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GORMAN ENGEL CURVES FOR INCOMPLETE DEMAND SYSTEMS

by Jeffrey T. LaFrance, Timothy K.M. Beatty, and Rulon D. Pope[*]

THE FOCUS OF THIS CHAPTER IS AGGREGATION across individuals to market–level behavior for a set of demand equations that does not exhaust the consumption budget. There are good reasons to consider this issue. The effects of government policies and preferences for goods and services both differ across individuals, and the theory of choice applies to individual decision makers. But the impacts of government policies and market intervention schemes on prices, quantities, taxes, benefits and so forth first unfold at the aggregate market level and then are transmitted to individuals. We therefore need to study demand and supply at the aggregate market level to determine equilibrium prices and quantities transacted, and then preference functions of individuals to identify the disaggregate impacts of those market outcomes. In addition, while aggregation theory has been addressed comprehensively for complete demand systems in Gorman (1953, 1961, 1981), Muellbauer (1975, 1976), Lewbel (1987, 1990), and van Daal and Merkies (1989), almost nothing has been written on aggregation for an incomplete set of demand equations.

Gorman (1953, 1961) developed the notion of exact aggregation and derived the quasi-homothetic polar form of indirect preferences that is necessary and sufficient for aggregation to a representative consumer with demands that depend on per capita income. Muellbauer (1975, 1976) extended the Gorman polar form to a nonlinear function of income to obtain the price independent generalized linear (PIGL) and price independent generalized logarithmic (PIGLOG) functional forms. The Almost Ideal Demand System (AIDS) of Deaton and Muellbauer (1980) implements Muellbauer's results to produce demands with budget shares expressed as functions of linear and quadratic terms in the logarithm of prices and a linear term in the logarithm of income. The AIDS and its linear approximation (LA-AIDS) have been linchpins in applied demand analysis since their introduction. Shortly thereafter, in an elegant contribution to the festschrift to Sir Richard Stone, Gorman (1981) derived the set of functional forms for demand models that can be written in terms of any additive set of functions of income. This forms the cornerstone for all subsequent aggregation theory in consumption.

The existing theory on the number and functional form of income terms in demand models relies critically on the properties of zero degree homogeneity and

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adding up for a complete system. But incomplete information is the rule, not the exception. In almost every case, we are faced with a subset of quantity, price, and income data – even with models that use consumption expenditure as the source of adding up. Consumption expenditure excludes borrowing, saving, and the labor-leisure trade-off, so that the periodic budget constraint is missing one or more goods. With borrowing, saving and time in consumption theory, the budget constraint is defined in terms of current wealth plus the discounted present value of expected potential future earnings relative to the discounted present value of current and expected future total consumption.

We also are almost always interested in a much smaller subset of goods consumed than the complete list of possible items that can be purchased and used by individuals in the economy. Usually we try to model a relatively small number of goods as a function of the prices of those goods, some measures of the costs of other goods, and either income or total expenditures on the goods of interest.

These considerations have two essential, related, and unavoidable impacts on the demand models we use in such cases. One is that the budget constraint becomes a strict inequality. The second is that the demand equations will not be homogeneous of degree zero in the prices of the goods of interest and income because zero degree homogeneity arises from adding up, which is defined over all prices and income.

Finally, since we most often only model *some* of the goods purchased and consumed, there is no reason to require the demand equations for the goods that we don't or can't model to have the same functional structure as those that we do model. They might have the same functional form, but then again, they might not; we have no way to know. We simply aren't in a position to measure, estimate, or predict the other demands with an incomplete information set. In almost all cases, therefore, we must admit that our demand systems are incomplete. Thus, for these reasons and probably others as well, incomplete systems are far more interesting than complete ones. An open question of considerable interest, then, is whether the results on aggregation for complete demand systems can be extended to incomplete demand systems. If so, what form might this extension take?

In this chapter, we extend Gorman's class of polynomial Engel curves to incomplete systems. Our extension permits any monotone and sufficiently smooth function of income. This greatly expands the class of feasible functional forms that can be used to model aggregate demand behavior in a theoretically consistent manner. But Gorman's maximum rank three result – the number of linearly independent functions of income is no more than three – follows purely from symmetry. Neither adding up nor homogeneity plays any role in the rank restriction.

Based on this fundamental result, we develop two distinct classes of incomplete demand models that nest the rank and functional form of the income variables. The second class of models can be globally restricted to satisfy *weak integrability* (LaFrance and Hanemann, 1989) through constraints on the model parameters. Both sets of models can be estimated with aggregate market data and accommodate inferences on the impacts of policies on consumption and economic welfare of identifiable groups of consumers.

The next section of the chapter reviews aggregation theory for complete demand systems and the third section presents and discusses our extension of Gorman's class of polynomial Engel curves to incomplete systems. The fourth section contains two classes of incomplete demand models that nest the rank and functional form of the demand equations for the goods of interest through Box-Cox transformations. The concluding section summarizes our results and briefly discusses our empirical experiences with these new models and the implications for demand analysis. To minimize the analytical burden of the chapter, the mathematical proofs and a set of detailed derivations are contained in a separate paper (LaFrance, Beatty, and Pope, 2003; hereafter LPB) that is freely available from the authors upon request.[1]

Aggregation Theory for Complete Demand Systems

WE BEGIN WITH A FAIRLY LARGE AMOUNT OF NOTATION. Let $\boldsymbol{p} \in \mathbb{R}^{n_q}_{++}$ be the vector of market prices for the goods of interest, $\boldsymbol{q} \in \mathbb{R}^{n_q}_{+}$, let $\tilde{\boldsymbol{p}} \in \mathbb{R}^{n_{\tilde{q}}}_{++}$ be the vector of market prices for other goods, $\tilde{\boldsymbol{q}} \in \mathbb{R}^{n_{\tilde{q}}}_{+}$, let $m \in \mathbb{R}_{++}$ be income, let $s = \tilde{\boldsymbol{p}}'\tilde{\boldsymbol{q}} = m - \boldsymbol{p}'\boldsymbol{q} > 0$ be expenditure on other goods, let $\boldsymbol{s} \in \mathbb{R}^{J}$ be a vector of demand shifters, let $\pi(\tilde{\boldsymbol{p}})$ be a 1° homogeneous function of $\tilde{\boldsymbol{p}}$,[2] let $\boldsymbol{x} = [g_1(p_1/\pi(\tilde{\boldsymbol{p}})) \cdots g_{n_q}(p_{n_q}/\pi(\tilde{\boldsymbol{p}}))]' \equiv \boldsymbol{g}(\boldsymbol{p}/\pi(\tilde{\boldsymbol{p}}))$ be a vector of twice continuously differentiable, strictly monotone functions, and let $y = f(m/\pi(\tilde{\boldsymbol{p}}))$ be a twice continuously differentiable, strictly monotone increasing transformation.

Suppose that we have a transformed demand system of the form

$$\frac{\partial y(\boldsymbol{x};\cdot)}{\partial \boldsymbol{x}} = \sum_{i=0}^{K} \alpha_i(\boldsymbol{x};\cdot) h_i\left(y(\boldsymbol{x};\cdot)\right), \tag{1}$$

where the "·" after the semicolon indicates that the system may not be complete and depends on other variables in addition to $\boldsymbol{x}$. However, for notational clarity and compactness, we will omit this set of unspecified arguments in most of what follows. By a simple change of variables from $\boldsymbol{p}$ and m to $\boldsymbol{x} = \ln(\boldsymbol{p})$ and $y = \ln(m)$, Gorman (1981) showed three things about all complete systems in this class:

(i) Normalizing for a unique representation, accounting for adding up, and for some of the implications of symmetry, the nonlinear partial differential equations can be trans-

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[1] The working paper can be downloaded from the University of California-Berkeley eScholarship Repository at http://repositories.cdlib.org/are_ucb/961.

[2] When the system is *complete*, $\tilde{\boldsymbol{p}}$ has no elements and in such a case we adopt the convention $\pi(\tilde{\boldsymbol{p}}) \equiv 1$.

formed into a system of linear, homogeneous, ordinary differential equations in functions of the natural logarithm of income. From the theory of differential equations, solutions to this system are of the form $h_i(m) = m^{\lambda_i}\left(\ln(m)\right)^i$, where each λ_i is a root of the characteristic polynomial for the linear ordinary differential equations. In general, such characteristic roots can be either real or complex, and complex roots come in conjugate pairs that may have both real and complex parts.

(ii) If the rank of the $n_q \times K+1$ coefficient matrix $\boldsymbol{A}(\boldsymbol{x}) \equiv [\alpha_{ij}(\boldsymbol{x})]$ is at least three, then symmetry implies that: (a) the characteristic roots are either purely real or purely complex (all roots of the form $\lambda_i = a_i + b_i\sqrt{-1}$ have $a_i = 0$ if $b_i \neq 0$ and conversely, $b_i = 0$ if $a_i \neq 0$); (b) if any roots are real, there are no complex roots, and conversely; and (c) for real roots, there are no product terms of the form $m^{\alpha}\left(\ln(m)\right)^{\beta}$ with both $\alpha \neq 1$ and $\beta \neq 0$.

(iii) Symmetry also implies that the rank of $\boldsymbol{A}(\boldsymbol{x})$ is at most three.

For rank three demand systems, this completely specifies the class of functional forms for the income terms. Only three mutually exclusive cases are possible: (a) $m\left(\ln(m)\right)^r$, r an integer; (b) $m^{1+\kappa}$, $\kappa \neq 0$; or (c) $m\sin\left(\tau\ln(m)\right)$ and $m\cos\left(\tau\ln(m)\right)$, $\tau > 0$, with both sine and cosine terms appearing as a conjugate complex pair. In other words, for rank three systems, the model must take one of the following three forms:

$$\boldsymbol{q} = \alpha_0(\boldsymbol{x})m + \sum_{j=1}^{K} \alpha_j(\boldsymbol{x})m[\ln(m)]^j ; \tag{2}$$

$$\boldsymbol{q} = \alpha_0(\boldsymbol{x})m + \sum_{\kappa\in S} \beta_\tau(\boldsymbol{x})m^{1-\kappa} + \sum_{\kappa\in S} \gamma_\tau(\boldsymbol{x})m^{1+\kappa} , \tag{3}$$

where S is a set of nonzero constants; or

$$\boldsymbol{q} = \alpha_0(\boldsymbol{x})m + \sum_{\tau\in T} \beta_\tau(\boldsymbol{x})m\sin\left(\tau\ln(m)\right) + \sum_{\tau\in T} \gamma_\tau(\boldsymbol{x})m\cos\left(\tau\ln(m)\right), \tag{4}$$

where T is a set of positive constants. This includes PIGLOG models and extensions that are polynomials in $\ln(m)$, simple polynomials in income, and PIGL models and extensions that are polynomials in m^{κ}.

Demand models that have *full rank* (Lewbel, 1990) are characterized by the property that the rank of the matrix $\boldsymbol{A}(\boldsymbol{x})$ is equal to the number of its columns, that is, the number of different income functions, $h_j(y)$. Full rank one complete systems are homothetic due to adding up,

$$\boldsymbol{q} = \alpha_0(\boldsymbol{x})m . \tag{5}$$

With budget shares on the left, all full rank one complete systems are zero-order polynomials in income. Muellbauer (1975, 1976) showed that all full rank two complete systems are either PIGL,

$$\boldsymbol{q} = \alpha_0(\boldsymbol{x})m + \alpha_1(\boldsymbol{x})m^{1-\kappa},\ \kappa \neq 0, \tag{6}$$

or PIGLOG,

$$\boldsymbol{q} = \alpha_0(\boldsymbol{x})m + \alpha_1(\boldsymbol{x})m\ln(m). \tag{7}$$

Appealing to Bernoulli's equation,

$$\frac{\partial\left[e(\boldsymbol{x};\cdot)^{\kappa}/\kappa\right]}{\partial\boldsymbol{x}} = e(\boldsymbol{x};\cdot)^{\kappa-1}\frac{\partial e(\boldsymbol{x};\cdot)}{\partial\boldsymbol{x}} = \beta_0(\boldsymbol{x}) + \beta_1(\boldsymbol{x})\left(\frac{e(\boldsymbol{x};\cdot)^{\kappa}}{\kappa}\right), \tag{8}$$

we can see that the full rank two PIGL model,

$$\frac{\partial e(\boldsymbol{x};\cdot)}{\partial\boldsymbol{x}} = \alpha_0(\boldsymbol{x})e(\boldsymbol{x};\cdot) + \alpha_1(\boldsymbol{x})e(\boldsymbol{x};\cdot)^{1-\kappa}, \tag{9}$$

is a first-order polynomial in the power function, $e(\boldsymbol{x};\cdot)^{\kappa}/\kappa$. Similarly, by appealing to the logarithmic transformation,

$$\frac{\partial\ln\left[e(\boldsymbol{x};\cdot)\right]}{\partial\boldsymbol{x}} = e(\boldsymbol{x};\cdot)^{-1}\frac{\partial e(\boldsymbol{x};\cdot)}{\partial\boldsymbol{x}} = \alpha_0(\boldsymbol{x}) + \alpha_1(\boldsymbol{x})\ln\left[e(\boldsymbol{x};\cdot)\right], \tag{10}$$

we can see that full rank two PIGLOG model,

$$\frac{\partial e(\boldsymbol{x};\cdot)}{\partial\boldsymbol{x}} = \alpha_0(\boldsymbol{x})e(\boldsymbol{x};\cdot) + \alpha_1(\boldsymbol{x})e(\boldsymbol{x};\cdot)\ln\left[e(\boldsymbol{x};\cdot)\right], \tag{11}$$

is a first-order polynomial in the natural logarithm of income.

Gorman (1981) conjectured that the quadratic is the most general nondegenerate full rank three complete system. A series of subsequent articles (Jerison 1993; Lewbel 1987, 1990; Russell 1982, 1996; Russell and Farris 1993, 1998; Van Daal and Merkies 1989) has since proven this conjecture – complete full rank three Gorman Engel curve systems are quadratic: a quadratic extension of the Bernoulli equation,

$$\frac{\partial[e(\boldsymbol{x};\cdot)/\kappa]^{\kappa}}{\partial\boldsymbol{x}} = e(\boldsymbol{x};\cdot)^{\kappa-1}\frac{\partial e(\boldsymbol{x};\cdot)}{\partial\boldsymbol{x}} = \beta_0(\boldsymbol{x}) + \beta_1(\boldsymbol{x})\left(\frac{e(\boldsymbol{x};\cdot)^{\kappa}}{\kappa}\right) + \beta_2(\boldsymbol{x})\left(\frac{e(\boldsymbol{x};\cdot)^{\kappa}}{\kappa}\right)^2, \tag{12}$$

is a generalized PIGL,

$$\frac{\partial e(\boldsymbol{x};\cdot)}{\partial\boldsymbol{x}} = \alpha_0(\boldsymbol{x})e(\boldsymbol{x};\cdot) + \alpha_1(\boldsymbol{x})e(\boldsymbol{x};\cdot)^{1-\kappa} + \alpha_2(\boldsymbol{x})e(\boldsymbol{x};\cdot)^{1+\kappa}; \tag{13}$$

a quadratic extension of the logarithmic transformation,

$$\frac{\partial\ln[e(\boldsymbol{x};\cdot)]}{\partial\boldsymbol{x}} = \frac{\partial e(\boldsymbol{x};\cdot)/\partial\boldsymbol{x}}{e(\boldsymbol{x};\cdot)} = \alpha_0(\boldsymbol{x}) + \alpha_1(\boldsymbol{x})\ln[e(\boldsymbol{x};\cdot)] + \alpha_2(\boldsymbol{x})\{\ln[e(\boldsymbol{x};\cdot)]\}^2, \tag{14}$$

is a generalized PIGLOG,

$$\frac{\partial e(\boldsymbol{x};\cdot)}{\partial\boldsymbol{x}} = \alpha_0(\boldsymbol{x})e(\boldsymbol{x};\cdot) + \alpha_1(\boldsymbol{x})e(\boldsymbol{x};\cdot)\ln[e(\boldsymbol{x};\cdot)] + \alpha_2(\boldsymbol{x})e(\boldsymbol{x};\cdot)\{\ln[e(\boldsymbol{x};\cdot)]\}^2; \tag{15}$$

and a quadratic complex exponential transformation,[3]

$$\frac{\partial}{\partial\boldsymbol{x}}\left[-(\iota\tau)^{-1}e(\boldsymbol{x};\cdot)^{-\iota\tau}\right] = e(\boldsymbol{x};\cdot)^{-\iota\tau-1}\frac{\partial e(\boldsymbol{x};\cdot)}{\partial\boldsymbol{x}}$$

$$= \alpha_0(\boldsymbol{x}) + \tfrac{1}{2}[\alpha_1(\boldsymbol{x}) - \iota\alpha_2(\boldsymbol{x})]e(\boldsymbol{x};\cdot)^{\iota\tau} + \tfrac{1}{2}[\alpha_1(\boldsymbol{x}) + \iota\alpha_2(\boldsymbol{x})]\left(e(\boldsymbol{x};\cdot)^{\iota\tau}\right)^2, \tag{16}$$

where $\iota = \sqrt{-1}$, is a trigonometric demand system,[4]

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[3] Since $\iota^{-1} = 1/\sqrt{-1} = -\iota$, including $-(\iota\tau)^{-1}$ on the left is innocuous. The right-hand-side can be multiplied by $-\iota\tau$ and absorbed into the complex price functions without changing the structure. Also, none of the authors discussed above derive or state (16)–(17) explicitly. But this can be deduced from Gorman (1981).

[4] We apply de Moivre's theorem to get the real-valued trigonometric expressions in (17) for the complex-valued terms in (16),

$$\begin{aligned} e^{\pm\iota\tau x} &= 1 + \tfrac{1}{1!}(\pm\iota\tau x) + \tfrac{1}{2!}(\pm\iota\tau x)^2 + \cdots \\ &= \left[1 - \tfrac{1}{2!}(\tau x)^2 + \tfrac{1}{4!}(\tau x)^6 + \cdots\right] \pm \iota\left[\tfrac{1}{1!}(\tau x) - \tfrac{1}{3!}(\tau x)^3 + \tfrac{1}{5!}(\tau x)^5 + \cdots\right] \\ &= \cos(\tau x) \pm \iota\sin(\tau x)\,. \end{aligned}$$

$$\frac{\partial e(\boldsymbol{x};\cdot)}{\partial \boldsymbol{x}} = e(\boldsymbol{x};\cdot)\left\{\alpha_0(\boldsymbol{x}) + \alpha_1(\boldsymbol{x})\sin\left[\tau \ln\left(e(\boldsymbol{x};\cdot)\right)\right] + \alpha_2(\boldsymbol{x})\cos\left[\tau \ln\left(e(\boldsymbol{x};\cdot)\right)\right]\right\}. \tag{17}$$

We therefore reach the following three conclusions. Full rank complete systems in the Gorman Engel curve class are polynomials of only one function of income. The order of the polynomial is at most two. And the class of functional forms is severely limited.

Aggregation Theory for Incomplete Demand Systems

WHEN THE GOODS OF INTEREST FORM AN INCOMPLETE demand system (Gorman 1965, Epstein 1982, LaFrance 1985, 1986, 1990, LaFrance and Hanemann 1989), the results detailed in the previous section cannot be applied. Very recently, LBP have shown that there is no restriction on the class of income functions that can meet Gorman's definition when the system is incomplete (also, see Russell and Farris 1993: p. 319). The main reason is that adding up plays two critical roles in Gorman's argument. First, a constant must be one of the income functions. Second, with a simple change of variables, the demand equations can be converted to a system of linear, homogeneous, ordinary differential equations with functions of the natural logarithm of income as the dependent variables.

A related, but slightly distinct, reason is that the functional form restrictions for full rank one and two complete systems are due to homogeneity. Homogeneity can be accommodated with the prices of other goods, adding up does not apply to a subset of goods, and the goods not modeled do not necessarily have the same functional form as the demands for the goods under study. These all increase the flexibility of an incomplete demand model.

Gorman Engel curves can be extended naturally to incomplete demand systems. Consider the class of incomplete demand models that are polynomials in *any* transformation of deflated income. This very large class of incomplete demand models is characterized in LBP as follows.

Proposition: *Let* $\mathcal{X} \subset \mathbb{R}^{n_q}$ *be the domain of definition for the deflated and transformed prices,* $\boldsymbol{x}$*, and let* $\mathcal{X}^{\circ}$ *be the interior of* $\mathcal{X}$*. If the incomplete demand system can be represented as,*

$$\frac{\partial y(\boldsymbol{x};\cdot)}{\partial \boldsymbol{x}} = \sum_{i=0}^{K} \alpha_i(\boldsymbol{x};\cdot)\, y(\boldsymbol{x};\cdot)^i ,$$

and has continuous derivatives with respect to $\boldsymbol{x}$ *over an open neighborhood* $\mathcal{N}(\boldsymbol{x}) \subset \mathcal{X}^{\circ}$*, then the rank of* $A(\boldsymbol{x}) = \left[\alpha_0(\boldsymbol{x};\cdot)\ \alpha_1(\boldsymbol{x};\cdot) \cdots \alpha_K(\boldsymbol{x};\cdot)\right]$ *is no greater than 3.*

The proof in LBP is constructive and uses continuity of the symmetry conditions only for the powers of y from K+1 to 2K–1. The proof leads immediately to the following.

Corollary 1: *If* $\alpha_i(\boldsymbol{x}^1;\cdot) \neq \boldsymbol{0} \, \forall \, \boldsymbol{x}^1 \in \mathcal{N}(\boldsymbol{x}) \forall i = 0,\ldots,K$ *, and* $rank\left[\boldsymbol{A}(\boldsymbol{x})\right] = K+1$*, then* $K \leq 2$.

There are two additional implications of this proposition. The first follows immediately from LBP's method of proof. The second makes use of the continuity of the remaining symmetry conditions for powers of y from 2 to K.

Corollary 2: *If* $rank\left[\alpha_0(\boldsymbol{x};\cdot)\ \alpha_1(\boldsymbol{x};\cdot)\ \alpha_2(\boldsymbol{x};\cdot)\right] = 3$*, then*

$$\alpha_i(\boldsymbol{x}^1;\cdot) = \varphi_i(\boldsymbol{x}^1)\alpha_2(\boldsymbol{x}^1;\cdot),\ \varphi_i : \mathbb{R}^{n_q} \to \mathbb{R},\ \varphi_i \in \mathcal{C}^1,\ \forall\, i \geq 3,\ \forall\, \boldsymbol{x}^1 \in \mathcal{N}(\boldsymbol{x}).$$

If the φ_i *are constant with respect to* ***x****, then* $\varphi_i = 0 \ \forall \ i \geq 3$.

In other words, a maximum rank of three follows purely from symmetry and a quadratic is the highest order full rank incomplete system in this class if either (a) none of the price vectors vanish or (b) the factors of proportionality for powers 2 through K do not depend explicitly on the transformed and deflated prices $\boldsymbol{x}$.

It is equally important to explain what we have not said (and can not say) about this class of demand models. In particular, these results do not preclude higher order polynomials. They only preclude more than three income terms and an associated *linearly independent* matrix of price functions. The best way to illustrate this is with an example. The one we present is motivated by Jerison (1993). Let the indirect utility function be given by

$$v(\boldsymbol{x},\tilde{\boldsymbol{p}},y,\boldsymbol{s}) = \upsilon\left[\left(\frac{\beta(\boldsymbol{x},\tilde{\boldsymbol{p}},\boldsymbol{s})}{\gamma(\boldsymbol{x},\tilde{\boldsymbol{p}},\boldsymbol{s})-y}\right)^{\eta} - \delta(\boldsymbol{x},\tilde{\boldsymbol{p}},\boldsymbol{s});\tilde{\boldsymbol{p}},\boldsymbol{s}\right], \tag{18}$$

where we assume $\gamma(\boldsymbol{x},\tilde{\boldsymbol{p}},\boldsymbol{s}) > y$ for monotonicity and let η be any real number in the interval [1,∞). Applying Roy's identity, we generate an incomplete demand system as

$$\left(\frac{dy}{dm}\right)\boldsymbol{q} = \mathbf{diag}\left[x_i'(p_i)\right]\left[\frac{\partial\gamma}{\partial\boldsymbol{x}} - \frac{\partial\beta}{\partial\boldsymbol{x}}\left(\frac{\gamma-y}{\beta}\right) + \left(\frac{\beta}{\eta}\right)\frac{\partial\delta}{\partial\boldsymbol{x}}\left(\frac{\gamma-y}{\beta}\right)^{\eta+1}\right]. \tag{19}$$

Under certain conditions on η, this has the form of the proposition and illustrates the nature of its implications. First note that there are three linearly independent functions of y on the right-hand-side of (19). When η = 1, we have a quadratic in y. But the parameter η can be any integer in [1,∞) and preferences will remain well-behaved with appropriate choices for the functions β, γ, δ. If η is an integer greater than one, expanding the last term in square brackets with the binomial formula implies that all powers of y from 0 to $\eta+1$ appear on the right. Each term is proportional to $y^{\eta+1}$ with factor of proportionality equal to $\binom{n+i}{i}(-\gamma)^{i}$. None of these factors can be constant or the demand model would have rank less

less than three. The model cannot be reduced to a quadratic for any integer value of $\eta > 1$. Finally, since the first two terms in square brackets only involve powers 0 and 1 in y, the sub-matrix of price functions for the powers of y from 2 through $\eta+1$ has rank at most equal to one. This completes the case of polynomials and a finite number of separable transformed and deflated price and income functions for this example.

However, η also can assume a rational or even an irrational value in $[1,\infty)$ and preferences will remain well-behaved with appropriate choices of the functions β, γ, δ. In such a case, the last term in square brackets on the right-hand-side of (19) is analytic with a convergent infinite Taylor series expansion over the set of positive values for $\gamma - y$. The vectors of price functions in this convergent series will continue to be proportional and the rank of the full matrix of price functions (with, of course, an infinite number of columns) continues to be at most three. Thus in this example, a finite number of terms that are functionally separable between the deflated transformed prices and the deflated transformed income variables requires a polynomial in y. But an extensive set of well-defined models exists beyond the set of quadratic polynomials. Each element of this set can be represented as an irreducible polynomial of higher order than a quadratic and may have an infinite number of terms in y.

Nesting Rank and Functional Form of Incomplete Demand Models

IN THIS SECTION, WE APPLY THE ABOVE RESULTS to the development of two sets of incomplete demand models. These new models can be used to nest both rank and functional form. The first extends the AIDS to a quadratic price independent generalized linear incomplete demand system (QPIGL-IDS). The second extends the quadratic direct utility model to a QPIGL-IDS that also includes an extension of translog indirect preferences (Christensen, Jorgenson, and Lau, 1975). Throughout this section, we assume that the model applies to n_q out of $N \geq n_q+1$ goods and define the Box-Cox transformations $y \equiv (m^{\kappa}-1)/\kappa$ and $x_i \equiv (p_i^{\lambda}-1)/\lambda,\ i=1,\ldots,n_q$ for our transformations of deflated prices and income.

A QPIGL-IDS Extension of AIDS

To generate our incomplete system extension of the AIDS, we write a class of indirect utility functions in the form

$$v(\boldsymbol{x}, y, \tilde{\boldsymbol{p}}, \boldsymbol{s}) = \upsilon\left\{-\left[\left(y - \alpha_0(\tilde{\boldsymbol{p}}, \boldsymbol{s}) - \boldsymbol{\alpha}(\tilde{\boldsymbol{p}}, \boldsymbol{s})'\boldsymbol{x} - \tfrac{1}{2}\boldsymbol{x}'\boldsymbol{B}\boldsymbol{x}\right)^{-1} + \boldsymbol{\delta}'\boldsymbol{x}\right] e^{\boldsymbol{\gamma}'\boldsymbol{x}}; \tilde{\boldsymbol{p}}, \boldsymbol{s}\right\}, \tag{20}$$

where $\boldsymbol{\alpha}(\tilde{\boldsymbol{p}}, \boldsymbol{s})$ is a vector of functions of the other prices and demographics, $\alpha_0(\tilde{\boldsymbol{p}}, \boldsymbol{s})$ is a scalar function of the other prices and demographics, both $\boldsymbol{\alpha}(\tilde{\boldsymbol{p}}, \boldsymbol{s})$ and $\alpha_0(\tilde{\boldsymbol{p}}, \boldsymbol{s})$ are 0° homogeneous in $\tilde{\boldsymbol{p}}$, $\boldsymbol{B}$ is an $n_q \times n_q$ matrix of parameters, and $\boldsymbol{\gamma}$ and $\boldsymbol{\delta}$ are both n_q–vectors of parameters. Due to 0° homogeneity of $\boldsymbol{\alpha}(\tilde{\boldsymbol{p}}, \boldsymbol{s})$ and $\alpha_0(\tilde{\boldsymbol{p}}, \boldsymbol{s})$ in $\tilde{\boldsymbol{p}}$, we can (and do) take $(\boldsymbol{p}, \tilde{\boldsymbol{p}}, m)$ all to be deflated by $\pi(\tilde{\boldsymbol{p}})$ without any loss in generality. Then, applying Roy's identity generates our QPIGL-IDS extension of the AIDS in budget share form,

$$\boldsymbol{w} = m^{-\kappa}\boldsymbol{P}^{\lambda}\left[\boldsymbol{\alpha} + \boldsymbol{B}\boldsymbol{x} + \boldsymbol{\gamma}(y - \alpha_0 - \boldsymbol{\alpha}'\boldsymbol{x} - \tfrac{1}{2}\boldsymbol{x}'\boldsymbol{B}\boldsymbol{x}) + (\boldsymbol{\delta} + \boldsymbol{\delta}'\boldsymbol{x}\boldsymbol{\gamma})(y - \alpha_0 - \boldsymbol{\alpha}'\boldsymbol{x} - \tfrac{1}{2}\boldsymbol{x}'\boldsymbol{B}\boldsymbol{x})^2\right], \tag{21}$$

where $\boldsymbol{P}^{\lambda} \equiv \mathbf{diag}[p_i^{\lambda}]$.

Next, assume that $\boldsymbol{\alpha}$ and $\boldsymbol{B}$ do not vanish simultaneously. Otherwise, this model can not attain rank three. Then $\boldsymbol{\gamma} \neq \mathbf{0}$ and $\boldsymbol{\delta} \neq \mathbf{0}$ are necessary and sufficient for a rank three QPIGL-IDS; $\boldsymbol{\gamma} \neq \mathbf{0}$ and $\boldsymbol{\delta} = \mathbf{0}$ are necessary and sufficient for a rank two quasi-homothetic PIGL-IDS,

$$\boldsymbol{w} = m^{-\kappa}\boldsymbol{P}^{\lambda}\left[\boldsymbol{\alpha} + \boldsymbol{B}\boldsymbol{x} + \boldsymbol{\gamma}(y - \alpha_0 - \boldsymbol{\alpha}'\boldsymbol{x} - \tfrac{1}{2}\boldsymbol{x}'\boldsymbol{B}\boldsymbol{x})\right]; \tag{22}$$

$\boldsymbol{\gamma} = \mathbf{0}$ and $\boldsymbol{\delta} \neq \mathbf{0}$ are necessary and sufficient for a rank two QPIGL-IDS that excludes the linear terms,

$$\boldsymbol{w} = m^{-\kappa}\boldsymbol{P}^{\lambda}\left[\boldsymbol{\alpha} + \boldsymbol{B}\boldsymbol{x} + \boldsymbol{\delta}(y - \alpha_0 - \boldsymbol{\alpha}'\boldsymbol{x} - \tfrac{1}{2}\boldsymbol{x}'\boldsymbol{B}\boldsymbol{x})^2\right]; \tag{23}$$

and $\boldsymbol{\gamma} = \boldsymbol{\delta} = \mathbf{0}$ is necessary and sufficient for a homothetic PIGL-IDS,

$$\boldsymbol{w} = m^{-\kappa}\boldsymbol{P}^{\lambda}(\boldsymbol{\alpha} + \boldsymbol{B}\boldsymbol{x}), \tag{24}$$

with common income elasticity equal to $1 - \kappa \;\forall\; \kappa \in \mathbb{R}$.[5] In each case, if $\kappa = \lambda = 0$ we obtain a generalization of the AIDS specification, while if $\kappa = \lambda = 1$ we obtain a generalization of a linear or quadratic expenditure system. We thus are able to directly nest both rank and functional form in this QPIGL-IDS class of generalized AIDS models.

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[5] For an incomplete demand system, homotheticity is defined by equality of the income elasticities of demand for the goods of interest. It is not necessary that the other demands have this income elasticity. In particular, the common income elasticity of demand for a homothetic subset of goods is not necessarily equal to one and may not be constant (LaFrance and Hanemann 1989). This is one among many ways that incomplete demand systems are more flexible and far richer than complete systems.

A QPIGL-IDS Extension of Quadratic Utility

In this subsection, we apply our nesting procedures to produce a rank three generalization of the quadratic direct and translog indirect utility models. We first define the functions

$$\varphi(\boldsymbol{x}) = \boldsymbol{x}'\boldsymbol{B}\boldsymbol{x} + 2\boldsymbol{\gamma}'\boldsymbol{x} + 1\,, \tag{25}$$

$$\theta(\boldsymbol{x}, \tilde{\boldsymbol{p}}, \boldsymbol{s}) = \alpha_0(\tilde{\boldsymbol{p}}, \boldsymbol{s}) + \boldsymbol{\alpha}(\tilde{\boldsymbol{p}}, \boldsymbol{s})'\boldsymbol{x}\,. \tag{26}$$

The starting point is the class of indirect utility functions defined by

$$v(\boldsymbol{x}, y, \tilde{\boldsymbol{p}}, \boldsymbol{s}) = \upsilon\left\{ -\left[\frac{\sqrt{\varphi(\boldsymbol{x})}}{[y - \theta(\boldsymbol{x}, \tilde{\boldsymbol{p}}, \boldsymbol{s})]} + \frac{\boldsymbol{\delta}'\boldsymbol{x}}{\sqrt{\varphi(\boldsymbol{x})}} \right]; \tilde{\boldsymbol{p}}, \boldsymbol{s} \right\}. \tag{27}$$

Again applying Roy's identity gives the demand equations for the goods $\boldsymbol{q}$ in budget share form as

$$\boldsymbol{w} = m^{-\kappa}\boldsymbol{P}^{\lambda}\left\{ \boldsymbol{\alpha} + \left[1 - \boldsymbol{\delta}'\boldsymbol{x}\left(\frac{y-\theta}{\varphi} \right) \right]\left(\frac{y-\theta}{\varphi} \right)(\boldsymbol{B}\boldsymbol{x} + \boldsymbol{\gamma}) + \frac{(y-\theta)^2}{\varphi}\boldsymbol{\delta} \right\}. \tag{28}$$

Members of the class of incomplete demand systems generated by this indirect utility function include rank two "translog indirect" and quadratic direct (quasi-)utility functions and rank three extensions that are quadratic in log-income and income, respectively. If $\kappa = \lambda = 0$, we obtain a rank three extension of a translog indirect utility model, while if $\kappa = \lambda = 1$, we obtain a rank three extension of the quadratic model. For all values of κ and λ, we again obtain a full rank three generalized quadratic QPIGL-IDS.[6] Rank two versions are obtained when $\boldsymbol{\delta} = \boldsymbol{0}$, while if $\theta(\boldsymbol{x}, \tilde{\boldsymbol{p}}, \boldsymbol{s}) \equiv 0$ and $\boldsymbol{\delta} = \boldsymbol{0}$, we obtain rank one homothetic versions, again nesting the rank and functional form of the income terms within a single unifying framework.[7]

A useful perspective of this incomplete system arises from noting that the demands for $\boldsymbol{q}$ satisfy the partial differential equations,

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[6] We use the terminology "generalized quadratic" to refer to the fact that the indirect utility function is defined in terms of deflated and transformed prices and income, $\boldsymbol{x}$ and y, respectively, rather than directly in terms of $\boldsymbol{p}$ and m.

[7] An advantage of both choices of preference functions is that the demands are conditionally linear in $\boldsymbol{\delta}$. This simplifies the interpretation, estimation and testing of the second-order income effects.

$$\frac{\partial y}{\partial \boldsymbol{x}} = \alpha + \left[1-\boldsymbol{\delta}'\boldsymbol{x}\left(\frac{y-\theta}{\varphi}\right)\right]\left(\frac{y-\theta}{\varphi}\right)(\boldsymbol{Bx}+\boldsymbol{\gamma}) + \frac{(y-\theta)^2}{\varphi}\boldsymbol{\delta}\,. \tag{29}$$

This functional form has the added advantage of permitting the determination of necessary and sufficient conditions for symmetry and sufficient conditions for concavity of y in $\boldsymbol{x}$, hence of e in $\boldsymbol{p}$, entirely from (29). Calculating the second-order partial derivatives and careful (and, of course, quite tedious) grouping, canceling, and algebraic manipulation of various terms gives

$$\begin{aligned}\frac{\partial^2 y}{\partial \boldsymbol{x}\partial \boldsymbol{x}'} &= \left[1-\boldsymbol{\delta}'\boldsymbol{x}\left(\frac{y-\theta}{\varphi}\right)\right]\left(\frac{y-\theta}{\varphi}\right)\left[\boldsymbol{B}-\left(\frac{1}{\varphi}\right)(\boldsymbol{Bx}+\boldsymbol{\gamma})(\boldsymbol{Bx}+\boldsymbol{\gamma})'\right] \\ &+ 2\frac{(y-\theta)^3}{\varphi^2}\left[\boldsymbol{I}-\left(\frac{1}{\varphi}\right)(\boldsymbol{Bx}+\boldsymbol{\gamma})\boldsymbol{x}'\right]\boldsymbol{\delta}\boldsymbol{\delta}'\left[\boldsymbol{I}-\left(\frac{1}{\varphi}\right)\boldsymbol{x}(\boldsymbol{Bx}+\boldsymbol{\gamma})'\right].\end{aligned} \tag{30}$$

From this expression, LBP show that symmetry of $\boldsymbol{B}$ is necessary and sufficient for Slutsky symmetry, and that $1-\boldsymbol{\delta}'\boldsymbol{x}(y-\theta)/\varphi>0$, $y-\theta<0$, $\varphi>0$, and $\boldsymbol{B}=\boldsymbol{LL}'+\boldsymbol{\gamma\gamma}'$ are both necessary and sufficient for $\partial^2 y/\partial\boldsymbol{x}\partial\boldsymbol{x}'$ to be symmetric and negative semidefinite. If y is convex in m and and $\boldsymbol{x}$ is concave in $\boldsymbol{p}$, these are in turn sufficient for *global weak integrability* of the demands for $\boldsymbol{q}$ throughout the open set

$$\Im \equiv \left\{(\boldsymbol{p},\tilde{\boldsymbol{p}},m,\boldsymbol{s})\in \mathbb{R}^{n_q}_{++}\times\mathbb{R}^{n_{\tilde q}}_{++}\times\mathbb{R}_{++}\times\mathbb{R}^{J} : \varphi>0, y-\theta<0, 1-\boldsymbol{\delta}'\boldsymbol{x}(y-\theta)/\varphi>0\right\}.$$

These curvature restrictions apply only to the parameters of the model and are straightforward to implement. We recently have experienced good success applying them to U.S. food consumption (Beatty and LaFrance, 2001; LaFrance and Beatty, 2004).

Conclusions

WE HAVE EXTENDED THE THEORY OF AGGREGATION in demand analysis to incomplete demand systems. In stark contrast to complete demand systems, there is no functional form restriction. But the maximal rank of the incomplete demand system is three. This result follows purely from Slutsky symmetry.

We also have used Box-Cox transformations of the prices of the goods of interest and a separate Box-Cox transformation on income to generate two large classes of nested functional forms. One makes it possible to test for the rank and functional form of generalized AIDS models. The other permits the same analysis to be applied to generalized translog indirect and quadratic direct utility functional forms. This latter class of models also permits the global imposition of parameter constraints implied by economic theory on the subset of demand equations under study.

In our empirical work, we have found both frameworks for nesting incomplete demand systems to be tractable as well as substantial improvements over the traditional alternatives (Beatty and LaFrance, 2001a, 2001b; LaFrance, Beatty, Pope and Agnew, 2000, 2002; LaFrance and Beatty, 2004). For both classes of functional forms, rank three often, but not always, appears to be essential. Estimates for the Box-Cox parameters on the price and income variables also tend to fall much closer to one than zero, although the restrictions $\kappa = \lambda = 1$ or 0, respectively, are rejected at all reasonable levels of significance for most of the data we have used to empirically investigate this question. To us, these empirical findings suggest that extending the logarithmic and linear functional forms and increasing the rank of our demand models to (at least) three are important advances in applied demand analysis. Hopefully, other applied researchers also will find these results and models to be useful and informative.

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