Solutions to Problem Set 5

ARE 261

November 4, 2001

Question 1

The Euler equation, using Calculus of Variations, is

$$\ddot{x} = 0 \tag{1}$$

If we integrate this equation twice and use the boundary conditions we get the following solution

$$x(t) = t + 2 \tag{2}$$

The necessary conditions using the Maximum Principle are

$$\frac{\partial H}{\partial u} = x - 2u + \lambda = 0$$
$$\frac{\partial^2 H}{\partial u^2} = -2 \leq 0$$
$$-\frac{\partial H}{\partial x} = \dot{\lambda} = -u + 2x - \lambda$$
$$\frac{\partial H}{\partial \lambda} = \dot{x} = x + u$$

Now guess that the co-state variable is linear in the state, or that, $\lambda = s(t)x$. Differentiate this equation with respect to time. This yields

$$\dot{\lambda} = x\dot{s} + s\dot{x} \tag{3}$$

From the first necessary condition we know that $u = \frac{\lambda + x}{2}$. Use this to get rid off u in the third and fourth necessary condition and then substitute these into equation (3). This gives the following ODE in s

$$2\dot{s} + 6s + s^2 - 3 = 0 \tag{4}$$

Question 2

The first thing we are asked to find is the stock level that maximizes steady state harvest. The equation for steady state level harvest is given by setting the equation of motion for the stock of fish to zero. Thus

$$h^{\text{ss}} = \alpha\beta x^{\text{ss}} - \alpha(x^{\text{ss}})^2 \tag{5}$$

where the superscript ss denotes steady state. To determine the maximum steady state level of harvest differentiate equation 5 with respect to x^{ss} and set the expression to zero. This implies that

$$h^m = \frac{\alpha\beta^2}{4}$$
$$x^m = \frac{\beta}{2}$$

Next we want to find the stock level that maximizes the steady state flow of net utility. Net utility at steady state is given by

$$U(h^{\mathrm{ss}}) - c(x^{\mathrm{ss}})h^{\mathrm{ss}} \tag{6}$$

where h^{ss} is defined by equation 5. Differentiate net utility with respect to the steady state level of stock and set the expression to zero. This gives the following

$$\alpha\beta - 2\alpha x^* = \frac{c_1(x^*)h^*}{U_1(h^*) - c(x^*)} \tag{7}$$

where subscripts denote derivatives. Equation 6 along with equation 5 determine h^* and x^*.

If $c_1(x^s) = 0$ then $x^m = x^*$ and $h^m = h^*$. However, if $c_1(x^s) < 0$ then $x^m > x^*$ and $h^m < h^*$. Agents increase the stock to decrease the cost of harvesting which is a decreasing function of the stock.

The socially optimal steady state harvest and steady state stock are determined by solving the following control problem

$$\max_h \int_{t=0}^{t=\infty} e^{-r\tau}\big(U(h) - c(x)h\big)d\tau \tag{8}$$

subject to

$$\dot{x} = \alpha\beta x - \alpha x^2 - h \tag{9}$$

The corresponding Hamiltonian is

$$H = U(h) - c(x)h + \lambda(\alpha\beta x - \alpha x^2 - h) \tag{10}$$

and the necessary conditions are

$$\frac{\partial H}{\partial h} = U_1(h) - c(x) - \lambda = 0$$
$$\frac{\partial^2 H}{\partial h^2} = U_{11}(h) \leq 0$$
$$-\frac{\partial H}{\partial x} = \dot{\lambda} - r\lambda = c_1(x)h - \alpha\beta\lambda + 2\alpha\lambda x$$
$$\frac{\partial H}{\partial \lambda} = \dot{x} = \alpha\beta x - \alpha x^2 - h$$

Totally differentiate the first necessary condition and substitute in the third and fourth necessary conditions to get a differential equation in h. Setting this and the equation of motion for the stock to zero gives the following system of equations that can be solved for the socially optimal steady state harvest and steady state stock

$$\alpha\beta - 2\alpha x^s = r + \frac{c_1(x^s)h^s}{U_1(h^s) - c(x^s)} \tag{11}$$

If $c_1(x) = 0$ then $x^s < x^m$ (compare equations (7) and (11)) and $h^s < h^m$ (from the shape of the steady state harvest function). Because the planner discounts the future he/she will drive the stock down. However, if $c_1(x)$ is large then $x^s > x^m$ and $h^s < h^m$. If driving the stock down increases cost of harvesting significantly then the planner will choose to maintain a larger stock.

Finally, under competitive open-access entry drives rents to zero and thus

$$p(h^c) - c(x^c) = 0 \tag{12}$$

This equation along with the equation of motion at the steady state determine the steady state harvest and stock under competitive open access. Additionally differentiate equation (12) with respect to the stock. This gives the following expression

$$U_{11}(h^c)(\alpha\beta - 2\alpha x^c) - c_1(x^c) = 0 \tag{13}$$

Since $U_{11}(h) \leq 0$ and $c_1(x) \leq$ it follows that $\alpha\beta - 2\alpha x^c \geq 0$. In order to compare h^*, h^s and h^c write the corresponding first order conditions for x^*, x^s and x^c as follows

$$\begin{aligned}
\alpha\beta - 2\alpha x^* &= \frac{c_1(x^*)h^*}{U_1(h^*) - c(x^*)} \\
\alpha\beta - 2\alpha x^s - r &= \frac{c_1(x^s)h^s}{U_1(h^s) - c(x^s)} \\
\alpha\beta - 2\alpha x^c &\geq 0
\end{aligned}$$

If $r = 0$ then it follows that $x^c \leq x^* = x^s$. Under competitive open access the agents drive the stock down.

Question 3

The social planner's control problem is given by

$$\max_y \int_{\tau=0}^{\tau=\infty} e^{-\delta t}\big(ay - \frac{by^2}{2} - \frac{cy}{qx}\big)dt \tag{14}$$

subject to $\dot{x} = rx - \frac{rx^2}{K} - y$. The corresponding current value Hamiltonian is

$$H = ay - \frac{by^2}{2} - \frac{cy}{qx} + \lambda\big(rx - \frac{rx^2}{K} - y\big) \tag{15}$$

The necessary conditions for optimality are

$$a - by - \frac{c}{qx} - \lambda = 0$$

$$-b \leq 0$$

$$\dot{\lambda} = \delta\lambda - \frac{cy}{qx^2} - \lambda r + \frac{2\lambda rx}{K}$$

$$\dot{x} = rx - \frac{rx^2}{K} - y$$

Differentiate the first necessary condition with respect to time and substitute in for $\dot{x}$ and $\dot{\lambda}$. This gives the following differential equation in $\dot{y}$

$$\dot{y} = \frac{c}{qb}\left(\frac{r}{K} + \frac{\delta}{x}\right) - \frac{a - by}{b}\left(\delta - r + \frac{2rx}{K}\right) \tag{16}$$

Now you have a system of differential equations in x and y. You can draw the phase portrait and the direction arrows to find out the slope of the saddle path near the steady state.

Some hints to solve the problem with MATLAB:

- Pick reasonable parameter values.
- Plot the isocline using the function “fplot”

  example:

  fplot(‘ydot’, [lowerbound upperbound])

  where function ‘ydot’ contains expression for the isocline ydot=0.
- Solve for the steady state: using the matlab function “roots”.

  Set xdot=0 and ydot=0. This should give you a fourth order polynomial in x. Use the matlab function “roots” (type help roots in matlab to get more information) to numerically solve for the four roots of the polynomial. Pick the root that is real and positive. This is the steady state value for x. Find the corresponding steady state value for y.
- Solve for the optimal feedback rule by solving the differential equation dy/dx.

  Before we can solve the differential equation we need a boundary condition. We obtain this as follows:

  1. Linearize the system around the steady state (first linearize the system and then evaluate at x=xstar and y=ystar).
  2. Find the converging separatrix for the linear system. Do this by assuming y=m+nx, and you’ll get two values for n. Pick the one consistent with the slope of the saddle path that you see from the phase portrait. Then you can solve for m.

- A boundary condition is given by (xstar+epsilon, m+n*(xstar+epsilon)).
- use the ODE to solve for dy/dx directly with the boundary condition. This step is the same as what you did in Problem set 2.