

Exam ARE 263  
Fall 2004

Answer all three questions.. Each question is worth the same number of points. (If you find yourself drowning in algebra and running out of time, step back and explain in words how you would proceed if there was no time pressure.)

1) This is a discrete time problem. A firm rents capital at the unit price of  $p$  per period. The function  $\pi(k)$  is the firm's maximum level of profit during a period, exclusive of rent, conditional on the level of capital  $k$  (the restricted profit function). The firm incurs an adjustment cost of changing the amount of capital that it rents. The cost incurred in period  $t$  is  $c(k_t - k_{t-1})$ , with  $c(0) = 0$ ,  $c'(0) = 0$  and  $c''(k_t - k_{t-1}) > 0$ . Total profits in a period equal  $\pi(k_t) - pk_t - c(k_t - k_{t-1})$ . The discount factor is  $\beta$ .

(a) Write the optimization problem for the firm that maximizes the present value of the stream of total profits; write the dynamic programming equation.

(b) Use the DPE to obtain the Euler equation.

(c) Give an intuitive interpretation of the Euler equation.

(d) Discuss an algorithm for solving this (discrete stage) control problem numerically.

(e) Write the condition that determines the steady state level of capital. How does the steady state depend on  $\beta$ ? Explain the relation between the steady state level of capital in this problem with adjustment costs, and the optimal level of capital in the corresponding static problem where adjustment costs are identically 0.

2) Consider the continuous time problem

$$\max_{\{c\}} \int_0^{\infty} e^{-rt} u(c_t, x_t) dt$$

s.t.  $\dot{x} = c - \delta x$   $x_0$  given

Assume that  $u$  is concave in the control variable  $c$  and the state variable  $x$ . Assume that there exists an interior steady state.

a) Using the current value Hamiltonian, write the necessary conditions for this problem.

b) Determine the comparative statics of the steady state with respect to the discount rate  $r$ .

c) At this level of generality, what (if anything) can you conclude about the monotonicity (with respect to time) of the control variable and the state variable? (Explain your answer briefly.)

d) Now suppose that  $u$  is additively separable, i.e.  $u(c, x) = A(c) + B(x)$ , with  $A'(c) > 0$  and  $B'(x) < 0$ , and both functions concave. Sketch the phase portrait.

e) For the additively separable case, what (if anything) can you conclude about the monotonicity (with respect to time) of the control variable and the state variable?

d) Explain how you can use the necessary conditions to obtain a numerical solution for the optimal control rule,  $c^*(x)$ .

3) Consider the optimization problem

$$\max_{\{u\}} \frac{1}{2} \int_0^T e^{-rs} (ax_s^2 + bu_s^2) ds$$

$$\text{subject to } \dot{x} = cx + u$$

with  $a < 0$ ,  $b < 0$ .

a) Write down the DPE for the case where  $T < \infty$ .

b) Obtain the solution to this problem when  $T < \infty$ .

c) Write down the DPE for the case where  $T = \infty$ .

d) Obtain the solution to the problem when  $T = \infty$  and briefly describe the relation between the solution for finite and infinite  $T$ .

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Sketch of (some) answers

1) This problem is almost exactly the same as the one that I went over in class, except that here I assume that the convex adjustment cost depends on the change in capital. In the problem I discussed in class, the convex adjustment cost depends on gross investment (the change in capital plus depreciation). For the problem in the exam, the steady state condition is exactly the same as the optimality condition in the static problem (i.e. the

problem without adjustment costs). For the problem in the exam, adjustment costs are minimized (and equal to zero) when adjustment is equal to 0, i.e. at the steady state. For this problem, the presence of adjustment costs affects the trajectory to the steady state, but adjustment costs have no effect on the steady state.

2) This problem is exactly like the one that we went over in section 3 of the notes, except that here I began with a general utility function. Since I told you to assume that there is an interior steady state, and since there is a single state variable, you know that the steady state is a saddle point. Therefore, you can conclude that the state trajectory is monotonic in time. However, you cannot conclude anything about the monotonicity of the control trajectory. In contrast, when you assume that the utility function is additively separable in the state and the control, you know that both the state and the control are monotonic in time.

3) In addition to making sure that you know what the correct DPE is for the two cases, I wanted to see that you knew how to solve a linear-quadratic problem. I also wanted to see that you understood that in the autonomous problem the control rule is a time-independent linear function of the state. In the finite horizon problem, the control rule is a linear function of the state, but the coefficient of the control rule depends on time-to-go.