Formalization of Economic Value Theory

Microeconomic Foundations

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Formally: Standard Demand/Welfare theory

Preference: a binary relation "$x_1 \succsim \mathbf{x}_2$", read as "$x_1$ at least as good as $\mathbf{x}_2$"
With the following traditional assumptions:

- **Completeness**

given $\mathbf{x}_1 = (x_1^1, ..., x_n^1)$ and $\mathbf{x}_2 = (x_1^2, ..., x_n^2)$ we have either

1. $\mathbf{x}_1 \succsim \mathbf{x}_2 \Leftrightarrow \mathbf{x}_1$ is at least as good as $\mathbf{x}_2$.
2. $\mathbf{x}_2 \succsim \mathbf{x}_1 \Leftrightarrow \mathbf{x}_2$ at least as good as $\mathbf{x}_1$.
3. $\mathbf{x}_1 \sim \mathbf{x}_2 \Leftrightarrow \mathbf{x}_1$ is indifferent $\mathbf{x}_2$.

Standard Demand/Welfare theory

- **Transitivity:** if $\mathbf{x}_1 \succsim \mathbf{x}_2$ and $\mathbf{x}_2 \succsim \mathbf{x}_3$, then $\mathbf{x}_1 \succsim \mathbf{x}_3$.

Other assumptions:

- **Continuity**

We can represent preferences using a utility function $U(\mathbf{x})$.

- **No satiation:** $U'(\mathbf{x}) > 0$.
- **Convexity of preferences or quasiconcavity of** $U(x)$.

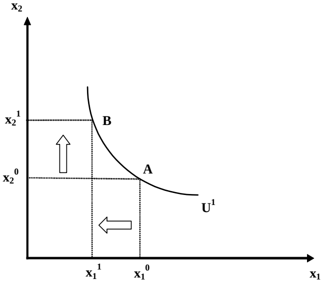
remember that for any monotonic function
$f()$ then $f[U(\mathbf{x}_1)] \geq f[U(\mathbf{x}_2)]$ if and only if $U(\mathbf{x}_1) \geq U(\mathbf{x}_2)$.

Substitutability

given
$\mathbf{x}_1 = (x_1^1, ..., x_i^1, x_j^1, ..., x_n^1)$ and $\mathbf{x}_2 = (x_1^1, ..., x_i^2, x_j^1, ..., x_n^1)$, with $x_i^2 < x_i^1$,
then there exist a bundle
$\mathbf{x}^* = (x_1^1, ..., x_i^2, x_j^*, ..., x_n^1)$ with
$x_j^* > x_j^1$ such that

$$U(\mathbf{x}_1) = U(\mathbf{x}^*)$$

this characteristic of preferences allows us to value different goods. In this case the individuals is willing to give up $(x_i^1 - x_i^2)$ units of x_i in order to gain $(x_j^* - x_j^1)$ units of x_j.
This can be applied to marketed goods or public goods.



Substitution

Consumer's problem

$$MAX \quad U(\mathbf{x})$$

$$\text{s.t. } m = \mathbf{px} = \sum_{i=1}^{n} p_i x_i.$$

Marshallian Demand Functions

$$\mathbf{x}(\mathbf{p}, m) = \begin{pmatrix} x_1 = x_1(\mathbf{p}, m) \\ \vdots \\ x_n = x_n(\mathbf{p}, m) \end{pmatrix}.$$

Properties of Ordinary demand Functions

1. Homogeneity $x_i = x_i(\theta \mathbf{p}, \theta m) = x_i(\mathbf{p}, m)$.
2. Adding up $\sum_{i=1}^{n} p_i x_i(\mathbf{p}, m) = m$.
3. Symmetry $x_j(\mathbf{p}, m)\frac{\partial x_i(\mathbf{p},m)}{\partial m} + \frac{\partial x_i(\mathbf{p},m)}{\partial p_j} = x_i(\mathbf{p}, m)\frac{\partial x_j(\mathbf{p},m)}{\partial m} + \frac{\partial x_j(\mathbf{p},m)}{\partial p_i}$ $\forall i \neq j$,
4. Negative Semi-definiteness of the matrix $\mathbf{S} = [\mathbf{S}_{ij}]$ including $S_{ii} = \frac{\partial h_i(\mathbf{p},U)}{\partial p_i} = x_i(\mathbf{p}, m)\frac{\partial x_i}{\partial m} + \frac{\partial x_i}{\partial p_i} \leq 0$.

Example

$$
\begin{aligned}
x_1 &= a_1 + b_{11} p_1 + b_{12} p_2 + b_{13} p_3 + c_1 m \\
x_2 &= a_2 + b_{21} p_1 + b_{22} p_2 + b_{23} p_3 + c_2 m \\
x_3 &= a_3 + b_{31} p_1 + b_{32} p_2 + b_{33} p_3 + c_3 m
\end{aligned}
$$

then

$$s_{11} = b_{11} + c_1 \left(a_1 + b_{11} p_1 + b_{12} p_2 + b_{13} p_3 + c_1 m \right) \leq 0$$

and

$$s_{12} = s_{21}, \; s_{23} = s_{32} \text{ and } s_{13} = s_{31},$$

Integrability

Given the equations satisfying conditions (1) to (4), does there exist a quasiconcave $U(x)$ that could generate them?
Yes. Hurwicz and Uzawa, 1971.

$$MIN\ m = \sum_{i=1}^{n} p_i x_i = \mathbf{px}$$
$$\text{s.t.: } U(\mathbf{x}) \geq U^0.$$

Hicksian or compensated demand functions.

$$\mathbf{h}(\mathbf{p}, U) = \begin{pmatrix} h_1(\mathbf{p}, U) \\ \vdots \\ h_n(\mathbf{p}, U) \end{pmatrix},$$

Expenditure Function

$$\sum_{i=1}^{n} p_i h_i(\mathbf{p}, U) = e(\mathbf{p}, U)$$

Indirect utility Function

$$U(\mathbf{x}(\mathbf{p}, m)) = v$$

Inverse relationships

$$\begin{aligned} u &= v(p, e(\mathbf{p}, U)) \\ m &= e(\mathbf{p}, U(\mathbf{p}, m)) \end{aligned}$$

hence

$$\begin{aligned} x_i &= x_i(\mathbf{p}, m) = h_i(\mathbf{p}, v(p, m)) \\ h_i &= h_i(\mathbf{p}, U) = h_i(\mathbf{p}, e(\mathbf{p}, U)) \end{aligned}$$

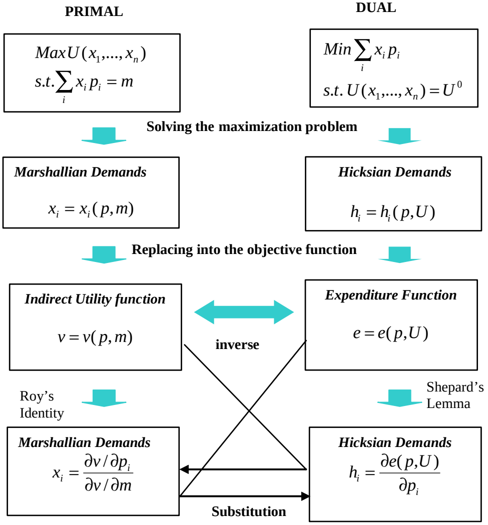
- Roy's Identity

$$x_i(p, m) = -\frac{\partial v/\partial p_i}{\partial v/\partial m}$$

- Shepard's Lemma

$$\frac{\partial e(\mathbf{p}, U)}{\partial p_i} = h_i(\mathbf{p}, U).$$

Summary:


PRIMAL
DUAL
$Max\, U(x_1,...,x_n)$
$s.t. \sum_i x_i p_i = m$
$Min \sum_i x_i p_i$
$s.t.\, U(x_1,...,x_n) = U^0$
Solving the maximization problem
Marshallian Demands
$x_i = x_i(p,m)$
Hicksian Demands
$h_i = h_i(p,U)$
Replacing into the objective function
Indirect Utility function
$v = v(p,m)$
inverse
Expenditure Function
$e = e(p,U)$
Roy's Identity
Shepard's Lemma
Marshallian Demands
$x_i = \frac{\partial v / \partial p_i}{\partial v / \partial m}$
Hicksian Demands
$h_i = \frac{\partial e(p,U)}{\partial p_i}$
Substitution

Hurwicz & Uzawa's integrability results

$$\frac{\partial e}{\partial p_i} = h_i(\mathbf{p}, U) = h_i(\mathbf{p}, v(\mathbf{p}, m)) = x_i(\mathbf{p}, m). \quad i = 1, ..., N$$

Partial Differential Equations with unknown $e(p, U)$ and Boundary $m = e(p, C)$ for some constant C.
Solving this equations we obtain $e(p, U)$ and inverting this results we have the indirect utility function $v(p, e(\mathbf{p}, U))$.
Exploit Homogeneity to find the direct utility function

$$v(\frac{p}{m}, 1) = \bar{v}(\frac{p}{m}) = \bar{v}(\pi)$$

then

$$U = \min_{\pi} \bar{v}(\pi) \; s.t. \; \pi x \leq 1$$

Welfare Measures

1. **Compensating Variation**: Amount of money that is necessary to make the individual indifferent between the original situation and the new price (quality) set. Alternatively, for a price reduction maximum amount of money that the individual is **willing to pay** for the opportunity to consume at the new price set. And for a price increase, Minimum amount of money that the individual is **willing to accept** in order to make the person indifferent to the price change.
2. **Equivalent variation**: minimum amount of money that the individual would have to receive (**WTA**) to induce that person to voluntary forgo the opportunity to purchase at the new price level (lower prices). And for a price increase, it is the maximum amount of money that the individual is **willing to pay** to avoid the change in prices.
3. **Compensating surplus**: same as CV but with quantities.
4. **Equivalent surplus**: same as EV but with quantities
5. **Consumer surplus**

Compensating Variation (CV)

$$CV = \int_{p^0}^{p^1} h_i(\mathbf{p}, U^0)\partial p_i = \int_{p^0}^{p^1} \frac{\partial e(\mathbf{p}, U^0)}{\partial p_i}\partial p_i = e\left(p^1, U^0\right) - e\left(p^0, U^0\right)$$

$$v(p^1, m - CV) = v(p^0, m) = U^0,$$

for a change in price and income

$$\begin{aligned} CV &= e\left(p^1, U^1\right) - e\left(p^0, U^0\right) + \left(m^0 - e\left(p^1, U^0\right)\right) \\ CV &= m^1 - m^0 + \left(m^0 - e\left(p^1, U^0\right)\right) \end{aligned}$$

several prices

$$CV = \int_{p^0}^{p^1} \sum h_i(\mathbf{p}, U^0)\partial p_i$$

Equivalent Variation

$$EV = \int_{p^0}^{p^1} h_i(\mathbf{p}, U^1)\partial p_i = \int_{p^0}^{p^1} \frac{\partial e(\mathbf{p}, U^1)}{\partial p_i}\partial p_i = e\left(p^1, U^1\right) - e\left(p^0, U^1\right).$$

$$v(p^0, m + EV) = v(p^1, m) = U^1$$

Relationship between Welfare Measure and WTP or WTA

Utility change	Comp. Variation	Equiv. Variation
$u' > u^0$	WTP	WTA
$u' < u^0$	WTA	WTP

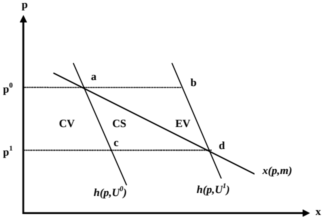
- Compensating variation: implied property right in the initial situation.
- Equivalent variation: implied property right in the change.

Consumer Surplus

$$CS = \int_{p^0}^{p^*} x_i(p, m)\partial p_i,$$

$$\Delta CS = \int_{p^0}^{p^1} -\frac{\partial v/\partial p_i}{\partial v/\partial m}\partial p_i = -\frac{1}{\partial v/\partial m}\int_{p^0}^{p^1} \frac{\partial v}{\partial p_i}\partial p_i = \frac{v(p^0, m) - v(p^1, m)}{\partial v/\partial m}$$

Willig's Approximations



if $\left|\frac{\eta^u CS}{2m^0}\right| \leq 0.05, \quad \left|\frac{\eta^l CS}{2m^0}\right| \leq 0.05 \quad y \quad \left|\frac{CS}{2m^0}\right| < 0.9$

$$\frac{\eta^l\,|CS|}{2m^0} \leq \frac{CV - CS}{|CS|} \leq \frac{\eta^u\,|CS|}{2m^0},$$

$$\frac{\eta^l\,|CS|}{2m^0} \leq \frac{CS - EV}{|CS|} \leq \frac{\eta^u\,|CS|}{2m^0},$$

Critics to Willig's approximation

1. For large income elasticities the bounds are wide
2. Poor approximation for deadweight loss
3. Does not work well for multiple price changes
4. Can do better: obtain exact results

Three ways to formulate demand Function

1. Formulate $U(x)$
2. Start with $v(p, e(\mathbf{p}, U))$ and use Roy's Identity.
3. Start with $x_i(\mathbf{p}, m)$ and use Hurwicz-Uzawa.

Example

$$x = \alpha + \beta p + \lambda m$$

$\beta < 0$, $\lambda > 0$ and $x_1^0 = x_i(p_1^0, \mathbf{p}, m) \leq -\frac{\beta}{\lambda}$.

$$\frac{\partial e}{\partial p} = \alpha + \beta p + \lambda e(p_1, \mathbf{p}, U),$$

$$e(p, U) = Ue^{\lambda p} - \frac{1}{\lambda}\left(\alpha + \beta p + \frac{\beta}{\lambda}\right)$$

with $e(p, U) = e^{-\int -\lambda \partial p}\left[A + \int (\alpha + \beta p) e^{-\int \lambda \partial p} \partial p\right]$

Marshallian Demand	$\mathbf{x} = \boldsymbol{\alpha} + \boldsymbol{\beta}\mathbf{p} + \boldsymbol{\lambda}\mathbf{m}$
Consumer surplus	$-\frac{x_1^2 - x_0^2}{2\beta}$
Expenditure	$Ue^{\lambda p} - \frac{1}{\lambda}(\alpha + \beta p + \frac{\beta}{\lambda})$
Indirect Utility	$e^{-\lambda p}\left[m + \frac{1}{\lambda}(\alpha + \beta p + \frac{\beta}{\lambda})\right]$
Hicksian Demand	$\lambda e^{\lambda p} U - \frac{\beta}{\lambda}$
Utility Function	$e^{\lambda(\frac{\alpha + \lambda x_2 - x_1}{\beta + \lambda x_1})}(\frac{\lambda x_1 + \beta}{\lambda^2})$
Compensating Variation	$e^{\lambda(p_1^1 - p_1^0)}(\frac{x_1^0}{\lambda} + \frac{\beta}{\lambda^2}) - (\frac{x_1^1}{\lambda} + \frac{\beta}{\lambda^2}).$
Equivalent Variation	$(\frac{x_1^0}{\lambda} + \frac{\beta}{\lambda^2}) - e^{\lambda(p_1^0 - p_1^1)}(\frac{x_1^1}{\lambda} + \frac{\beta}{\lambda^2})$

Other Approach

- Start with the definition of a utility function
- Jorgensen, Lau and Stoker (1982)

$$\ln v(p,m) = \alpha' \ln \mathbf{p} + \frac{1}{2} \ln \mathbf{p}' \mathbf{B}_{pp} \ln \mathbf{p} - \mathbf{D}(\mathbf{p}) \ln \mathbf{M}_k + \ln \mathbf{p}' \mathbf{B}_{pA} \mathbf{A}_k$$

$$\mathbf{D}(\mathbf{p}) = -\mathbf{1} + \iota \mathbf{B}'_{pp} \ln \mathbf{p}$$

with share equations

$$w_k = \frac{1}{\mathbf{D}(\mathbf{p})} \left(\alpha + \mathbf{B} \ln \mathbf{p} - \iota \mathbf{B}_{\mathbf{pp}} \ln \mathbf{M}_k + \mathbf{B}_{pA} \mathbf{A}_k \right)$$

Expenditure Function

$$\ln \mathbf{M}_k = \frac{1}{\mathbf{D}(\mathbf{p})} \left(\alpha' \ln \mathbf{p} + \frac{1}{2} \ln \mathbf{p}' \mathbf{B}_{pp} \ln \mathbf{p} - \ln \mathbf{V} + \ln \mathbf{p}' \mathbf{B}_{pA} \mathbf{A}_k \right)$$

Vartia (1983) algorithm

$$e(t_k) - e(t_{k-1}) = \sum_{i=1}^{n} \int_{t_{k-1}}^{t_k} h^i(\mathbf{p}(t), e(t)) dp_i(t).$$

$$\begin{aligned} e(t_k) - e(t_{k-1}) &= \sum_{i=1}^{n} \frac{1}{2} \left[h^i(\mathbf{p}(t_k), e(t_k)) + h^i(\mathbf{p}(t_{k-1}), e(t_{k-1})) \right] \\ &\quad * \left[\mathbf{p}_i(t_k) - \mathbf{p}_i(t_{k-1}) \right]. \end{aligned}$$

He suggests several algorithms to solve this problem.
Hausman and Newey (1995) Combine this with a non-parametric estimation of a demand function.

Several Goods

- System of Differential equations
- LaFrance & Hanemann (1989): Use a incomplete demand system

$$x = \alpha + \beta p_x + \gamma p_y + \lambda m,$$

$$e(p_x, p_y, m) = U \exp(\lambda p_x) - \frac{1}{\lambda}(\alpha + \beta p_x + \gamma p_y + \frac{\beta}{\lambda}),$$

- From LaFrance and Hanemann (1989):
  - "..it is generally impossible to measure unequivocally welfare changes from non-market effects using incomplete demand systems of market demand functions..."
  - New suggestions : Specifying prices as quality-adjusted repacking functions (Willig, 1978), Von Haefen and Phaneuf (2003) shows it is possible to link the non-market good to the private good.
- Hanemann & Morey, (1992): Partial demand system
- Aggregation of consumption into a composite good

Composite good

given p_1, p_2 and p_3 where p_2 and p_3 move in the same proportion θ with respect to a baseline period p_2^0 and p_3^0 such that $p_2 = \theta p_2^0$ and $p_3 = \theta p_3^0$. Then the ratio p_2/p_3 remains constant at p_2^0/p_3^0. Then θ represents a new "price" for the group of goods or composite good with quantity $q_2 p_2^0 + q_3 p_3^0$. The expenditure function $e(p_1, p_2, p_3, U)$ will be $e(p_1, \theta p_2^0, \theta p_3^0, U)$ or

$$e^*(p_1, \theta, U) = e(p_1, \theta p_2^0, \theta p_3^0, U).$$

and

$$\frac{\partial e^*}{\partial \theta} = \frac{\partial e}{\partial p_2}\frac{\partial p_2}{\partial \theta} + \frac{\partial e}{\partial p_3}\frac{\partial p_3}{\partial \theta},$$

$$\frac{\partial e^*}{\partial \theta} = q_2 p_2^0 + q_3 p_3^0.$$

which is the Shepard's lemma.

Separability

$$U = U(U_g(x, y), s, z)$$

Subutility $U_g(x, y)$. Three groups: $(x, y), s, z$. Decision tree in two steps. First budget to each group and then allocation within a group.

$$MAX \quad U \ [U_g(x, y), s, z]$$
$$\text{st.} m = p_x x + p_y y + p_s s + z$$

FOC

$$\frac{\frac{\partial U}{\partial U_g}\frac{\partial U_g}{\partial x}}{\frac{\partial U}{\partial U_g}\frac{\partial U_g}{\partial y}} = \frac{p_x}{p_y}$$

Which does not depend on other goods. $m_g = m - p_s s - z = p_x x + p_y y$. And the demand functions are

The Maler-Lancaster model

- The theoretical formalization of what was involved in non-market valuation was due to Maler (1971, 1973), building on the more limited model of Lancaster (1966).
- Maler's utility function is $u = u(x, q)$ where the individual gets satisfaction from q but does not control the level of q, and does not pay for q, only for the x's

The Maler-Lancaster model

$$U = U(x, q)$$

q can be:

- Exogenous attributes of the individual (eg. age, health)
- Exogenous attributes of commodities (q_{ik} is the amount of the kth attribute associated with one unit of consumption of the ith good)
- Exogenous availability of public good (road system, air quality, water quality)

The Maler-Lancaster model

- The notion is that there is a set of characteristics or attributes associated with each commodity. Suppose there are K relevant characteristics (attributes), and let q_{ik} denote the amount or level of the k-th characteristic associated with one unit of consumption of commodity i.
- The characteristics of each commodity are taken as given by the consumer who is free to vary only the quantity of the commodity; quality variation is accomplished through quantity variation.
- If there are N separate differentiated commodities the utility function takes the form $y = f(x_1, x_2, ..., x_n, q_1, q_2, ..., q_n, z)$; demands now depend on the attributes as well as the prices.

The Maler-Lancaster model

- Offers an explicit account of why the $x's$ are viewed as separate commodities, based on their specific characteristics.
- Provides a framework for analyzing the effect of characteristics on demand – a model of the demand for attributes (i.e., for q).
- Provides the formalism for extending the Hicksian welfare measures from changes in p to changes in q – formalizes notion of WTP and WTA for a change in q.

Welfare measures for changes in environmental quality

- NonMarket valuation is generally interested in value changes in welfare due to changes in environmental quality
- The utility function can be defined as

$$U = U(\mathbf{x}, \mathbf{q}),$$

$\mathbf{q}$ is a vector of environmental amenities. This Utility function has associated an indirect utility function and an expenditure function given by $v(p, q, m)$ and $e(p, q, U)$. As in the price case we can use these functions to calculate welfare measures for changes in q. For example if we move from q^0 to q^1, with $q^1 > q^0$, then

$$CV = e(p, q^0, U^0) - e(p, q^1, U^0) = \int_{q^0}^{q^1} \frac{\partial e}{\partial q}(p, q, U^o)\partial q,$$

$$EV = e(p, q^0, U^1) - e(p, q^1, U^1) = \int_{q^0}^{q^1} \frac{\partial e}{\partial q}(p, q, U^1)\partial q.$$

or

$$v(p, q^0, m) = v(p, q^1, m - VC) = U^0,$$

$$v(p, q^0, m + VE) = v(p, q^1, m) = U^1.$$

$$\begin{aligned} EV &= q^0 q^1 ba \\ CS &= q^0 q^1 bc \\ CV &= q^0 q^1 dc \end{aligned}$$

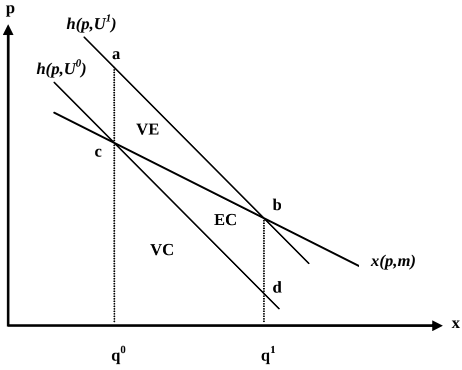


Figure:

Differences between WTP and WTA: Quantity Space

Randall & Stoll (1980) find similar results to those suggested by Willig

$$\frac{\xi^l |CS^*|}{2m^*} \leq \frac{CS^* - WTP}{|CS^*|} \leq \frac{\xi^u |CS^*|}{2m^*},$$
$$\frac{\xi^l |CS^*|}{2m^*} \leq \frac{WTA - CS^*}{|CS^*|} \leq \frac{\xi^u |CS^*|}{2m^*}.$$

Therefore

$$WTA - WTP \approx \frac{\xi^l |CS^*|^2}{m^*}.$$

ξ is the price flexibility of price with respect to changes in income $\left(\frac{\partial p/\partial m}{p/m}\right)$.

CS^* represents the consumer surplus as the area below the demand curve, between the two quantities.

m^* is the quantity of the good used as a numeraire.

For perfectly divisible goods and no transaction cost $CV = CS = EV = p(q^1 - q^0)$.

for a decrease in q $CV > CS > EV$

Hanemann (1991) extends this results suggesting that $\xi = \frac{\eta}{\sigma}$, where η is the price elasticity and σ is the elasticity of substitution.

If $\eta = 0$ or $\sigma = \infty$ then the welfare measures are identical but if $\sigma = 0$ they differ significantly

Integrability and change in quality

LaFrance & Hanemann (1989) show it is not possible to calculate Hicksian welfare measures for a change in q using an ordinary demand function.
The constant of integration will have both Utility and environmental quality. $C = C(U, q)$.
We need additional information or assumptions in order to solve this problem.
Larson (1991) suggested to use the week complementarity assumption suggested by Maler (1974)

Integrability and change in quality

Hicks suggestion

$$\frac{\partial h_i}{\partial q}(p, q, U) > 0.$$

For a normal good this implies $\frac{\partial x}{\partial q} > 0$. For example

$$x = \alpha + \beta p + \lambda m + \gamma q, \Rightarrow \frac{\partial x}{\partial q} = \gamma > 0.$$

Maler (1974) The marginal willingness to pay for quality is zero if the demand for the private good (complementary to q) is zero. That is:

$$\frac{\partial e}{\partial q}(\hat{p},\ q,\ U^0) = 0,$$

Where $\hat{p}$ is the choke price.

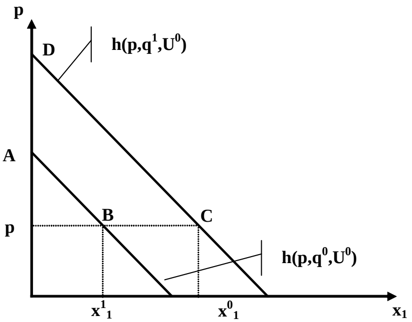


Figure:

The figure shows the welfare change associated with a higher level of quality as the area *ABCD*
PBA is the initial surplus while *PCD* is the final surplus.

Using the expenditure function we have

$$CV = e(p, q^0, U^0) - e(p, q^1, U^0).$$

But this could be transformed a little bit using the following definitions

$$CV(q^0) = e(\hat{p}, q^0, U^0) - e(p^0, q^0, U^0),$$
$$CV(q^1) = e(\hat{p}, q^1, U^0) - e(p^0, q^1, U^0).$$

with $\hat{p}$ the choke price.

$$VC(q^1) - VC(q^0) = e(\hat{p}, q^1, U^0) - e(p^0, q^1, U^0) - \left[e(\hat{p}, q^0, U^0) - e(p^0, q^0, U^0) \right],$$

$$VC(q^1) - VC(q^0) = e(p^0, q^0, U^0) - e(p^0, q^1, U^0).$$

$e(\hat{p}, q^1, U^o)$ and $e(\hat{p}, q^0, U^o)$ are equal to zero since $\left(\frac{\partial e}{\partial q}(\hat{p}, q, m) = 0 \right)$ by the Weak complementary assumption.
Therefore

$$CV(q) = \int_{p_i}^{\hat{p}_i^1} h_i(p_i, q^1, U) \partial p - \int_{p_i}^{\hat{p}_i^0} h_i(p_i, q^0, U) \partial p, \qquad (1)$$

We already know this condition

$$\frac{\partial e(p,q,U)}{\partial p} = x_i(p,q,e(p,q,U)),$$

given a demand function $x_i(p,q,m)$ we could integrate back to obtain a a quasi-expenditure function $\tilde{e}(p,q,\theta(q,U))$. which satisfies

$$e(p,q,U) = \tilde{e}(p,q,\theta(q,U))$$

where $\tilde{e}$ is known until a constant of integration θ.
$\theta(.)$ is not equal to the utility level as before because it depends on q.
but using $\frac{\partial e(\hat{p}_i(0,q,U),q,U)}{\partial q} = 0$. We could recover the expenditure function.

Steps for recovering the expenditure function

1. $e(p, q, U) = \tilde{e}(p, q, \theta(q, U))$,
2. $\dfrac{\partial e(\hat{p}_i(0, q, U), q, U)}{\partial q} = 0$
3. $\dfrac{\partial \tilde{e}(\hat{p}_i, q, \theta(q, U))}{\partial q} + \dfrac{\partial \tilde{e}(\hat{p}_i, q, \theta(q, U))}{\partial \theta} \dfrac{\partial \theta(q, U)}{\partial q} = 0.$

From 1 we have $\theta(q, U) = \tilde{\theta}(q, \phi(U))$, with $\phi(U)$ a constant of integration.

Example:

$x_i = \alpha + \beta p + \lambda m + \gamma q$
Hausman (1981): $\tilde{e}(p, q, \theta(p, U)) = \theta(q, U)e^{\lambda p} - \frac{1}{\lambda}(\alpha + \beta p + \gamma q + \frac{\beta}{\lambda})$.
With $0 \leq x \leq -\beta/\lambda$.

- find the choke price

$$\frac{\partial \tilde{e}(p, q, \theta(q, U))}{\partial p} = \lambda\theta(q, U)e^{\lambda p} - \frac{\beta}{\lambda} = 0,$$

$$\theta(q, U)e^{\lambda p} = \frac{\beta}{\lambda^2},$$

$$\hat{p} = \frac{1}{\lambda}\ln(\frac{\beta}{\lambda^2\theta(q, U)}).$$

- replacing in the expenditure function

$$\tilde{e}\left(p, q, \theta(p, U)\right) = \frac{\beta}{\lambda^2} - \frac{1}{\lambda}(\alpha + \frac{\beta}{\lambda}\ln(\frac{\beta}{\lambda^2\theta(q, U)}) + \gamma q + \frac{\beta}{\lambda}).$$

Using weak complementarity

$$\frac{\partial e(\hat{p}_i(0, q, U), q, U)}{\partial q} = -\frac{\beta}{\lambda^2}\left\{\frac{\lambda^2\theta(.)}{\beta}\left(\frac{-\beta\lambda^2(\partial\theta(.)/\partial q)}{\left[\lambda^2\theta(.)\right]^2}\right)\right\} - \frac{\gamma}{\lambda} =,$$

$$\frac{1}{\theta(q, U)}\partial\theta(q, U) = \frac{\gamma\lambda}{\beta}\partial q.$$

whose solution is

$$\ln \theta(q, U) = \frac{\gamma\lambda}{\beta} q + A, \Rightarrow \theta(q, U) = \phi e^{(\gamma\lambda/\beta) q},$$

where A is the new constant of integration and $\phi = e^{A}$ is a constant depending on U but not p nor q. Finally
The expenditure function

$$e(p, q, U) = \phi(U) e^{\frac{\lambda}{\beta}(\gamma q + \beta p)} - \frac{1}{\lambda}(\alpha + \beta p + \gamma q + \frac{\beta}{\lambda}),$$

$$v(p, q, m) = [m + (\frac{1}{\lambda})(\alpha + \beta p + \gamma q + \frac{\beta}{\lambda})] e^{-(\frac{\lambda}{\beta})(\gamma q + \beta p)}.$$