A foundation for the solution of consumption-saving behavior with a borrowing constraint and unbounded marginal utility

Eugenio S.A. Bobenrieth H.,

Departamento de Economía, Universidad de Concepción, Concepción, Chile.

Juan R.A. Bobenrieth H.,

Departamento de Matemática, Universidad del Bío-Bío, Concepción, Chile.

and

Brian D. Wright,*

Department of Agricultural and Resource Economics, University of California, Berkeley.

Abstract

Models of precautionary saving or storage include cases where the marginal value of accumulated balances is unbounded, with an invariant distribution with infinite mean. Based on a uniform continuity argument, we show that a model of saving with bounded marginal value can be used to approximate the unbounded marginal value function, and the quantiles of its invariant distribution, arbitrarily accurately. These results offer a foundation for a strategy for numerical solution of marginal values in cases where they are unbounded, and for derivation of the quantiles of their invariant distributions.

KEYWORDS: Saving, invariant distribution, convergence.

JEL Classification Numbers: C63, E21, E22, Q11.

* Corresponding Author. 207 Giannini Hall, University of California, Berkeley, CA 94720-3310; wright@are.berkeley.edu, phone (510)642-9213, fax (510)643-3075.

A foundation for the solution of consumption-saving behavior with a borrowing constraint and unbounded marginal utility

Eugenio S.A. Bobenrieth H., Juan R.A. Bobenrieth H., and
Brian D. Wright

1. Introduction

The individual and aggregate effects of income shocks in the presence of borrowing constraints are issues in the study of economic implications of precautionary saving for the macroeconomy. (See for example Deaton 1991, Krusell and Smith 1998, Lucas 2003). If unemployment induces an atom at zero in the distribution of income realizations, and the marginal utility is infinite at zero consumption, then expected marginal value of saving in the invariant distribution is infinite. Standard strategies[1] for numerical solution of the marginal value function do not apply.

We prove ergodicity and the Strong Law of Large Numbers in a model of saving that extends the model of Deaton (1991) to cases in which marginal value is unbounded. Based on a uniform continuity argument, we show that a model of saving with bounded marginal value can approximate the unbounded marginal value function and the quantiles of its invariant distribution (but not its moments[2]), arbitrarily accurately, apart from any numerical error.[3] Our results offer a foundation of a strategy for numerical solution of marginal values in cases where they are unbounded, and for numerical derivation of the distributional implications of precautionary savings in cases of this type.

[1]See for example Gustafson (1958), Wright and Williams (1984), Marcet (1988), Taylor and Uhlig (1990), Judd (1998), Christiano and Fisher (2000), Miranda and Fackler (2002).

[2]Santos and Peralta-Alva (2005) obtain results on approximation of moments under additional restrictions that include boundedness of the corresponding dynamical system.

[3]Our focus is on the convergence of the marginal value function and of the quantiles of its invariant distribution in a sequence of models, rather than on the numerical approximation of the marginal value or the quantiles of its invariant distribution, for a given model. Santos and Vigo-Aguiar (1998) and Santos (2000) address numerical approximation with respect to a given model under strong assumptions regarding concavity and interiority of the solution.

2. The model

We address a standard model of intertemporal utility maximization by individuals with income in the form of a single storable consumption commodity. Time is discrete.

The utility function of the representative individual $U : \mathbb{R}_+ \rightarrow \mathbb{R}_+$, is continuous, once continuously differentiable, strictly increasing and strictly concave, and satisfies $U(0) = 0$, $U'(0) = \infty$, and $\lim_{c\rightarrow\infty} U(c) < \infty$. Define $f \equiv U'$. Note that $U(c) = \int_0^c f(v)\, dv$.

Individual income is the exogenous i.i.d. sequence of random variables $\{\omega_t\}_{t\in\mathbb{N}}$ with finite mean μ. The support of ω_t is either bounded, or unbounded above. Specifically, ω_t has a mixed discrete-continuous distribution with an atom at 0, and its support is either $K \equiv [0, \bar{\mu}]$, with $0 < \bar{\mu} < \infty$, or $K \equiv [0, \infty)$. The distribution of ω_t is of the form $\alpha L_d + (1 - \alpha)L_c$, where $\alpha \in (0, 1)$, L_d is a discrete distribution that has an atom at 0, and L_c is an absolutely continuous distribution, with continuous and strictly positive derivative m on the interior of K. We denote by $(S, \mathcal{F}, P)$ the underlying probability space for the random variables $\{\omega_t\}_{t\in\mathbb{N}}$.

The individual has rational expectations and has a constant one period discount factor δ, $0 < \delta < 1$. The total amount saved is $x \geq 0$. Accumulated balances depreciate at rate d, $0 \leq d < 1$. Given savings x, total resources available to the individual in the next period are $z' \equiv (1 - d)x + \omega'$, where ω' is next period's income.

Given an infinite horizon, the Bellman equation for the surplus problem is:

$$\nu(z) = \max_x \{U(z - x) + \delta E[\nu(z')]\}, \tag{1}$$

subject to

$$z' = (1 - d)x + \omega',$$

$$x \geq 0, \ z - x \geq 0,$$

where $E[.]$ denotes the expectation with respect to next period's income ω'.

Standard results imply that ν is continuous and strictly increasing, and that the optimal policy function $x(z)$ is single valued, strictly increasing, and continuous. By Bobenrieth, Bobenrieth and Wright (2006), ν is strictly concave. Consumption and marginal value are given by the functions $c(z) \equiv z - x(z)$, $p(z) \equiv f(c(z))$.

The policy function satisfies the Euler condition:

$$p(z) \geq \delta(1-d)E[p((1-d)x(z)+\omega')], \quad \text{with equality if } x(z) > 0 \tag{2}$$

Given initial available resources $z > 0$, condition (2) implies that $z' > 0$ and $x(z') > 0$, and this arbitrage condition holds with equality in the current period and for the indefinite future. Note that $p(0) = f(0) = \infty$.

Define available resources at time t as z_t. Since $x(z_t) \geq 0$ and $\omega_{t+1} \geq 0$, a suitable state space for available resources is $Z \equiv [0, \infty)$. Let $\mathcal{B}$ be the $\sigma-$field of Borel subsets of Z. For the case where the support of ω_t is bounded, let $\bar{z}$ be the unique fixed point of $g_{\bar{\mu}}(z) \equiv (1-d)x(z) + \bar{\mu}$.

If L_d has mass points of sizes $\{\alpha_k\}_{k\in A}$ with support $\{a_k\}_{k\in A}\,(A \subseteq \mathbb{N})$, the transition probability of available resources is given by:

$$Q = \alpha \sum_{k\in A} \alpha_k Q_k + (1-\alpha) Q_c,$$

where:

$$Q_k(z, B) \equiv \begin{cases} 1, & \text{if} \quad (1-d)x(z) + a_k \in B \\ 0, & \text{if} \quad (1-d)x(z) + a_k \notin B \end{cases} \qquad \text{for each } k \in A, \quad \text{and}$$

$$Q_c(z, B) \equiv \int_B m(z' - (1-d)x(z))\, dz', \qquad z \in Z,\ B \in \mathcal{B}.$$

For $t \in \mathbb{N}$, we define $Q^t(z, B) \equiv \int_B Q^{t-1}(y, B)\, Q(z, dy)$.

Let Φ be the Markov process of available resources, $\mathcal{P}(Z)$ the set of (Borel) probability measures on Z, $||\cdot||$ the total variation norm on $\mathcal{P}(Z)$, and given any initial probability measure $\gamma_0 \in \mathcal{P}(Z)$, let

$$\gamma_0 Q^t(B) \ \equiv \ \int_Z Q^t(z, B)\gamma_0(dz), \qquad B \in \mathcal{B},$$

so that $\gamma_0 Q^t$ denotes the distribution of z_t when $z_0 \sim \gamma_0$.

3. Results

We first establish ergodic properties of Φ, using definitions and results of Meyn and Tweedie (1993) and Stachurski (2006).

Theorem.

(i) Φ *is aperiodic and positive Harris recurrent. As a consequence, it is ergodic. Precisely,* Q *has a unique invariant probability measure* γ_*, *and for any initial condition* $\gamma_0 \in \mathcal{P}(Z)$,

$$||\gamma_0 Q^t - \gamma_*|| \to 0 \qquad as \quad t \to \infty.$$

(ii) Φ *satisfies the Strong Law of Large Numbers. That is, if* h *is a Borel measurable real valued function on* Z *and* $\int_Z |h(z)|\gamma_*(dz) < \infty$, *then*

$$\frac{1}{N}\sum_{t=0}^{N-1} h(z_t) \;\to\; \int_Z h(z)\gamma_*(dz) \qquad as \quad N \to \infty \qquad P - a.s.,$$

where $\{z_t\}_{t\geq 0}$ *is the process starting at* $z_0 \sim \gamma_0$.

Proof of the Theorem. Appendix A.

Let $\pi_* \equiv \gamma_* c^{-1} f^{-1}$ be the invariant probability measure for marginal value, and F_* the corresponding invariant distribution. Define $p_t \equiv p(z_t)$.

Given a sequence of realizations $\{p_t\}_{t\geq 0}$, consider the empirical distributions:

$$F_N(p, \{p_t\}_{t\geq 0}) \equiv \frac{\#\{0 \leq t \leq N-1 \;:\; p_t \in [0,p]\}}{N}, \quad p \geq 0,\; N \in \mathbb{N}.$$

Given $q,\; 0 \leq q \leq 1$, define the empirical qth quantile corresponding to F_N :

$$y_{q,N}(\{p_t\}_{t\geq 0}) \equiv \min\{p \geq 0 \;:\; F_N(p, \{p_t\}_{t\geq 0}) \geq q\},$$

and for the invariant distribution F_*, define the qth quantile:

$$y_{q,*} \equiv \min\{p \in [0,\infty] \;:\; F_*(p) \geq q\}.$$

Consider a family of models parametrized by $\varepsilon \geq 0$, the minimum of the support of income. The income sequence $\{\omega_t^{(\varepsilon)}\}_{t\in\mathbb{N}}$ is defined by $\omega_t^{(\varepsilon)} \equiv \omega_t + \varepsilon$, with the common discount factor δ and the common utility function U.

For the discussion that follows, it is convenient to use notation that recognizes the dependence of the relevant elements upon given ε. Let $x^{(\varepsilon)} = x^{(\varepsilon)}(z)$ denote the optimal policy function in the corresponding Bellman equation (1), and $c^{(\varepsilon)}(z) \equiv z - x^{(\varepsilon)}(z)$. For any given $s \in S$, let $\{p_t^{(\varepsilon)}(s)\}_{t\geq 0}$ be the marginal value sequence induced by s. Furthermore, define $y_{q,N}^{(\varepsilon)}(s) \equiv y_{q,N}(\{p_t^{(\varepsilon)}(s)\}_{t\geq 0})$.

For each $\varepsilon \geq 0$, there is a unique stationary probability measure $\gamma_*^{(\varepsilon)}$, which is a global attractor, and the corresponding Markov process of available resources $\Phi^{(\varepsilon)}$ satisfies the strong law of large numbers. $\pi_*^{(\varepsilon)} \equiv \gamma_*^{(\varepsilon)}(c^{(\varepsilon)})^{-1}f^{-1}$ denotes the invariant probability measure for marginal value, $F_*^{(\varepsilon)}$ the corresponding distribution, and $y_{q,*}^{(\varepsilon)}$ the qth quantile.

Standard results imply that the correspondence $(z, \varepsilon) \mapsto x^{(\varepsilon)}(z)$ is continuous. Therefore, given any sequence of positive numbers $\{\varepsilon_n\}_{n\in\mathbb{N}}$ that converges to zero, $\{x^{(\varepsilon_n)}(z)\}_{n\in\mathbb{N}}$ converges uniformly to $x^{(0)}(z)$, for z in any compact subset of $\mathbb{R}$.

Define the inverse marginal value function $z^{(\varepsilon)} : (0, \infty) \to \mathbb{R}_+,\ z^{(\varepsilon)}(p) \equiv (c^{(\varepsilon)})^{-1}(f^{-1}(p))$. From the continuity of $x^{(\varepsilon)}(z)$ in (z, ε), for any sequence $\{\varepsilon_n\}_{n\in\mathbb{N}} \downarrow 0$, the corresponding sequence $\{z^{(\varepsilon_n)}(p)\}_{n\in\mathbb{N}}$ converges uniformly to $z^{(0)}(p)$ in $[\underline{p}, \infty)$, for any $\underline{p} > 0$. For $\varepsilon > 0$, the Euler condition (2) allows for the possibility that savings are completely depleted, and $\sup\{z : x^{(\varepsilon)}(z) = 0\} > \varepsilon$.

Using the continuity of $(z, \varepsilon) \mapsto x^{(\varepsilon)}(z)$ we conclude that for any given sequence $\{\varepsilon_n\}_{n\in\mathbb{N}} \downarrow 0$, the corresponding sequence $\{p^{(\varepsilon_n)}(z)\}_{n\in\mathbb{N}}$ converges uniformly to $p^{(0)}(z)$ in $[\underline{z}, \infty)$, for any given $\underline{z} > 0$. Further, we have the following Proposition regarding the convergence of the empirical quantiles of the marginal value distribution to the quantiles of the corresponding invariant distribution with $\varepsilon = 0$.

Proposition 1. *Assume that the support of ω_t is bounded. Given any $\beta > 0$ and any $q_0, q_1,\ 0 < q_0 < q_1 < 1$, there exists $\varepsilon = \varepsilon(\beta, q_1) > 0$ and a set $S_\varepsilon \subseteq S$, with $P(S_\varepsilon) = 1$ such that given any $s \in S_\varepsilon$ there exists $N_1 = N_1(\beta, \varepsilon, s, q_0) \in \mathbb{N}$ such that:*

$$N \geq N_1 \ \Rightarrow \ \sup_{q\in[q_0,q_1]} \left| y_{q,N}^{(\varepsilon)}(s) \ - \ y_{q,*}^{(0)} \right| \ < \ \beta.$$

The same result holds if the support of ω_t is unbounded, for $d > 0$.

Proof of Proposition 1. Appendix B.

Proposition 1 implies that it is possible to find $\varepsilon > 0$ such that a sufficiently large sample of realizations of marginal values yields an empirical distribution with quantiles that approximate those of the invariant distribution of the model with unbounded marginal value, with uniform degree of accuracy β.

Note that the strong law of large numbers for Markov chains and the facts that for a given $\varepsilon > 0$ the expectation of the invariant distribution, $E_{\pi_*^{(\varepsilon)}}(p)$, is finite, and that $E_{\pi_*^{(0)}}(p) = \infty$, imply that the sample average of $\{p_t^{(\varepsilon)}\}_{t\geq 0}$ does not converge on $E_{\pi_*^{(0)}}(p)$.

Proposition 2. *For any fixed $\varepsilon > 0$ there exists a set S_ε, $S_\varepsilon \subseteq S$, with $P(S_\varepsilon) = 1$, such that for any given $s \in S_\varepsilon$, and for sufficiently large sample size $N \in \mathbb{N}$, the difference:*

$$\frac{\sum_{t=0}^{N-1} p_t^{(0)}(s)}{N} - \frac{\sum_{t=0}^{N-1} p_t^{(\varepsilon)}(s)}{N} \qquad \textit{becomes arbitrarily large.}$$

Proof of Proposition 2. Appendix B.

The continuity results presented above furnish a foundation for a strategy for the numerical solution of the model with a mass point at 0 and $f(0) = \infty$. One can approximate $p^{(0)}(z), x^{(0)}(z)$, and $c^{(0)}(z)$ arbitrarily accurately on $[\underline{z}, \infty)$, $\underline{z} > 0$, for some $\varepsilon > 0$ as follows. One can select a standard numerical specification of the model with a discretized version of the income distribution, translated such that the first mass point is at $\varepsilon > 0$. To obtain a numerical solution for $p^{(0)}(z), x^{(0)}(z)$, $c^{(0)}(z)$, one can use the fact that the solution of this model with $\varepsilon > 0$ approximates the solution of its counterpart with $\varepsilon = 0$. The solution is arbitrarily exact, in the sense that the saving function $x^{(0)}(z)$ and the inverse marginal value function $z^{(0)}(p)$ can be approximated uniformly, with arbitrary degree of accuracy, apart from any numerical error associated with approximating the solution of the model for $\varepsilon > 0$.

By Proposition 1, simulation of a numerical model following the above strategy yields an empirical distribution that approximates the quantiles of the invariant distribution of $p^{(0)}(z)$ arbitrarily exactly.

By Proposition 2, for any fixed $\varepsilon > 0$, with probability one the difference $\dfrac{\sum_{t=0}^{N-1} p_t^{(0)}(s)}{N} - \dfrac{\sum_{t=0}^{N-1} p_t^{(\varepsilon)}(s)}{N}$ becomes arbitrarily large for large $N \in \mathbb{N}$. However, the expectation $E_{\pi_*^{(\varepsilon)}}(p)$ goes to $E_{\pi_*^{(0)}}(p)$, as ε goes to zero. In fact, $\lim_{\varepsilon \to 0} E_{\pi_*^{(\varepsilon)}}(p) = \infty = E_{\pi_*^{(0)}}(p)$. The function $\varepsilon \mapsto E_{\pi_*^{(\varepsilon)}}(p)$ is continuous at $\varepsilon = 0$ with respect to the chordal metric[4] in the codomain $[0, \infty]$.

EXAMPLE: We now present an illustrative numerical example, with bounded income support.[5] Marginal utility is given by $f = 1/\sqrt{c}$, with $\delta = 1/1.05$ and $d = 0$. The distribution of ω_t has one atom, at 0, of size 0.1. The continuous part of the distribution of ω_t is uniform, discretized by 90 equally spaced and equally weighted points, in $(0, 1]$. We consider 6 values of ε : 0.1, 0.01, 0.001, 0.0001, 0.00001, and 0.000001, with $\omega_t^{(\varepsilon)} \equiv \omega_t + \varepsilon$.

For ε in a neighborhood of zero, the marginal value of savings for $x > 0$ is a highly nonlinear function of z. To approximate the latter function (for a given $\varepsilon > 0$) accurately using numerical methods, we express the logarithm of the marginal value of savings as a Chebyshev polynomial function of the logarithm of $x + \varepsilon$, where the order of the polynomial is equal to the number of points at which the function is evaluated, minus one.

Basic statistics for our results are reported in Table 1. The threshold of the marginal value function for positive saving is given by $\underline{p}^{(\varepsilon)} \equiv \delta E p^{(\varepsilon)}[\omega^{(\varepsilon)}]$. We calculate a lower bound for $\underline{p}^{(\varepsilon)}$ as $\delta E f[\omega^{(\varepsilon)}]$. For each value of ε, we simulate the model for 100,000 time periods (starting at marginal value equal to the threshold $\underline{p}^{(\varepsilon)}$) obtaining marginal net returns on savings with averages all less than 0.12 percent (in absolute value).[6] Figures 1 and 2 report the marginal

[4]The real line $\mathbb{R}$ can be embedded in (S^1-(north pole)) by a stereographic projection, and the unit circle S^1 is the one-point compactification of $\mathbb{R}$. Using this projection, the chordal metric in the extended real line is given by $d(z_1, z_2) = \dfrac{2|z_1 - z_2|}{[(1+z_1^2)(1+z_2^2)]^{1/2}}$, $z_1, z_2 \in \mathbb{R}$, $d(z, \infty) = \dfrac{2}{(1+z^2)^{1/2}}$, $z \in \mathbb{R}$. See for example, Dugundji (1966, pp. 242-246), and Conway (1978, p.9).

[5]The boundedness of income implies that we can restrict our attention to a bounded space for available resources, without loss of generality.

[6]We calculate the averages considering only observations associated with positive savings.

value functions and the empirical distributions, respectively.[7] The cases corresponding to $\varepsilon \leq 0.001$ are indistinguishable in the figures.

4. Conclusion

In this paper we have provided the basis for a numerical strategy for approximation of the marginal value, and approximation of its invariant distribution, for a model of saving in which marginal value is unbounded. We have focused on the case in which discounted marginal value is a submartingale. We conjecture that the same approach can be implemented for stationary cases with bounded income support where the accumulated balance appreciates (for examples, see Schechtman and Escudero 1977, Deaton 1991), and that extensions of the approach presented in this paper might be applied to a broader class of models, using the methods described in Mirman, Morand and Reffett (2002).

Appendix A.

Proof of the Theorem.

(i) First we prove that Φ is $\varphi-$irreducible for some non-trivial measure φ. For the case in which the support of ω_t is $K = [0, \bar{\mu}]$, note that for each $z \in Z$ and for each $t \in \mathbb{N}$ the support of z_t conditional on $z_0 = z$ is the interval $[g_0^t(z), g_{\bar{\mu}}^t(z)]$, where $g_0(z) \equiv (1-d)x(z)$ and $g_{\bar{\mu}}(z) \equiv (1-d)x(z) + \bar{\mu}$. Let λ be the Lebesgue measure on $(Z, \mathcal{B})$. Since $g_0^t(z) \to 0$ and $g_{\bar{\mu}}^t(z) \to \bar{z}$ (as $t \to \infty$), we conclude that if $\lambda(B \cap [0, \bar{z}]) > 0,\ B \in \mathcal{B}$, then given $z \in Z$, there exists $n \in \mathbb{N}$ such that $\text{Prob}[z_n \in B | z_0 = z] > 0$. For the case where the support of ω_t is $K = [0, \infty)$, observe that for each $z \in Z$ and for each $t \in \mathbb{N}$ the support of z_t, conditional on $z_0 = z$, is $[g_0^t(z), \infty)$. It follows that given $\lambda(B) > 0,\ B \in \mathcal{B}$, and $z \in Z$, there exists $n \in \mathbb{N}$ such that $\text{Prob}[z_n \in B | z_0 = z] > 0$. In both cases, we use the assumption that the derivative m of the absolutely continuous part of the distribution of ω_t is strictly positive on the interior of K.

[7]To be able to show the marginal utility function in Figure 1, we truncate its domain from a strictly positive value. Given the differences in marginal values for the cases we consider, the domain in Figure 2 is adjusted in order to show together all of the cases.

Second we prove that Φ possesses a small set C. (See for example Meyn and Tweedie, 1993, p.106). Moreover, we prove that the drift condition

$$\int_Z V(z')Q(z,dz') - V(z) \leq -1 + b\mathbb{1}_C(z), \qquad \forall\ z \in Z, \tag{3}$$

holds for some $b < \infty$ and some nonnegative Borel measurable function V which is bounded on C. The proof for the case of unbounded support of ω_t is taken from Stachurski (2006). Consider the unique fixed point $\hat{z}$ of the function $g_{\mu+\zeta}(z) \equiv (1-d)x(z)+\mu+\zeta$, where μ is the first moment of ω_t, and ζ is a strictly positive number which, in the bounded case, is such that $\mu+\zeta < \bar{\mu}$. Let $C \equiv [0,\hat{z}]$, and take $V(z) \equiv z/\zeta$. If $z \notin C$ then $(1-d)x(z)+\mu+\zeta < z$, and therefore $\int_Z V(z')Q(z,dz') - V(z) < -1$. Since C is compact and $\int_Z V(z')Q(z,dz') - V(z)$ is continuous in z, there exists $b < \infty$ such that $\int_Z V(z')Q(z,dz') - V(z) \leq -1 + b$. This proves the drift condition (3). It remains to verify that $C = [0,\hat{z}]$ is $\varrho-$small for some nontrivial measure ϱ. Pick any $\xi > 0$ such that $(1-d)x(\hat{z}) + \xi < \hat{z}$. If we denote $\sigma(z_t|z_0 = z)$ as the support of z_t conditional on $z_0 = z$, by the argument for irreducibility there exists $n \in \mathbb{N}$ such that $\sigma(z_t|z_0 = z)$ strictly contains $I \equiv [(1-d)x(\hat{z}) + \xi, \hat{z}]$, $\forall\ t \geq n$, $\forall\ z \in C$, where n does not depend on z. For the case where the support of ω_t is bounded we use the fact that there exists $n \in \mathbb{N}$ such that $g^t_{\bar{\mu}}(0) > \hat{z}$, $\forall\ t \geq n$, and therefore $g^t_{\bar{\mu}}(z) > \hat{z}$, $\forall\ t \geq n$, $\forall\ z \in C$. For the case where the support of ω_t is unbounded, $n = 1$. Furthermore, if $\upsilon_z = \upsilon_z(z_n)$ is the derivative of the absolutely continuous part of z_n conditional on $z_0 = z$, then $\upsilon_z(z_n) > 0$, $\forall\ z_n \in I, \forall\ z \in C$. For the case where the support of ω_t is unbounded, $\upsilon_z(z_n) = \upsilon_z(z_1) = m(z_1 - (1-d)x(z))$. For the case where the support of ω_t is bounded:

$$\upsilon_z(z_n) = \int_Z \cdots \int_Z m(z_n - (1-d)x(z_{n-1})) \cdots m(z_1 - (1-d)x(z))\, dz_{n-1} \cdots dz_1.$$

Since $\upsilon_z(z_n) > 0$, $\forall\ (z,z_n) \in C \times I$ and $\upsilon_z(z_n)$ is continuous in $(z,z_n) \in C \times I$, there exists a real constant $\kappa_n > 0$ such that $\upsilon_z(z_n) \geq \kappa_n$, $\forall\ (z,z_n) \in C \times I$. Therefore, for any $z \in C$ and for any $B \in \mathcal{B}$,

$$\begin{aligned}\text{Prob}[z_n \in B|z_0 = z] &\geq (1-\alpha)^n \int_B \upsilon_z(z_n)\, dz_n \geq (1-\alpha)^n \int_{B\cap I} \upsilon_z(z_n)\, dz_n \\ &\geq (1-\alpha)^n \kappa_n \lambda(B \cap I).\end{aligned}$$

Let be ϱ_n the non-trivial measure in $(Z,\mathcal{B})$ defined by $\varrho_n(B) \equiv (1-\alpha)^n \kappa_n \lambda(B \cap I)$. Then C is ϱ_n-small. Note that C is also ϱ_t-small, for all $t \geq n$, where $\varrho_t = d_t \varrho_n$, for some $d_t > 0$.

Therefore Φ is aperiodic (see Meyn and Tweeedie, 1993, pp. 116-118). Furthermore, by Theorem 11.3.4 in Meyn and Tweeedie (1993, p. 265), Φ is positive Harris recurrent. As a consequence, by Theorem 13.3.3 in Meyn and Tweeedie (1993, p. 323), Φ is ergodic.

(ii) Since Φ is a positive Harris chain, by Theorem 17.1.7 in Meyn and Tweeedie (1993, p. 416), Φ satisfies the Strong Law of Large Numbers. *Q.E.D.*

Appendix B.

For the proof of Proposition 1, we have four preliminary results.

Lemma 1. *Given any* $\varepsilon \geq 0$, *and given any measurable set* $D \subseteq f(c^{(\varepsilon)}(Z))$,

$$\lim_{N\to\infty} \frac{\#\{0 \leq t \leq N-1 \ : \ p_t^{(\varepsilon)}(s) \in D\}}{N} \ = \ \pi_*^{(\varepsilon)}(D), \quad a.s.\ s \in S.$$

Proof. The result follows immediately from the Strong Law of Large Numbers (Theorem, part (ii)). *Q.E.D.*

Lemma 2. *Given any* $\varepsilon \geq 0$,

$$\lim_{N\to\infty} \left[\sup_{p\in[0,\infty]} \left| F_N(p, \{p_t^{(\varepsilon)}(s)\}_{t\geq 0}) \ - \ F_*^{(\varepsilon)}(p) \right| \right] \ = \ 0, \text{ a.s. } s \in S.$$

Proof. Since $\varepsilon \geq 0$ is fixed in this lemma, we will not use notation that recognizes the dependence on ε in this proof. By Lemma 1, for any $p \geq 0$, there exists a set $S(p) \subseteq S$ of $P-$measure one, such that $\lim_{N\to\infty} F_N(p, \{p_t(s)\}_{t\geq 0}) = F_*(p), \ \forall \ s \in S(p)$. Given any $s \in \bigcap_{p\in\mathbb{Q}^+} S(p)$, the sequence of distributions $p \mapsto F_N(p, \{p_t(s)\}_{\ t\geq 0})$ $(N \in \mathbb{N})$ converges pointwise to the distribution $p \mapsto F_*(p)$ for any rational number p. We conclude that $\{F_N(\cdot, \{p_t(s)\}_{t\geq 0})\}_{N\in\mathbb{N}}$ converges pointwise to F_* for each continuity point of F_*. If $\varepsilon = 0,\ F_*$ has no discontinuity points. If $\varepsilon > 0,\ F_*$ has a countable set of discontinuity points. Let $\mathcal{D}$ be the set of atoms of F_*. For each atom $d \in \mathcal{D}$, by Lemma 1 we conclude that:
$F_N(d^\pm, \{p_t(s)\}_{t\geq 0}) \to F_*(d^\pm)$ (as $N \to \infty$) for all s in a set $\hat{S}_d \subseteq S$ of $P-$measure one. Therefore, $\forall \ s \in \left(\bigcap_{p\in\mathbb{Q}^+} S(p)\right) \bigcap \left(\bigcap_{d\in\mathcal{D}} \hat{S}_d\right)$,

$$\lim_{N\to\infty}\left[\sup_{p\in[0,\infty]}|F_N(p,\{p_t(s)\}_{t\ge 0})\;-\;F_*(p)|\right]\;=\;0. \qquad Q.E.D.$$

Lemma 3. *For* $\varepsilon>0,$ *and for any* $q_0,\ 0<q_0<1,$

$$\lim_{N\to\infty}\left[\sup_{q\in[q_0,1]}\left|y^{(\varepsilon)}_{q,N}(s)\;-\;y^{(\varepsilon)}_{q,*}\right|\right]\;=\;0,\ \ a.s.\ s\in S,$$

and for $\varepsilon=0,$ *and for any* $q_0,q_1,\ 0<q_0<q_1<1,$

$$\lim_{N\to\infty}\left[\sup_{q\in[q_0,q_1]}\left|y^{(0)}_{q,N}(s)\;-\;y^{(0)}_{q,*}\right|\right]\;=\;0,\ \ a.s.\ s\in S.$$

Proof. Since $\varepsilon\ge 0$ is fixed in this lemma, we will not use notation that recognizes the dependence on ε in this proof. Let $q_0,q_1,\ 0<q_0<q_1<1$. Given any $s\in S$ the functions $\varphi_N:[q_0,1]\to\mathbb{R},\ \varphi_N(q)\equiv y_{q,N}(s)$ $(N\in\mathbb{N})$ are increasing, and the function $\varphi_*:[q_0,1]\to[y_{q_0,*},f(\varepsilon)],\ \varphi_*(q)\equiv y_{q,*}$, is continuous. To conclude that for $\varepsilon>0$,

$$\lim_{N\to\infty}\left[\sup_{q\in[q_0,1]}|\varphi_N(q)\;-\;\varphi_*(q)|\right]\;=\;0,$$

and for $\varepsilon=0$,

$$\lim_{N\to\infty}\left[\sup_{q\in[q_0,q_1]}|\varphi_N(q)\;-\;\varphi_*(q)|\right]=0,$$

it suffices to prove that the sequence $\{\varphi_N\}_{N\in\mathbb{N}}$ converges pointwise to φ_*.

We now prove pointwise convergence. If $q=1$, then $y_{1,*}=f(\varepsilon)$, and $y_{1,N}(s)=\max\{p_0(s),p_1(s),\cdots,p_{N-1}(s)\}$, $\forall N\in\mathbb{N}$. Since with probability one there exists a subsequence $\{p_{t_k}(s)\}_k$ such that $p_{t_k}(s)\uparrow f(\varepsilon)$, we conclude that $\lim_{N\to\infty}y_{1,N}(s)=y_{1,*}$, a.s. $s\in S$.

Let $q\in[q_0,1)$, and $\beta>0$. Since the support of F_* is an interval, $F_*(y_{q,*}+\beta)>q$. By Lemma 2, for any s in a set of $P-$measure one, $\lim_{N\to\infty}F_N(p,\{p_t(s)\}_{t\ge 0})\;=\;F_*(p),\ \forall\,p\ge 0.$

In particular, $\lim_{N\to\infty} F_N(y_{q,*}+\beta,\{p_t(s)\}_{t\geq 0}) = F_*(y_{q,*}+\beta)$. Therefore, $\exists\ N_1 \in \mathbb{N}$ such that:

$N \geq N_1 \ \Rightarrow\ F_N(y_{q,*}+\beta,\{p_t(s)\}_{t\geq 0}) > q$, implying that $y_{q,N}(s) \leq y_{q,*}+\beta$. Similarly, since $F_*(y_{q,*}-\beta) < q$ we conclude that $\exists\ N_2 \in \mathbb{N}$ such that:

$N \geq N_2 \ \Rightarrow\ y_{q,N}(s) > y_{q,*}-\beta$. Taking $N_3 \equiv \max\{N_1, N_2\}$, we have $N \geq N_3 \ \Rightarrow\ |y_{q,N}(s) - y_{q,*}| \leq \beta$. *Q.E.D.*

Lemma 4. *Assume that the support of ω_t is bounded. Let $\{\varepsilon_n\}_{n\in\mathbb{N}}$ be any sequence of positive numbers that converges to zero. The sequence of the corresponding invariant measures $\{\pi_*^{(\varepsilon_n)}\}_{n\in\mathbb{N}}$ converges weakly to $\pi_*^{(0)}$, as $n\to\infty$. The corresponding sequence of distributions $\{F_*^{(\varepsilon_n)}\}_{n\in\mathbb{N}}$ converges uniformly to $F_*^{(0)}$. As a consequence, for any $q_1 \in [0,1)$,*

$$\lim_{\varepsilon_n\to 0}\left[\sup_{q\in[0,q_1]}\left|y_{q,*}^{(\varepsilon_n)} - y_{q,*}^{(0)}\right|\right] = 0.$$

The same result holds for the case in which the support of ω_t is unbounded, for $d > 0$.

Proof. First consider the case where the support of ω_t is bounded. Take $0 < z_0 < \infty$ fixed (the same for all $\varepsilon \geq 0$). Then the function that yields the supremum of the support of $z_{t+1}^{(\varepsilon)}$ is $g_{\bar{\mu}+\varepsilon}(z_t) \equiv (1-d)x^{(\varepsilon)}(z_t) + \bar{\mu} + \varepsilon$. From the facts that given any $\varepsilon \geq 0$ there exists a unique fixed point $\overline{z}^{(\varepsilon)}$ of $g_{\bar{\mu}+\varepsilon}$, and that $g_{\bar{\mu}+\varepsilon}(z) < z$ for all $z > \overline{z}^{(\varepsilon)}$, we conclude that $z_t^{(\varepsilon)} \leq \max\{z_0, \overline{z}^{(\varepsilon)}\}$ for all $t \geq 0$. Then a suitable state space is $\tilde{Z}^{(\varepsilon)} \equiv [0, \max\{z_0, \overline{z}^{(\varepsilon)}\}]$. Standard results imply that the correspondence $(z,\varepsilon) \mapsto x^{(\varepsilon)}(z)$ is continuous. Therefore, for ε in a neighborhood of zero, we can define a common compact state space $\tilde{Z}$. Observe that if $h : \tilde{Z} \to \mathbb{R}$ is continuous, then $\int_{\tilde{Z}} h(z')\, Q^{(\varepsilon)}(z, dz') = \alpha \sum_{k\in A} \alpha_k h((1-d)x^{(\varepsilon)}(z) + a_k + \varepsilon) + (1-\alpha)\int_0^{\bar{\mu}} h((1-d)x^{(\varepsilon)}(z) + \varepsilon + \omega) m(\omega)\, d\omega$ depends continuously on (z,ε), where $Q^{(\varepsilon)}$ denotes the transition probability of available resources, for the model with $\varepsilon \geq 0$. Indeed, the series $\sum_{k\in A} \alpha_k h((1-d)x^{(\varepsilon)}(z) + a_k + \varepsilon)$ is uniformly convergent in (z,ε) since $\left|\sum_{k=m}^{n} \alpha_k h(x^{(\varepsilon)}(z) + a_k + \varepsilon)\right| \leq \sum_{k=m}^{n} \alpha_k \cdot \max\{|h(z)| : z \in \tilde{Z}\}$. Theorem 12.13 in Stokey, Lucas, with Prescott (1989, p.384) implies that $\{\gamma_*^{(\varepsilon_n)}\}_{n\in\mathbb{N}}$ converges weakly to $\gamma_*^{(0)}$, and hence $\{\pi_*^{(\varepsilon_n)}\}_{n\in\mathbb{N}}$ converges weakly to $\pi_*^{(0)}$, as $n\to\infty$. Since the corresponding distribution $F_*^{(0)}$ has no atoms, $\{F_*^{(\varepsilon_n)}\}_{n\in\mathbb{N}}$ converges uniformly to $F_*^{(0)}$. By an argument similar to the proof of Lemma 3, we conclude that

$$\lim_{\varepsilon_n \to 0} \left[\sup_{q \in [0, q_1]} \left| y_{q,*}^{(\varepsilon_n)} \; - \; y_{q,*}^{(0)} \right| \right] \; = \; 0.$$

The case where the support of ω_t is unbounded (i.e. $K = [0, \infty)$) and $d > 0$ is proved in Le Van and Stachurski (2006, pp. 11-13). *Q.E.D.*

Proof of Proposition 1. The proposition is a consequence of Lemma 3 and Lemma 4.

For the proof of Proposition 2, we have a preliminary result:

Lemma 5. *Given any $\varepsilon \geq 0$,*

$$\lim_{N \to \infty} \frac{\sum_{t=0}^{N-1} p_t^{(\varepsilon)}(s)}{N} \;\; = \;\; E_{\pi_*^{(\varepsilon)}}(p), \;\; a.s. \; s \in S.$$

Proof. Since $\varepsilon \geq 0$ is fixed in this lemma, we will not use notation that recognizes the dependence on ε in this proof. If $\varepsilon > 0$, for any $s \in S$, the marginal value sequence $\{p_t(s)\}_{t \geq 0}$ is uniformly bounded. Hence, by the Strong Law of Large Numbers (Theorem, part (ii)),

$$\lim_{N \to \infty} \frac{\sum_{t=0}^{N-1} p_t(s)}{N} \;\; = \;\; E_{\pi_*}(p), \quad \text{a.s. } s \in S.$$

If $\varepsilon = 0$, the sequence $\{p_t(s)\}_{t \geq 0}$ is not uniformly bounded. For each $n \in \mathbb{N}$ consider a continuous function $\phi_n : f(c(Z)) \to \mathbb{R}$ with $\phi_n(p) = p$ if $p \leq n$, and $\phi_n(p) = 0$ if $p \geq n + 1$. By the Strong Law of Large Numbers (Theorem, part (ii)),

$$\lim_{N \to \infty} \frac{\sum_{t=0}^{N-1} \phi_n(p_t(s))}{N} \;\; = \;\; \int_{f(c(Z))} \phi_n(p) \, \pi_*(dp), , \quad \text{a.s. } s \in S.$$

The argument of the proof of Corollary 2 in Bobenrieth, Bobenrieth and Wright (2002, p.1218) implies that:

$$\lim_{n \to \infty} \int_{f(c(Z))} \phi_n(p) \, \pi_*(dp) \;\; = \;\; \int_{f(c(Z))} p \, \pi_*(dp) \;\; = \infty.$$

For each $n \in \mathbb{N}$ and $N \geq 1$, $\frac{\sum_{t=0}^{N-1} p_t(s)}{N} \geq \frac{\sum_{t=0}^{N-1} \phi_n(p_t(s))}{N}$, and therefore

$$\lim_{N \to \infty} \frac{\sum_{t=0}^{N-1} p_t(s)}{N} = \infty = E_{\pi_*}(p), \text{ a.s. } s \in S. \qquad Q.E.D.$$

Proof of Proposition 2. The proposition is a consequence of Lemma 5, and the facts that $E_{\pi_*^{(0)}}(p) = \infty$ and $E_{\pi_*^{(\varepsilon)}}(p) < \infty$ for $\varepsilon > 0$. *Q.E.D.*

References

Bobenrieth, E.S.A., Bobenrieth, J.R.A., Wright, B.D., 2002. A commodity price process with a unique continuous invariant distribution having infinite mean. Econometrica 70, 1213-1219.

———— , 2006. Strict concavity of the value function for a family of dynamic accumulation models. Manuscript, Universidad de Concepción, Universidad del Bío-Bío, University of California at Berkeley.

Conway, J.B., 1978. Functions of One Complex Variable. Springer-Verlag.

Christiano, L.J., Fisher, J.D.M., 2000. Algorithms for solving dynamic models with occasionally binding constraints. Journal of Economic Dynamics & Control 24, 1179-1232.

Deaton, A., 1991. Saving and liquidity constraints. Econometrica 59, 1221-1248.

Dugundji, J., 1966. Topology. Allyn and Bacon.

Gustafson, R.L., 1958. Carryover levels for grains. USDA Technical Bulletin 1178, Washington, DC: United States Department of Agriculture.

Judd, K.L., 1998. Numerical Methods in Economics. The MIT Press.

Krusell, P., Smith, A. A., 1998. Income and wealth heterogeneity in the macroeconomy. The Journal of Political Economy 106, 867-896.

Le Van, C., Stachurski, J., 2006. Parametric continuity of stationary distributions. Economic Theory. In press.

Lucas Jr., R.E., 2003. Macroeconomic priorities. The American Economic Review 93(1), 1-14.

Marcet, 1988. Solution of nonlinear models by parametrizing expectations. Manuscript, GSIA, Carnegie Mellon University.

Meyn, S.P., Tweedie, R.L., 1993. Markov chains and stochastic stability. Springer-Verlag.

Miranda, M.J., Fackler, P.L., 2002. Applied Computational Economics and Finance. The MIT Press.

Mirman, L.J., Morand, O.F., Reffett, K., 2002. A qualitative approach to markovian equilibrium in infinite horizon economies with capital. Working Paper, University of Virginia, University of Connecticut, Arizona State University.

Santos, M.S., 2000. Accuracy of numerical solutions using the Euler equation residuals. Econometrica 68, 137-1402.

Santos, M.S., Peralta-Alva A., 2005. Accuracy of simulations for stochastic dynamic models. Econometrica 73, 1939-1976.

Santos, M.S., Vigo-Aguiar J., 1998. Analysis of a numerical dynamic programming algorithm applied to economic models. Econometrica 66, 409-426.

Schechtman, J., Escudero V.L.S., 1977. Some results on "An income fluctuation problem." Journal of Economic Theory 16, 151-166.

Stachurski, J., 2006. Stability of BBW model. Manuscript. University of Melbourne.

Stokey, N.L., Lucas Jr., R.E., with Prescott, E.C., 1989. Recursive Methods in Economic Dynamics. Cambridge, MA: Harvard University Press.

Taylor, J.B., Uhlig, H., 1990. Solving nonlinear stochastic growth models: a comparison of alternative solution methods. Journal of Business and Economic Statistics 8, 1-18.

Wright, B.D., Williams, J.C., 1984. The welfare effects of the introduction of storage. Quarterly Journal of Economics 99, 169-192.

Table 1. Numerical Example.

Minimum of the Support of Income, ε	Threshold Marginal Value, $\underline{p}^{(\varepsilon)}$	Lower Bound for $\underline{p}^{(\varepsilon)}$, $\delta E f[\omega^{(\varepsilon)}]$	Threshold Resources $\left(p^{(\varepsilon)}\right)^{-1}(\underline{p}^{(\varepsilon)})$	Average Marginal Net Return on Savings (%)
0.1	1.6552	1.5466	0.3650	0.0598
0.01	2.6446	2.4650	0.1430	0.0439
0.001	4.7877	4.5894	0.0436	0.0481
0.0001	11.3183	11.1099	0.0078	0.1111
0.00001	31.9502	31.7039	0.0010	0.1137
0.000001	97.1326	96.8252	0.0001	-0.0808

Figure 1. Marginal Utility and Marginal Value Functions

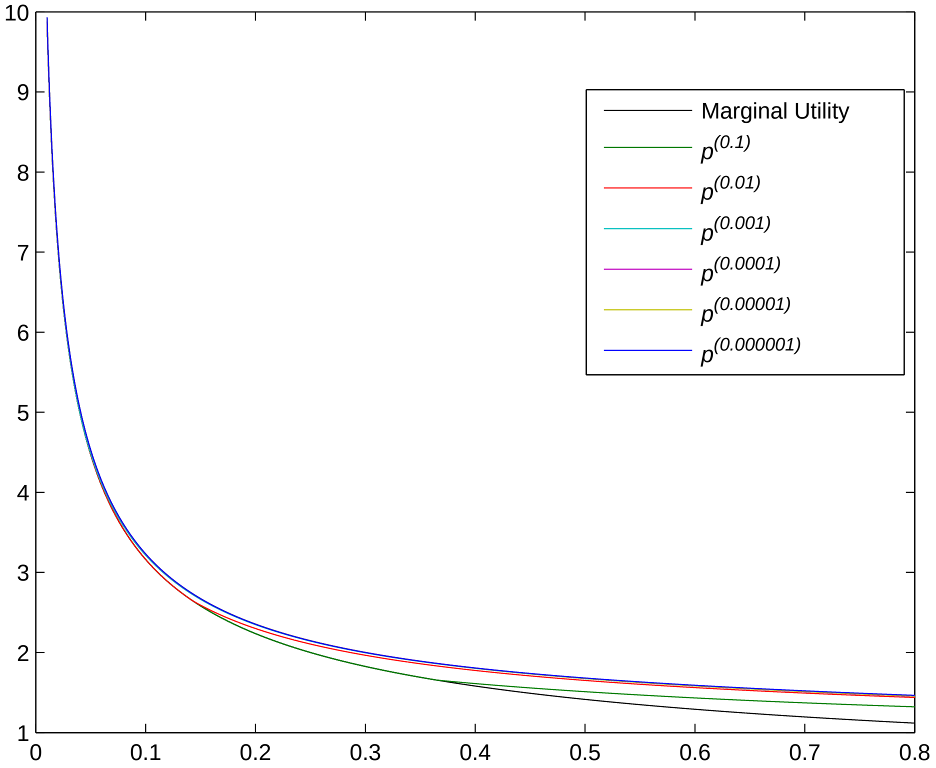

Figure 2. Empirical Marginal Value Distributions