \left(\frac{p}{(1+\varepsilon_1)^2+\tau^2}\right)\left(\frac{p}{\alpha_1(\boldsymbol{w})}\right)^{\varepsilon_1}\left[(1+\varepsilon_1+\tau)\sin\left(\tau\ln\left(\frac{p}{\alpha_2(\boldsymbol{w})}\right)\right)\right.$$

$$\left. + (1+\varepsilon_1-\tau)\cos\left(\tau\ln\left(\frac{p}{\alpha_2(\boldsymbol{w})}\right)\right)\right] - \beta(\boldsymbol{w});$$

(*c*) *K*=3

iii.a. $\varepsilon_1 \neq -1$

$$\pi(p,\boldsymbol{w}) = \frac{p}{(1+\varepsilon_1)^3}\left(\frac{p}{\alpha_1(\boldsymbol{w})}\right)^{\varepsilon_1}\left\{1 + (1+\varepsilon_1)^2\alpha_2(\boldsymbol{w}) + \left[(1+\varepsilon_1)\ln\left(\frac{p}{\alpha_3(\boldsymbol{w})}\right) - 1\right]^2\right\} - \beta(\boldsymbol{w});$$

iii.b. $\varepsilon_1 = -1$

$$\pi(p,\boldsymbol{w}) = \alpha_1(\boldsymbol{w})\left[\alpha_2(\boldsymbol{w})\ln\left(\frac{p}{\beta(\boldsymbol{w})}\right) + \tfrac{1}{3}\left(\ln\left(\frac{p}{\alpha_3(\boldsymbol{w})}\right)\right)^3\right] - \gamma(\boldsymbol{w}).$$

In each case, $\beta(\mathbf{w})$ *and* $\gamma(\mathbf{w})$ *are positively linearly homogeneous functions of* $\mathbf{w}$.

Proof: Throughout the proof, omit the input prices as arguments to simplify the notation.

(a) K=1 $\varepsilon_1 > 0$

$$q = \left(\frac{p}{\alpha_1}\right)^{\varepsilon_1}. \tag{A.41}$$

Direct integration leads to

$$\pi = \left(\frac{p}{1+\varepsilon_1}\right)\left(\frac{p}{\alpha_1}\right)^{\varepsilon_1} - \beta. \tag{A.42}$$

(b) K=2 *i.a.* $\varepsilon_1, \varepsilon_2 \neq -1$

$$q = \left(\frac{p}{\alpha_1}\right)^{\varepsilon_1} + \left(\frac{p}{\alpha_2}\right)^{\varepsilon_2}. \tag{A.43}$$

This is equivalent to the previous case with two power functions, so that

$$\pi = \left(\frac{p}{1+\varepsilon_1}\right)\left(\frac{p}{\alpha_1}\right)^{\varepsilon_1} + \left(\frac{p}{1+\varepsilon_2}\right)\left(\frac{p}{\alpha_2}\right)^{\varepsilon_2} - \beta. \tag{A.44}$$

i. b. $\varepsilon_1 \neq -1,\ \varepsilon_2 = -1$.

$$q = \left(\frac{p}{\alpha_1}\right)^{\varepsilon_1} + \left(\frac{\alpha_2}{p}\right). \tag{A.45}$$

Direct integration now leads to

$$\pi = \left(\frac{p}{1+\varepsilon_1}\right)\left(\frac{p}{\alpha_1}\right)^{\varepsilon_1} + \alpha_2 \ln\left(\frac{p}{\beta}\right) - \gamma. \tag{A.46}$$

Here, the constant of integration must take the form $-(\alpha_2 \ln \beta + \gamma)$, β, γ 1° homogeneous if π is to be 1° homogeneous.

ii. b. $\varepsilon_1 = -1$.

$$q = \frac{\alpha_1}{p} \ln\left(\frac{p}{\alpha_2}\right). \tag{A.47}$$

Once again, direct integration gives,

$$\pi = ½\alpha_1 \left[\ln\left(\frac{p}{\alpha_2}\right) \right]^2 - \beta, \tag{A.48}$$

which follows from

$$\frac{\partial}{\partial p} \left[\ln\left(\frac{p}{\alpha_2}\right) \right]^2 = \frac{2}{p} \ln\left(\frac{p}{\alpha_2}\right). \tag{A.49}$$

iii.

$$q = \left(\frac{p}{\alpha_1}\right)^{\varepsilon_1} \left[\sin\left(\tau \ln\left(\frac{p}{\alpha_2}\right)\right) + \cos\left(\tau \ln\left(\frac{p}{\alpha_2}\right)\right) \right]. \tag{A.50}$$

Use the complex definitions for sine and cosine in Abramowitz and Stegun (1972, p. 71),

$$\begin{aligned} \sin x &= \frac{1}{2\imath}\left(e^{\imath x} - e^{-\imath x}\right), \\ \cos x &= \frac{1}{2}\left(\mathrm{e}^{\imath x} + e^{-\imath x}\right), \end{aligned} \tag{A.51}$$

where $\imath = \sqrt{-1}$, to rewrite the supply function in the form

$$\begin{aligned} q &= \left(\frac{p}{\alpha_1}\right)^{\varepsilon_1} \left[\frac{1}{2\imath}\left(e^{\imath\tau \ln(p/\alpha_2)} - e^{-\imath\tau \ln(p/\alpha_2)}\right) + \frac{1}{2}\left(e^{\imath\tau \ln(p/\alpha_2)} + e^{-\imath\tau \ln(p/\alpha_2)}\right) \right] \\ &= ½\alpha_1^{-\varepsilon_1} \left[(1-\imath)\alpha_2^{-\imath\tau} p^{\varepsilon_1 + \imath\tau} + (1+\imath)\alpha_2^{\imath\tau} p^{\varepsilon_1 - \imath\tau} \right], \end{aligned} \tag{A.52}$$

using the algebraic identity $1/\imath = -\imath^2/\imath = -\imath$ in the second line. Integrating yields

$$\pi = \tfrac{1}{2}\alpha_1^{-\varepsilon_1}\left[\left(\frac{1-\iota}{1+\varepsilon_1+\iota\tau}\right)\alpha_2^{-\iota\tau}p^{1+\varepsilon_1+\iota\tau} + \left(\frac{1+\iota}{1+\varepsilon_1-\iota\tau}\right)\alpha_2^{\iota\tau}p^{1+\varepsilon_1-\iota\tau}\right] - \beta. \quad \text{(A.53)}$$

Now eliminate the complex terms in the denominator by using $(a+\iota b)(a-\iota b) = a^2 + b^2$,

$$\begin{aligned}\pi = \tfrac{1}{2}\left(\frac{p}{(1+\varepsilon_1)^2+\tau^2}\right)\left(\frac{p}{\alpha_1}\right)^{\varepsilon_1}&\left[(1+\varepsilon_1-\tau-\iota(1+\varepsilon_1+\tau))\left(\frac{p}{\alpha_2}\right)^{\iota\tau}\right.\\ &\left. + \ (1+\varepsilon_1-\tau-\iota(1+\varepsilon_1+\tau))\left(\frac{p}{\alpha_2}\right)^{-\iota\tau}\right] - \beta,\end{aligned} \quad \text{(A.54)}$$

applying the algebraic identity $\iota^2 = -1$. Group terms in ι, again using $\iota = -1/\iota$,

$$\begin{aligned}\pi = \left(\frac{p}{(1+\varepsilon_1)^2+\tau^2}\right)\left(\frac{p}{\alpha_1}\right)^{\varepsilon_1}&\left\{(1+\varepsilon_1-\tau)\cos\left(\tau\ln\left(\frac{p}{\alpha_2}\right)\right)\right.\\ &\left. + \ (1+\varepsilon_1+\tau)\sin\left(\tau\ln\left(\frac{p}{\alpha_2}\right)\right)\right\} - \beta.\end{aligned} \quad \text{(A.55)}$$

(c) K=3 *iii. a.* $\varepsilon_1 \neq -1$

$$q = \left(\frac{p}{\alpha_1}\right)^{\varepsilon_1}\left[\alpha_2 + \left(\ln\left(\frac{p}{\alpha_3}\right)\right)^2\right]. \quad \text{(A.56)}$$

Integrate the second term by parts twice, first using $u = \left[\ln(p/\alpha_3)\right]^2$ and $v' = (p/\alpha_2)^{\varepsilon_1}$,

$$\begin{aligned}\pi = &\left(\frac{p}{1+\varepsilon_1}\right)\left(\frac{p}{\alpha_1}\right)^{\varepsilon_1}\alpha_2 + \left(\frac{p}{1+\varepsilon_1}\right)\left(\frac{p}{\alpha_1}\right)^{\varepsilon_1}\left[\ln\left(\frac{p}{\alpha_3}\right)\right]^2\\ &-\left(\frac{2}{1+\varepsilon_1}\right)\int\left(\frac{p}{\alpha_1}\right)^{\varepsilon_1}\ln\left(\frac{p}{\alpha_3}\right)dp - \beta,\end{aligned} \quad \text{(A.57)}$$

and then using $u = \ln(p/\alpha_3)$ and $v' = (p/\alpha_2)^{\varepsilon_1}$,

$$\pi = \frac{p}{(1+\varepsilon_1)^3}\left(\frac{p}{\alpha_1}\right)^{\varepsilon_1}\left\{1+(1+\varepsilon_1)^2\alpha_2+\left[(1+\varepsilon_1)\ln\left(\frac{p}{\alpha_3}\right)-1\right]^2\right\}-\beta. \qquad \text{(A.58)}$$

iii. b. $\varepsilon_1 = -1$

$$q = \left(\frac{\alpha_1}{p}\right)\left[\alpha_2+\left(\ln\left(\frac{p}{\alpha_3}\right)\right)^2\right]. \qquad \text{(A.59)}$$

Distribute the first term on the right-hand-side of the supply function and integrate,

$$\pi = \alpha_1\alpha_2 \ln p + \tfrac{1}{3}\alpha_1\left[\ln\left(\frac{p}{\alpha_3}\right)\right]^3 - \tilde{\beta}, \qquad \text{(A.60)}$$

which follows from

$$\frac{\partial}{\partial p}\left[\ln\left(\frac{p}{\alpha_3}\right)\right]^3 = \frac{3}{p}\left[\ln\left(\frac{p}{\alpha_3}\right)\right]^2. \qquad \text{(A.61)}$$

The constant of integration must be such that the sum $\alpha_1\alpha_2 \ln p - \tilde{\beta}$ is 1° homogeneous. Set $\tilde{\beta} = \alpha_1\alpha_2 \ln \beta + \gamma$, where β, γ are arbitrary positive 1° homogeneous functions of ***w***,

$$\pi = \alpha_1\left[\alpha_2 \ln\left(\frac{p}{\beta}\right)+\tfrac{1}{3}\left(\ln\left(\frac{p}{\alpha_3}\right)\right)^3\right]-\gamma. \qquad \text{(A.62)}$$

The remaining K=3 cases are linear combinations of the solutions for K=1 and 2. ■

Remark: Sufficiency in each case can be shown simply by differentiating the profit function with respect to p.

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