Answer Key 8

6. $f: \mathbb{R} \to \mathbb{R}$ continuous, one-to-one and onto

$\Rightarrow$ ① f is strictly increasing: $x<y \Rightarrow f(x)<f(y)$

or ② f is strictly decreasing: $x<y \Rightarrow f(x)>f(y)$

let $a,b,c \in \mathbb{R}$ be s.t. $a<b<c$. we'll show $f(a)<f(b)<f(c)$ or $f(a)>f(b)>f(c)$:

1. Suppose $f(b) > \max\{f(a),f(c)\}$.

   claim: Then $\exists\, z \in (\max\{f(a),f(c)\}, f(b))$ and $\exists\, x_1 \in (a,b)$, $\exists\, x_2 \in (b,c)$ s.t. $z = f(x_1) = f(x_2)$.

   (this follows from the Intermediate Value Theorem:
   Since f is continuous, then $a<b$, $f(a)<f(b)$ implies $\forall y \in (f(a), f(b))$, $\exists\, x \in (a,b)$ s.t. $f(x)=y$.)

   but then we have $f(x_1)=f(x_2)$ and $x_1 \neq x_2$, which contradicts f one-to-one.

2. Suppose $f(b) = \max\{f(a),f(c)\}$. But then either $f(b)=f(a)$ or $f(b)=f(c)$, while $a \neq b \neq c$ so f one-to-one again contradicted.

3. Suppose $f(b) = \min\{f(a),f(c)\}$. Same reasoning as 2: contradiction.

4. Suppose $f(b) < \min\{f(a),f(c)\}$.

   Then, like in 1, $\exists\, z \in (f(b), \min\{f(a),f(c)\})$ and $\exists\, x_1 \in (a,b)$ and $\exists\, x_2 \in (b,c)$ s.t. $z = f(x_1) = f(x_2)$, violating one-to-one.

Therefore we've shown $\min\{f(a),f(c)\} < f(b) < \max\{f(a),f(c)\}$.

Since a,b,c were arbitrary, we have the following two possibilities:

1) $x<y<z \Rightarrow f(x)<f(y)<f(z) \Rightarrow f$ strictly increasing.

or, 2) $x<y<z \Rightarrow f(x)>f(y)>f(z) \Rightarrow f$ strictly decreasing.

Q.E.D.