Answer Key 10

2. $f(x,y) = (x^2 - x^3)\dfrac{\log(y)}{y}$ $\quad x \in \mathbb{R},\ y > 0.$

i) FOCs

1) $(2x - 3x^2)\dfrac{\log(y)}{y} = 0 \quad \Rightarrow x=0,\ x=\frac{2}{3} \text{ or } y=1$

2) $(x^2 - x^3)\dfrac{1-\log(y)}{y^2} = 0 \quad \Rightarrow x=0,\ x=1 \text{ or } y=e$

$\Rightarrow$ critical points: $\left\{ (0,y),\ y\in\mathbb{R},\ \left(\tfrac{2}{3}, e\right),\ (1,1) \right\}$

ii)
$$H_f(x,y) = \begin{bmatrix} (2-6x)\dfrac{\log(y)}{y} & (2x-3x^2)\dfrac{1-\log(y)}{y^2} \\ (2x-3x^2)\dfrac{1-\log(y)}{y^2} & (x^2-x^3)\dfrac{2\log(y)-3}{y^3} \end{bmatrix}$$

$$H_f(0,y) = \begin{bmatrix} 2\dfrac{\log(y)}{y} & 0 \\ 0 & 0 \end{bmatrix}$$

P.S.D. on $\{0\} \times [1,\infty)$

N.S.D. on $\{0\} \times (0,1]$

either way, can't tell.

$$H_f(1,1) = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

Indefinite $\Rightarrow$ saddle point

$$H_f(2/3, e) = \begin{bmatrix} -2/e & 0 \\ 0 & -\dfrac{4}{27e^3} \end{bmatrix}$$

N.D. $\Rightarrow$ local max

ii) function is unbounded on $\mathbb{R} \times (0,\infty)$:

$$\lim_{y\to 0} f(x,y) = +\infty \quad \text{when } x > 1$$
$$= -\infty \quad \text{when } x < 1$$

so there is no global max or min, and thus none are.

3. $f: \mathcal{U} \to \mathbb{R}$, $g: \mathcal{U} \to \mathbb{R}$ $\quad \mathcal{U} \subseteq \mathbb{R}^n$

f concave, C^2, g convex, C^2

let $h(x) = f(x) + \lambda[b - g(x)]$

Prove: h is concave. From Thm 17.8, h is concave on $\mathcal{U}$

<u>iff</u>: ① $h(y) - h(x) \leq Dh(x)(y-x) \quad \forall x, y \in \mathcal{U}$

<u>iff</u>: ② $D^2 h(x)$ is N.S.D. $\forall x \in \mathcal{U}$.

①

$$\begin{aligned}
h(y) - h(x) &= \left[f(y) + \lambda(b - g(y))\right] - \left[f(x) + \lambda(b - g(x))\right] \\
&= f(y) - f(x) - \lambda[g(y) - g(x)] \\
&\leq Df(x)(y-x) - \lambda[g(y) - g(x)] && \text{by } f \text{ concave.} \\
&\leq Df(x)(y-x) - \lambda Dg(x)(y-x) && \text{by } g \text{ convex} \\
&= [Df(x) - \lambda Dg(x)](y-x) \\
&= Dh(x)(y-x)
\end{aligned}$$

$\Rightarrow h(y) - h(x) \leq Dh(x)(y-x)$ $\forall x, y \in \mathcal{U}$

So h is concave.

② $D^2 h = D^2 f - \lambda D^2 g$

let v be nonzero vector in $\mathbb{R}^n$.

$$\begin{aligned}
\text{Then} \quad v^T D^2 h(x) v &= v^T [D^2 f(x) - \lambda D^2 g(x)] v \\
&= v^T D^2 f(x) v - \lambda v^T D^2 g(x) v && \text{by distributive law of matrix algebra and commutative law of scalar mult.} \\
&\leq 0 - \lambda (v^T D^2 g(x) v) && \text{by } f \text{ concave} \\
&\leq 0 - \lambda 0 && \text{by } g \text{ convex} \\
&= 0
\end{aligned}$$

So $v^T D^2 h(x) v \leq 0 \quad \forall v \in \mathbb{R}^n, v \neq 0, \forall x \in \mathcal{U}$

$\Rightarrow D^2 h(x)$ is N.S.D. on $\mathcal{U} \Rightarrow h$ concave.