Programming Model

x_{ij} acres of crop i planted with technology j
y_{ij} yield/acre
p_i price
a_{ijk} amount of input k per acre of crop i with technology j

$c_{ij} = \sum_k a_{ijk} p_k$ total (variable) cost of producing crop i

$\pi_{ij} = p_i y_{ij} - c_{ij}$ profit per acre with crop i technology

$$\sum_{i,j} x_{ij} \le L$$

$$\max_{\{x\}} \sum_{ij} \pi_{ij} x_{ij} \quad \text{st} \quad \sum_{ij} a_{ijk} x_{ij} \le \bar{L}_k \text{ availability of input k}$$

$$\left.\begin{array}{l} \sum_{j} x_{ij} \le \bar{x}_i \text{ constraint on production of crop } i \\ x_{ij} \ge 0 \end{array}\right\} \quad (1)$$

Let the solution be x_{ij}^*

Output $X_i^* = \sum_j x_{ij}^*$ (supply of crop i) Input demand $w^* = \sum a_{ijw} x_{ij}^*$

$$y_i^* = \sum_j x_{ij}^* y_{ij}$$

The maximization problem in (1) is what is known as a LINEAR PROGRAMMING problem

It is so called because it involves

(a) an objective function (i.e. what is maximized) that is LINEAR in the choice variables (the x_{ij})

(b) constraints that are LINEAR in the choice variable

Because of (b) the FEASIBLE REGION – i.e., the set of x_{ij} values that satisfy all the constraints – has a distinctive shape. It has linear surfaces, with corners (vertices).

Dantzig developed a computationally simple algorithm for solvi Linear Programming problems – the SIMPLEX METHOD – base on the fact that there is a finite number of vertices and a solution can always be found at one vertex.

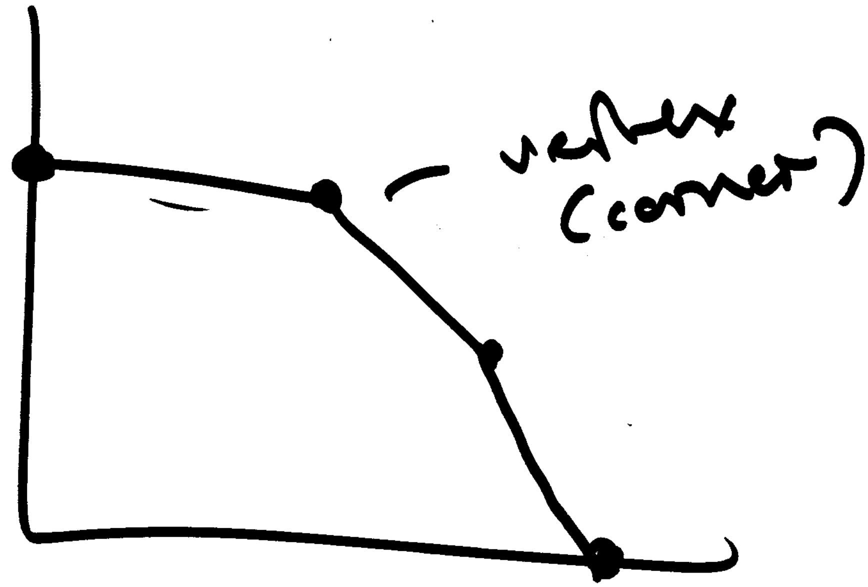
A LINEAR CONSTRAINT

$$a_{11}x_1 + a_{12}x_2 \le L_1$$

$$x_1 \ge 0, \quad x_2 \ge 0$$

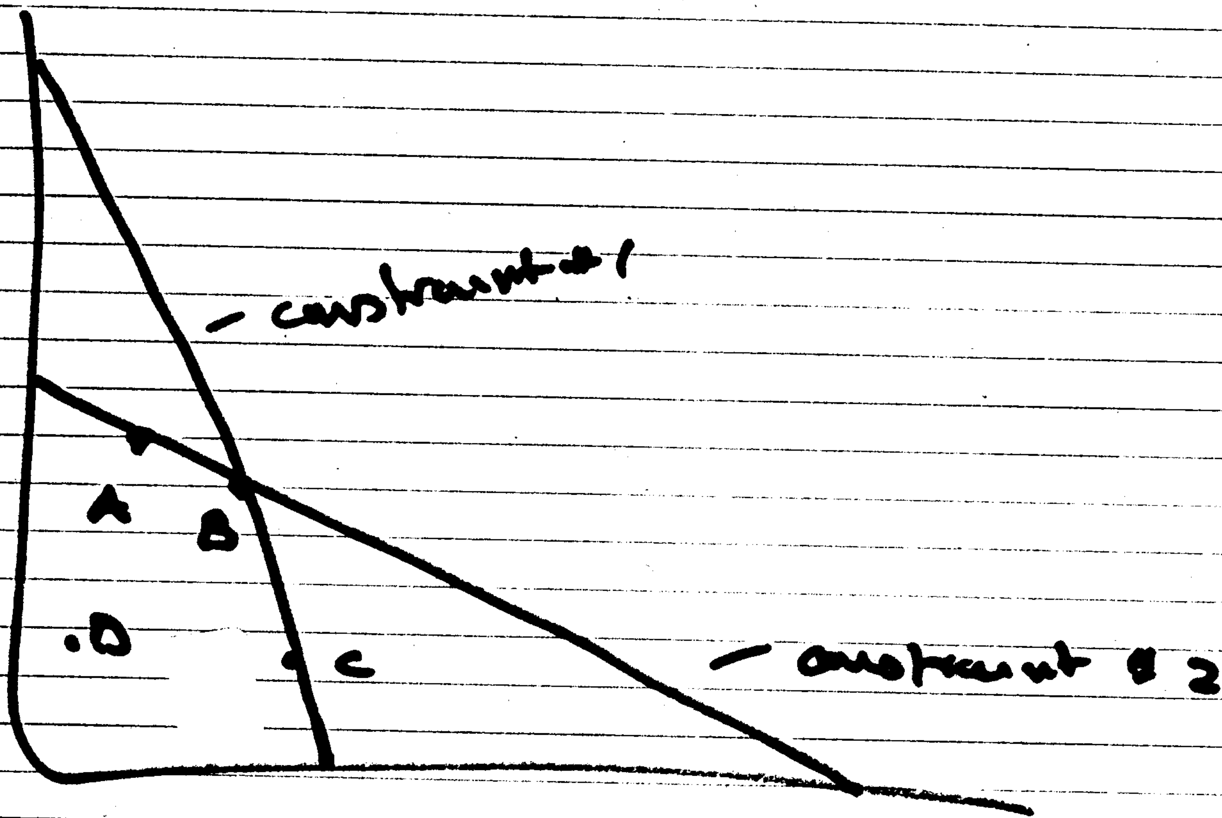
$$x_2 \le \frac{L_1}{a_{12}} - \frac{a_{11}}{a_{12}}x_1$$

with linear constraints feasible region is the intersection of a set of planes



as opposed to

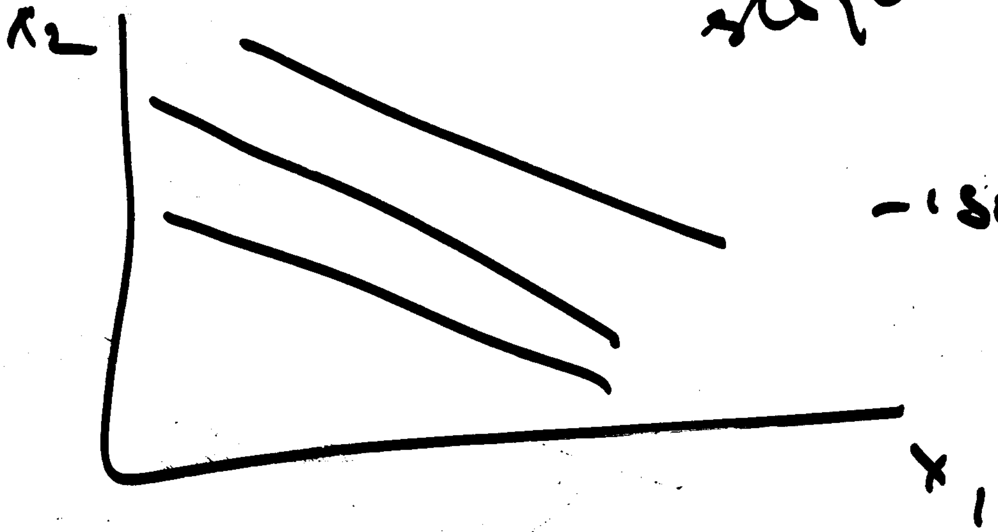
A CONSTRAINT MAY OR MAY NOT BE BINDING



At A, constraint 2 is binding, but not 1
At C, " 1 " 2

At B both constraints are binding

At D neither constraint is binding



slope = $-\frac{\pi_1}{\pi_2}$

– iso profit lines

$$\pi = \pi_1 x_1 + \pi_2 x_2$$

$$x_2 = \frac{\pi}{\pi_2} - \frac{\pi_1}{\pi_2} x_1$$

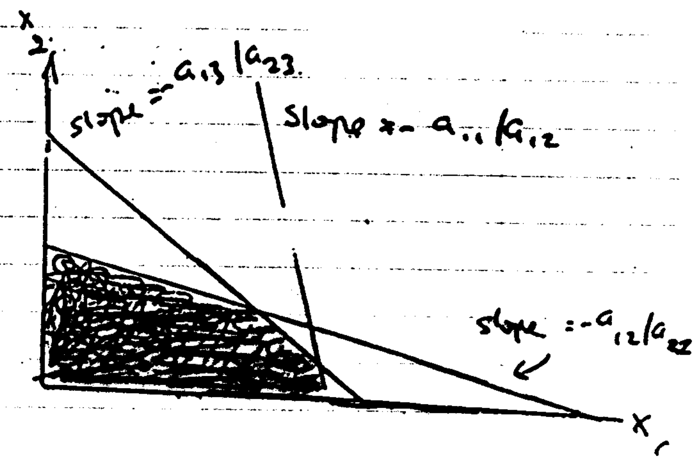
As an example, we consider a problem with 2 choice alternatives (x_1 and x_2) and 3 linear constraints

$$a_{11} x_1 + a_{12} x_2 \leq L_1$$

$$a_{21} x_1 + a_{22} x_2 \leq L_2$$

A constraint is said to be BINDING if it holds as an EQUALITY (=) and NON-BINDING if it holds as a strict INEQUALITY (<).

can have no more than NJ binding constraints if problem is to be feasible.



Solution Procedure

(1) Identify the feasible region, corresponding to the shaded area.

(2) **Superimpose iso profit lines with slope** $-\frac{\pi_1}{\pi_2}$

Identify the vertex on the highest iso-profit line (i.e., at which profit is maximized.

Given the solution, how does one obtain a demand function for water

There are two ways to do this.

Two ways to get a demand function for an input

(i) Vary p_k, recompute π_i's (if $a_{ik} \neq a_{jk}$; $\frac{\pi_i}{\pi_j}$ will change when raise p_k)

resolve LP for x_i^* — change in price, p_k, affects production cost plus + ∴ profit π_i

compute new demand function $w^* = \sum_i z_{wi} = \sum_i a_{iw} x_i^k$.

$$x_i = g^i(w,p) = \max_x pf(x) - \sum w p_i$$

(Note: we are focusing on unconditional input demand function here.)

(ii) Use Shadow Prices

The above implicitly assumes that if there is no constraint on $\bar{L}_k$, total use of input changes. But, if there is a constraint and it is binding, one could get no change in w_k as p_k changes. What to do in that case:

Shadow prices are associated with a constraint

$$\max pf(x) - \sum_i^N wp \quad \text{st } x_N = \bar{x}_N \text{ fixed}$$

If x_N is a fixed input, input demand functions look like $x_i = \bar{g}^i(w_1,\dots,w_{N-1},\bar{x}_N,p)$.

variable profit look like $\bar{\pi} = \sum \bar{g}^i(w_1,\dots,w_{N-1},\bar{x}_N,p)\cdot w_i$

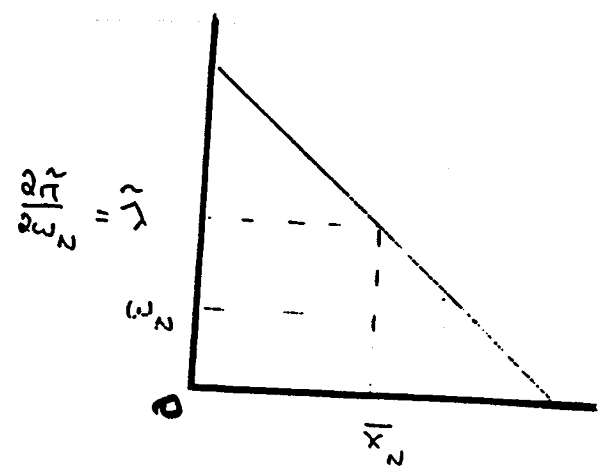
total profit $\equiv \tilde{\pi}(w_1,\dots,w_N,x_N,p) = \bar{\pi}(w_1,\dots,w_{N-1},\bar{x}_N,p) - w_N\bar{x}_N$.

GROSS $\bar{\lambda} \equiv \frac{\partial \bar{\pi}}{\partial x_N}(w_1,\dots,w_{N-1},\bar{x}_N,p)$ value (marginal profit) associated with relaxing the constraint $x_N = \bar{x}_N$

NET $\hat{\lambda} = \bar{\lambda} - w_N$ where $\bar{\lambda} = \frac{\partial \bar{\pi}}{\partial x_N}(w_1,\dots,w_{N-1},x_N,p)$

$\hat{\lambda}(w_1,\dots,w_{N-1},w_N,\bar{x}_N,p)$

$\bar{\lambda}(w_1,\dots,w_{N-1},\bar{x}_N,p)$ are both shadow prices



Then procedure is fix $\bar{x}_N$, compute $\bar{\lambda}$, vary $\bar{x}_N$, recompute $\bar{\lambda}$

Plot $\bar{\lambda}(w_1,\dots,w_{N-1},w_N,\bar{x}_N,p)$ as function of $\bar{x}_N$ —take this as the demand curve.

Can prove that these two demand functions are equivalent

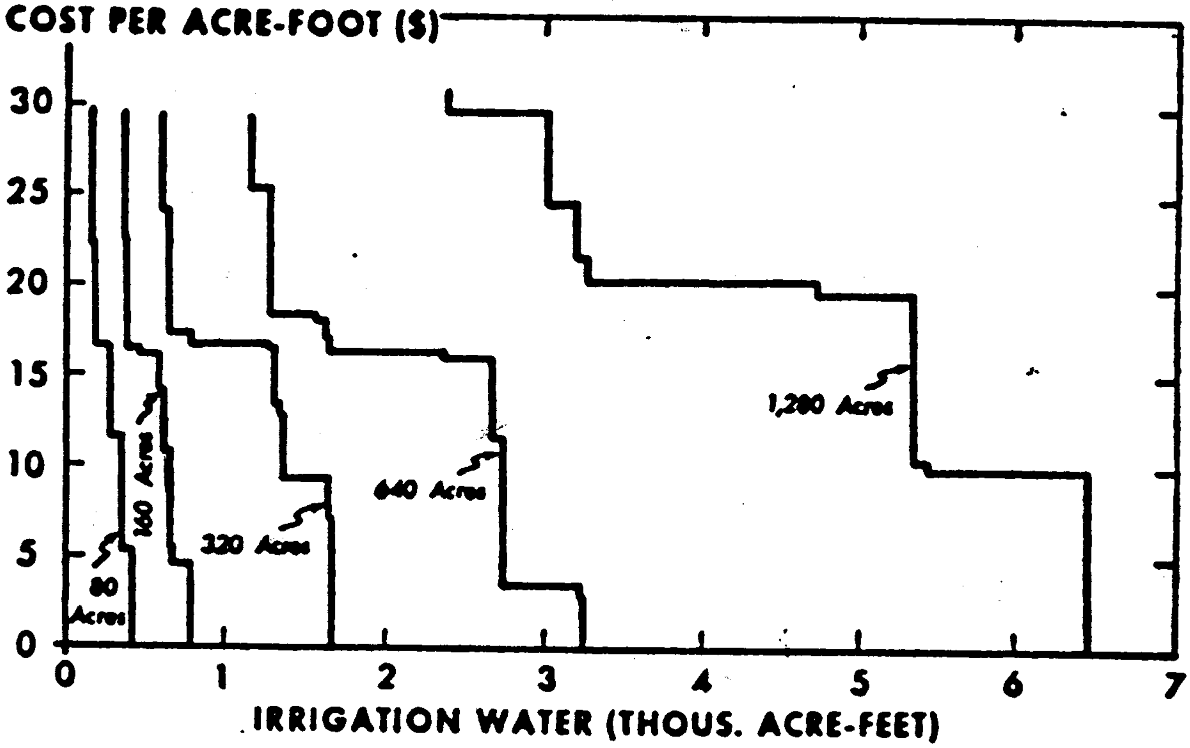


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FIGURE 1.—*Farm demand for irrigation water at various prices, five farm sizes.*

Moore + Hedges (1963)

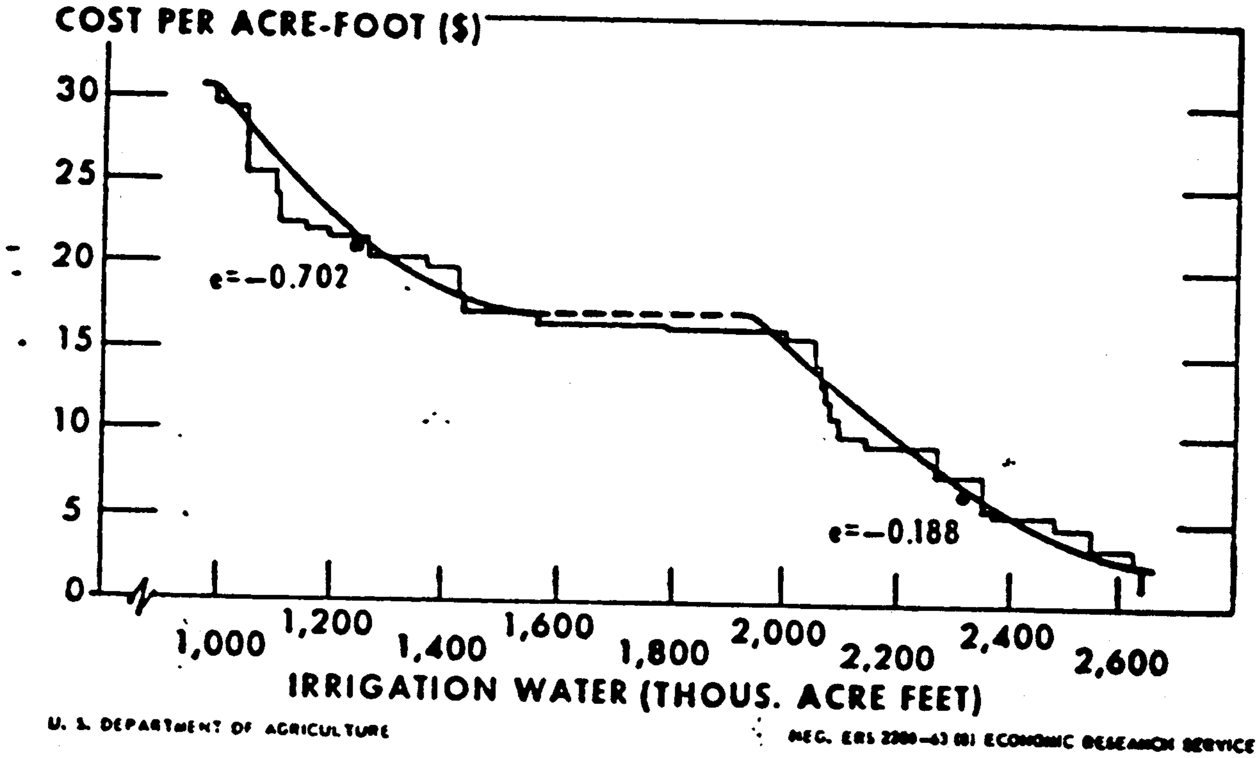


FIGURE 2.—*Demand for irrigation water at various prices, cash crop farms, with 70 percent Grade I soil.*

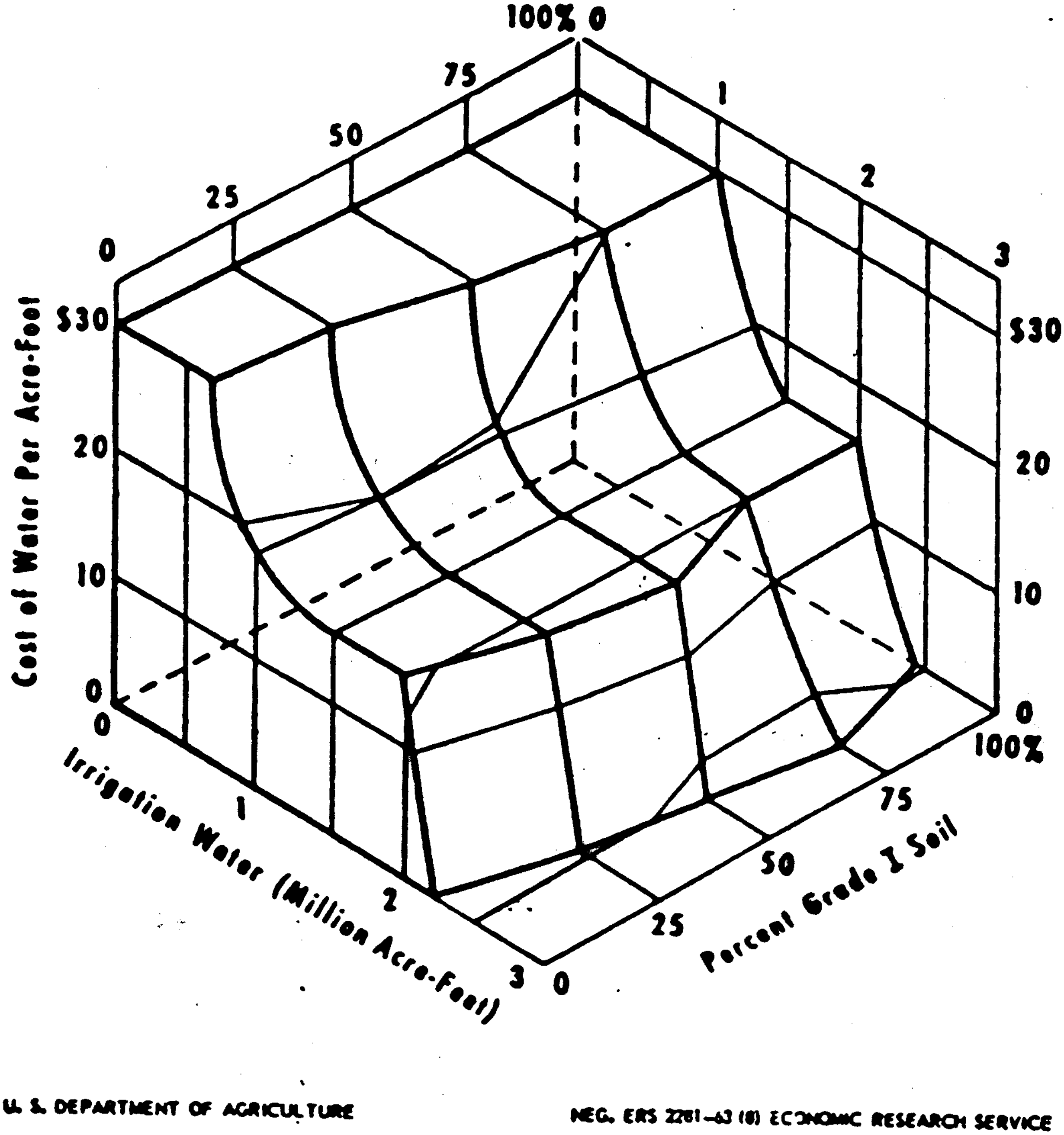


Figure 3.—*Short-run demand surface for irrigation water, cash crop farms, two soil grades.*

<u>QP vs LP</u>

What about if there is a downward sloping $p = \theta(y)$

$$\max\ p_y \cdot f(x) - \Sigma w_i x_i$$

$$\Rightarrow \quad p_y f_i(x) = w_i \quad i=1,\dots, N$$
$$p_y = \theta[f(x)]$$

i.e., $\theta[f(x)]\, f_i(x) = w_i$

solve for x_i and P_y

Resulting input demand curve is $x_i = \hat{g}^i(w)$.
Effect of incorporating a non-zero demand elasticity is to <u>reduce</u> effect of price change on demand for an input output → to $\hat{y}$ instead of y'
∴ a smaller change in output supply
∴ a smaller change in input demand

Moreover, the more inelastic the demand for y the smaller the change in input demand.

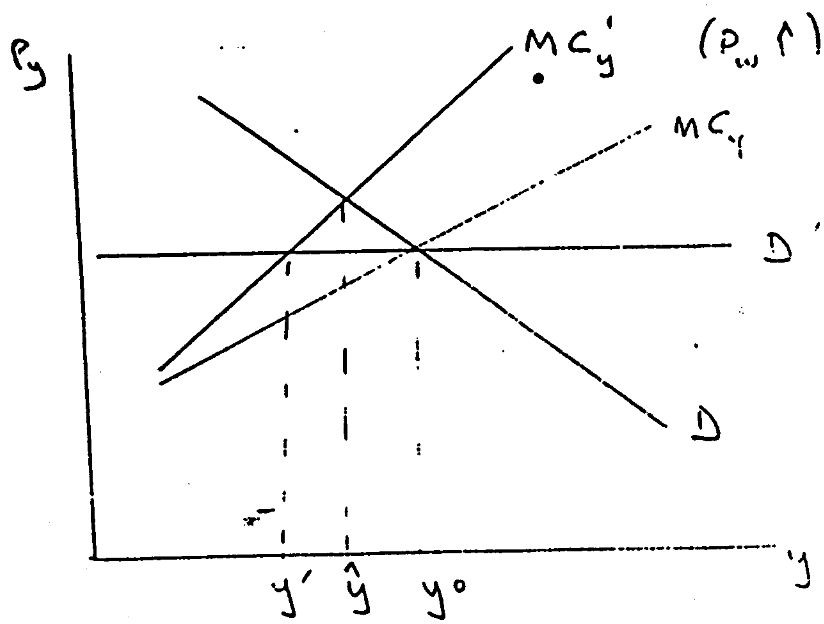
(opposite of Howitt)

To see how this relates to distinction between LP and QP

LP: max $\Sigma \pi_i y_i$ $\qquad \pi_i = p_i - c_i$ $\qquad P_i = a_i - b_i P_i$

QP: max $\Sigma \pi_i(y_i)\, y_i$ [redacted]

$= \Sigma (a_i - c_i)\, y_i - \Sigma b_i y_i^2$ $\qquad$ a quadratic objective

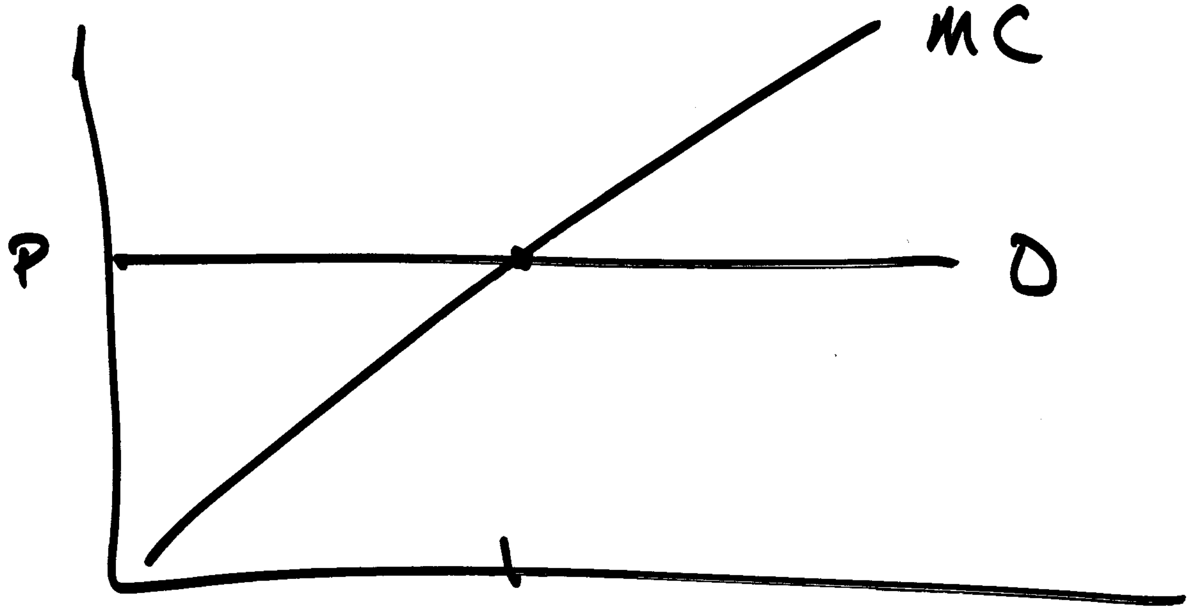


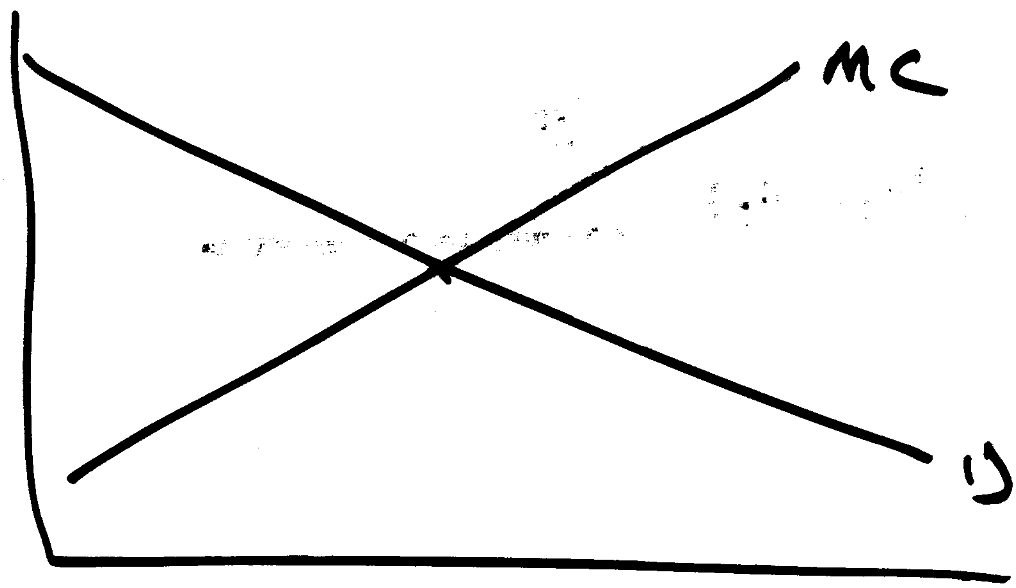
$c_i y_i$

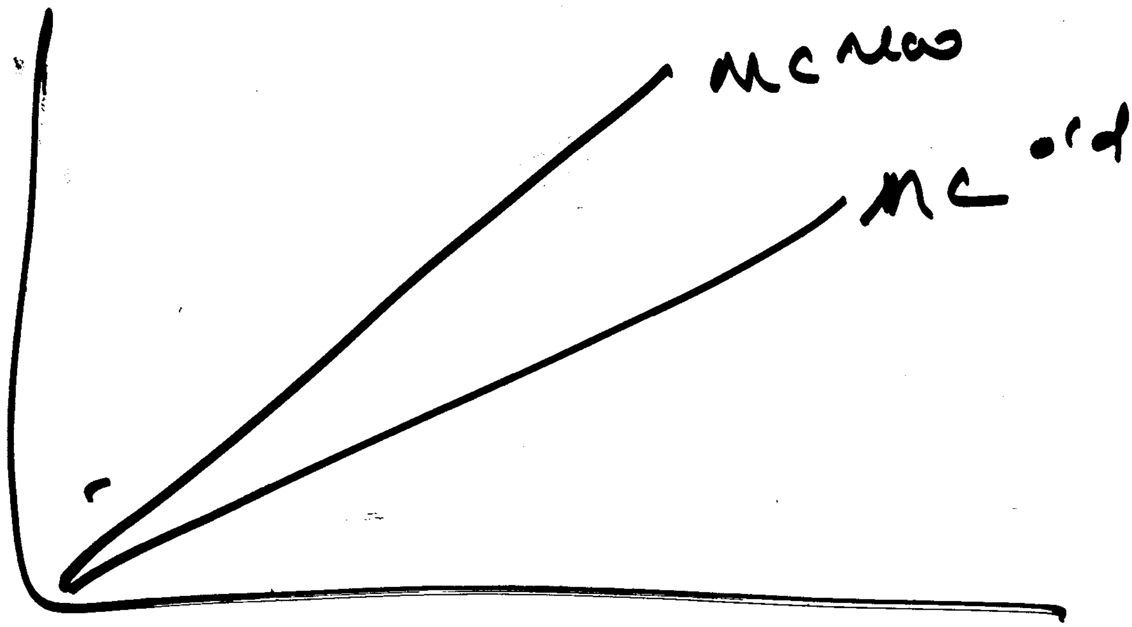
$$p_i(y_i)\, y_i = a_i y - b_i y_i^2$$

$$p_i = a_i - b_i y_i$$

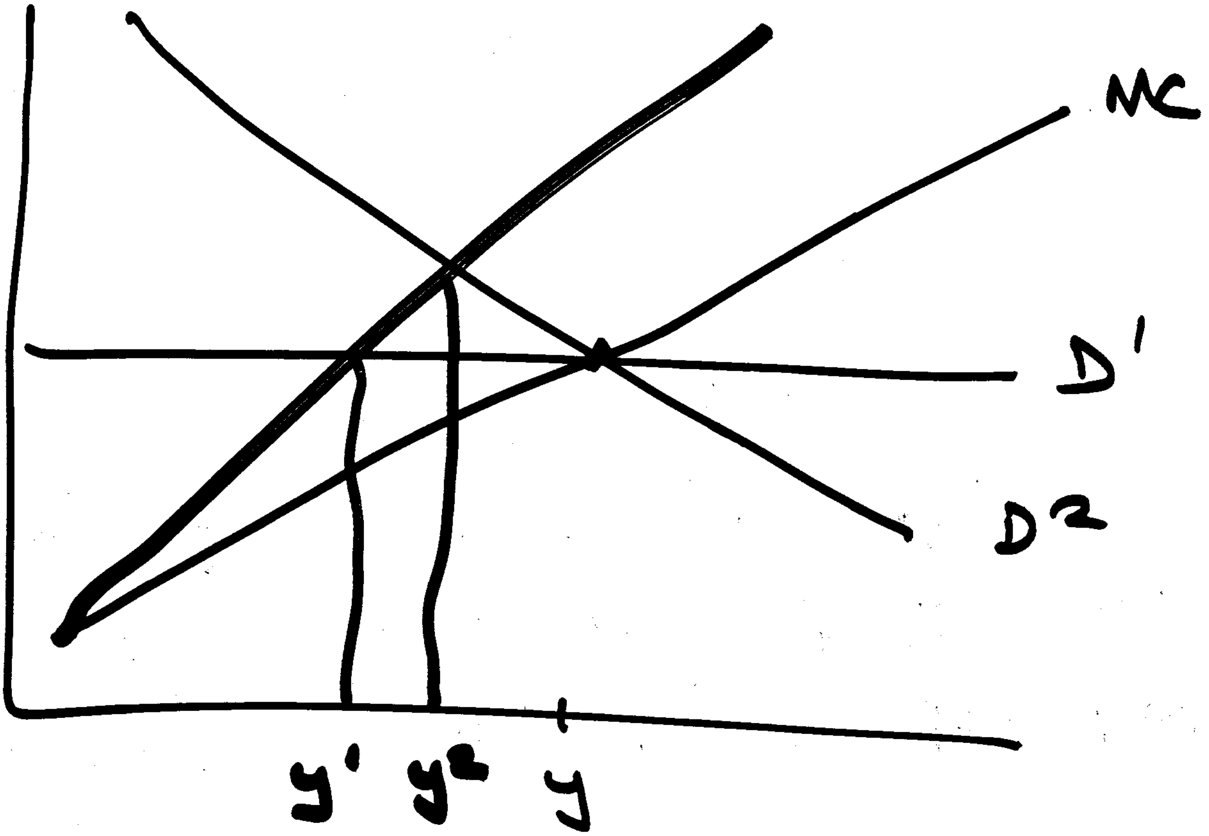
For output as a whole







increase in an input price
$\Rightarrow$ MC of production rises



Other things equal, QP formulation should give a less elastic output demand curve than CD