Solution to problem set 5
ARE 263, 2008

Problems 1 and 2
The following two files contain MATLAB code for these problems: (i) predator_prey2.m; and (ii) prey2.m.

Problem 3
To find the isoclines where the slope of the trajectory is either 0 or $\infty$ we set the differential equations to zero. The isocline where the slope of the trajectory is 0 is given by
\[ y_2 = \frac{1}{3} y_1 \]  
while the isocline where the slope of the trajectory is $\infty$ is given by
\[ y_2 = \frac{5}{3} y_1 \]
The steady state(s) occurs where these two isoclines intersect. For this system of differential equations there is one steady state at $y_1 = 0$ and $y_2 = 0$.

To find the separatrix we use the facts that the separatrix is a line and that its slope is equal to the slope of the trajectory (which is given by $\frac{dy_2}{dy_1} = \frac{\dot{y}_2}{\dot{y}_1}$). Say the equation for the separatrix is
\[ y_2 = k + Ky_1 \]  
then the separatrix is found by solving the following expression
\[ K = \frac{-2y_1 + 10y_2}{-5y_1 + 3y_2} \]  
To solve equation (4) we first substitute equation (3) into equation (4). This gives the following
\[ (3K^2 - 15K + 2)y_1 + 3kK - 10k = 0 \]  
Equating the coefficients of $y_1$ on either side gives a quadratic in K and implies that $k = 0$. There are thus two separatrices, $y_2 = 4.86y_1$ and $y_2 = 0.14y_1$.

The steady state is a saddle point. Initial values of $y_1$ and $y_2$ that lie precisely along the line $y_2 = 0.14y_1$ (one of the separatrices) are the only ones that would converge to the equilibrium; all other initial values diverge from the equilibrium point.

Problem 4
Part 1
The control is $u_i$, the state is $y_i$ and the co-state variable is $\lambda_i$. The first
order conditions with respect to each of these variables are given by

\[
\frac{\partial L}{\partial u_i} = \left( \frac{\partial F}{\partial u_i} + \lambda_i \right) \Delta x = 0, \quad i = 0, 1, \ldots, n - 1 \quad (6)
\]

\[
\frac{\partial L}{\partial y_i} = \left\{ \begin{array}{ll}
\frac{\partial F}{\partial y_i} \Delta x + \frac{(\lambda_i - \lambda_{i-1})}{\Delta x} = 0, & \quad i = 1, 2, \ldots, n - 1 \\
-\frac{\lambda}{\Delta x} = 0, & \quad i = n
\end{array} \right.
\quad (7)
\]

\[
\frac{\partial L}{\partial \lambda_i} = (y_{i+1} - y_i) - u_i \Delta x = 0 \quad i = 0, 1, \ldots, n - 1 \quad (8)
\]

The first and the third equation hold for \( i = 0, \ldots, n - 1 \) while the second equation is different for \( y_i \) (\( i = 1, 2, \ldots, n - 1 \)) and \( y_n \). (Since \( y_0 \) is given there is no first order condition for this variable.)

**Part 2**

The Lagrangian can be written as

\[
L = \sum_{i=0}^{n-1} (H_i \Delta x - \lambda_i (y_{i+1} - y_i))
\quad (9)
\]

The first order conditions in terms of the Hamiltonian are

\[
\frac{\partial L}{\partial u_i} = \frac{\partial H_i}{\partial u_i} \Delta x = 0 \quad \Rightarrow \quad \frac{\partial H_i}{\partial u_i} = 0 \quad i = 0, 1, \ldots, n - 1 \quad (10)
\]

\[
\frac{\partial L}{\partial y_i} = \left( \frac{\partial H_i}{\partial y_i} \Delta x + (\lambda_i - \lambda_{i-1}) \right) = 0 \quad i = 1, 2, \ldots, n - 1 \quad \Rightarrow
\]

\[
-\frac{\partial H_i}{\partial y_i} = \frac{(\lambda_i - \lambda_{i-1})}{\Delta x}
\]

\[
\frac{\partial L}{\partial y_n} = -\lambda_{n-1} = 0
\quad (12)
\]

\[
\frac{\partial L}{\partial \lambda_i} = \left( -(y_{i+1} - y_i) + \frac{\partial H_i}{\partial \lambda_i} \Delta x \right) = 0 \quad \Rightarrow
\]

\[
\frac{\partial H_i}{\partial \lambda_i} = \frac{u_i}{\Delta x}
\quad (13)
\]

Equation (12) implies that \( \lambda_{n-1} = 0 \) when \( y_n \) is free.

**Part 3**

When there is a scrap function the Lagrangian is

\[
L^* = L + f(y_n)
\]

where \( L \) is given by equation (24). Equation (12) is replaced by

then the derivative of the Lagrangian with respect to \( y_n \) changes to

\[
\frac{\partial L^*}{\partial y_n} = \frac{\partial f}{\partial y_n} - \lambda_{n-1} = 0
\quad (14)
\]

Hence

\[
\lambda_{n-1} = \frac{\partial f}{\partial y_n}
\quad (15)
\]

**Part 4**
If I want to change the length of each stage, $\Delta x$, and keep the length of the problem constant, I need to define the length of the problem as $(n-1) \Delta x \equiv T$, so $n = \frac{T}{\Delta x}$. As $\Delta x \to 0$, $n \to \infty$. (Each period becomes short, and there are many periods.) The summation in the equation (1) becomes an integral and the difference equations become differential equations. Formally taking the limit of the necessary conditions gives

$$\frac{\partial H}{\partial u_i} = 0 \quad (16)$$
$$\frac{\partial H}{\partial y_i} = -\dot{\lambda} \quad (17)$$
$$\frac{\partial H}{\partial \lambda_i} = \dot{y} \quad (18)$$

Note that in the discrete time problem, the Hamiltonian and the necessary conditions are defined for $i = 0, 1, 2, \ldots$. In the continuous time problem, the Hamiltonian is defined at every point in time between 0 and $T$, and the necessary conditions must hold at every point. (It is as if there were “uncountably many first order conditions” in the continuous limit.)

**Problem 5**

The current value Hamiltonian for the control problem is given by

$$H = u - \frac{u^2}{2} - \frac{x^2}{2} + \lambda(u - bx) \quad (19)$$

And the necessary conditions are

$$\frac{\partial H}{\partial u} = 1 - u + \lambda = 0 \quad (20)$$
$$\frac{\partial^2 H}{\partial u^2} = -1 < 0 \quad (21)$$
$$\dot{\lambda} = r\lambda - \frac{\partial H}{\partial x} = r\lambda + x + b\lambda \quad (22)$$
$$\dot{x} = \frac{\partial H}{\partial \lambda} = u - bx \quad (23)$$

The terminal conditions are replaced by the steady state conditions.

Next we want to write a system of differential equations in the state and the control which means that we have to get rid of $\lambda$ from the necessary conditions. But first differentiate the first necessary condition with respect to time. This gives

$$\dot{u} = \dot{\lambda} \quad (24)$$

Using this along with the other first order conditions gives the following two differential equations in the state and control

$$\dot{u} = (r + b)(u - 1) + x \quad (25)$$
$$\dot{x} = u - bx \quad (26)$$
To find the steady states (which substitute for the terminal conditions) we set the two differential equations equal to zero. This implies that the steady state values of $u^*$ and $x^*$ are given by

$$u^* = \frac{b(r+b)}{1 + b(r+b)}$$  \hspace{1cm} (27) \\
$$x^* = \frac{r + b}{1 + b(r+b)}$$  \hspace{1cm} (28)

The two isoclines we are interested in, namely $\dot{u} = 0$ and $\dot{x} = 0$ are given by

$$u = 1 - \frac{x}{r+b} \quad (\dot{u} = 0)$$  \hspace{1cm} (29) \\
$$u = bx \quad (\dot{x} = 0)$$  \hspace{1cm} (30)

To determine the directional arrows for the $\dot{u} = 0$ isocline differentiate the equation for the isocline with respect to $x$. This gives

$$\frac{d\dot{u}}{dx} = 1$$  \hspace{1cm} (31)

This implies that to the right of the isocline $u$ increases and to the left it decreases. Similarly to determine the directional arrows for the $\dot{x} = 0$ isocline we differentiate the equation for the isocline which gives

$$\frac{d\dot{x}}{du} = 1$$  \hspace{1cm} (32)

So above the isocline $x$ increases and decreases below the isocline.

We determine the stability of the steady state by looking at the $A$ matrix which for this problem is given by

$$(A) = \begin{pmatrix} r + b & 1 \\ 1 & -b \end{pmatrix}$$

The determinant of matrix $A$ is negative which implies that the steady state is a saddle point.

Finally, we need to determine the optimal control rule. For a linear quadratic system with one state the optimal control rule is given by the converging separatrix. We know that the separatrix is a line whose slope is equal to the slope of the trajectory. The slope of the optimal trajectory is given by

$$\frac{du}{dx} = \frac{\dot{u}}{\dot{x}} = \frac{(r+b)(u - 1) + x}{u - bx}$$  \hspace{1cm} (33)

Let $u = k_1 + k_2x$ denote the separatrix. We know that $\frac{du}{dx} = k_2$. Substituting for $u$ from the equation of the separatrix and equating the coefficients of $x$ gives the following two equations for $k_1$ and $k_2$

$$k_1k_2 = (r + b)(k_1 - 1)$$  \hspace{1cm} (34) \\
$$k_2^2 - bk_2 - 1 - (r + b)k_2 = 0$$  \hspace{1cm} (35)
These in turn imply that $k_2 = \frac{\alpha + \sqrt{\alpha^2 + 4}}{2}$ or $k_2 = \frac{\alpha - \sqrt{\alpha^2 + 4}}{2}$, and $k_1 = \frac{r + b}{r + k_2}$. Since the optimal path is given by the converging separatrix we can show that the correct root is $k_2 = \frac{\alpha - \sqrt{\alpha^2 + 4}}{2}$. With this root the optimally controlled system converges to the steady state. The other root gives you the formulae for the diverging separatrix – a path that diverges from the steady state. Therefore, the optimal control rule is

$$u = \frac{r + b}{r + b - k_2} + k_2 x$$

(36)

where $k_2 = \frac{\alpha - \sqrt{\alpha^2 + 4}}{2}$.

Problem 6

The necessary conditions using the Maximum Principle are

$$\frac{\partial H}{\partial u} = x - 2u + \lambda = 0$$
$$\frac{\partial^2 H}{\partial u^2} = -2 \leq 0$$
$$-\frac{\partial H}{\partial x} = \dot{\lambda} = -u + 2x - \lambda$$
$$\frac{\partial H}{\partial \lambda} = \dot{x} = x + u$$

Now guess that the co-state variable is linear in the state, or that, $\lambda = s(t)x$. Differentiate this equation with respect to time. This yields

$$\dot{\lambda} = xs + s\dot{x}$$

(37)

From the first necessary condition we know that $u = \frac{\lambda + x}{2}$. Use this to get rid off $u$ in the third and fourth necessary condition and then substitute these into equation (37). This gives the following ODE in $s$

$$2\dot{s} + 6s + s^2 - 3 = 0$$

(38)

Problem 7

The first thing that you are asked to find is the stock level that maximizes steady state harvest. The equation for steady state level harvest is given by setting the equation of motion for the stock of fish to zero. Thus

$$h^{ss} = \alpha \beta x^{ss} - \alpha (x^{ss})^2$$

(39)

where the superscript $ss$ denotes steady state. To determine the maximum steady state level of harvest differentiate equation 39 with respect to $x^{ss}$ and set the expression to zero. This implies that

$$h^m = \frac{\alpha \beta^2}{4}$$
$$x^m = \frac{\beta}{2}$$
Next we want to find the stock level that maximizes the steady state flow of net utility. Net utility at steady state is given by

$$U(h^{ss}) - c(x^{ss})h^{ss}$$

where $h^{ss}$ is defined by equation 39. Differentiate net utility with respect to the steady state level of stock and set the expression to zero. This gives the following

$$\alpha\beta - 2\alpha x^* = \frac{c_1(x^*)h^*}{U_1(h^*) - c(x^*)}$$

where subscripts denote derivatives. Equation 40 along with equation 39 determine $h^*$ and $x^*$.

If $c_1(x^*) = 0$ then $x^m = x^*$ and $h^m = h^*$. However, if $c_1(x^*) < 0$ then $x^m > x^*$ and $h^m < h^*$. Agents increase the stock to decrease the cost of harvesting which is a decreasing function of the stock.

The socially optimal steady state harvest and steady state stock are determined by solving the following control problem

$$\max_h \int_{t=0}^{t=\infty} e^{-r\tau} (U(h) - c(x)h) d\tau$$

subject to

$$\dot{x} = \alpha\beta x - \alpha x^2 - h$$

The corresponding Hamiltonian is

$$H = U(h) - c(x)h + \lambda(\alpha\beta x - \alpha x^2 - h)$$

and the necessary conditions are

$$\frac{\partial H}{\partial h} = U_1(h) - c(x) - \lambda = 0$$

$$\frac{\partial^2 H}{\partial h^2} = U_{11}(h) \leq 0$$

$$\frac{\partial H}{\partial x} = \dot{\lambda} - r\lambda = c_1(x)h - \alpha\beta\lambda + 2\alpha\lambda x$$

$$\frac{\partial H}{\partial \lambda} = \dot{x} = \alpha\beta x - \alpha x^2 - h$$

Totally differentiate the first necessary condition and substitute in the third and fourth necessary conditions to get a differential equation in $h$. Setting this and the equation of motion for the stock to zero gives the following system of equations that can be solved for the socially optimal steady state harvest and steady state stock

$$\alpha\beta - 2\alpha x^* = r + \frac{c_1(x^*)h^*}{U_1(h^*) - c(x^*)}$$

If $c_1(x) = 0$ then $x^* < x^m$ (compare equations (41) and (45)) and $h^s < h^m$ (from the shape of the steady state harvest function). Because the planner
discounts the future he/she will drive the stock down. However, if $c_1(x)$ is large then $x^* > x^m$ and $h^s < h^m$. If driving the stock down increases cost of harvesting significantly then the planner will choose to maintain a larger stock.

Finally, under competitive open-access entry drives rents to zero and thus

\[ p(h^c) - c(x^c) = 0 \] (46)

This equation along with the equation of motion at the steady state determine the steady state harvest and stock under competitive open access. Additionally differentiate equation (46) with respect to the stock. This gives the following expression

\[ U_{11}(h^c)(\alpha \beta - 2 \alpha x^c) - c_1(x^c) = 0 \] (47)

Since $U_{11}(h) \leq 0$ and $c_1(x) \leq 0$. In order to compare $h^*$, $h^s$ and $h^c$ write the corresponding first order conditions for $x^*$, $x^s$ and $x^c$ as follows

\[
\begin{align*}
\alpha \beta - 2 \alpha x^* &= \frac{c_1(x^*)h^s}{U_1(h^s) - c(x^*)} \\
\alpha \beta - 2 \alpha x^s - r &= \frac{c_1(x^*)h^s}{U_1(h^s) - c(x^*)} \\
\alpha \beta - 2 \alpha x^c &\geq 0
\end{align*}
\]

If $r = 0$ then it follows that $x^c \leq x^* = x^s$. Under competitive open access the agents drive the stock down.

**Problem 8**

The social planner’s control problem is given by

\[
\max_y \int_{\tau=0}^{\tau=\infty} e^{-\delta \tau} \left( ay - \frac{by^2}{2} - \frac{cy}{qx} \right) d\tau
\]

subject to $\dot{x} = rx - \frac{rx^2}{K} - y$. The corresponding current value Hamiltonian is

\[ H = ay - \frac{by^2}{2} - \frac{cy}{qx} + \lambda \left( rx - \frac{rx^2}{K} - y \right) \] (49)

The necessary conditions for optimality are

\[ a - by - \frac{c}{qx} - \lambda = 0 \]

\[ -b \leq 0 \]

\[ \dot{\lambda} = \delta \lambda - \frac{cy}{qx^2} - \lambda r + \frac{2\lambda rx}{K} \]

\[ \dot{x} = rx - \frac{rx^2}{K} - y \]

Differentiate the first necessary condition with respect to time and substitute in for $\dot{x}$ and $\lambda$. This gives the following differential equation in $\dot{y}$

\[ \dot{y} = \frac{c}{qy} \left( \frac{r}{K} + \frac{\delta}{x} \right) - \frac{a - by}{b} \left( \delta - r + \frac{2rx}{K} \right) \] (50)
Now you have a system of differential equations in x and y. You can draw the phase portrait and the direction arrows to find out the slope of the saddle path near the steady state.

Some hints to solve the problem with MATLAB:

- Pick reasonable parameter values.
- Plot the isocline using the function “fplot”
  example:
  \[
  \text{fplot('ydot', [lowerbound upperbound])}
  \]
  where function ‘ydot’ contains expression for the isocline ydot=0.
- Solve for the steady state: using the matlab function “roots”.
  Set xdot=0 and ydot=0. This should give you a fourth order polynomial in x. Use the matlab function “roots” (type help roots in matlab to get more information) to numerically solve for the four roots of the polynomial. Pick the root that is real and positive. This is the steady state value for x. Find the corresponding steady state value for y.
- Solve for the optimal feedback rule by solving the differential equation dy/dx.
  Before we can solve the differential equation we need a boundary condition. We obtain this as follows:

  1. Linearize the system around the steady state (first linearize the system and then evaluate at x=xstar and y=ystar).
  2. Find the converging separatrix for the linear system. Do this by assuming y=m+nx, and you’ll get two values for n. Pick the one consistent with the slope of the saddle path that you see from the phase portrait. Then you can solve for m.

- A boundary condition is given by ( xstar+epsilon, m+n*(xstar+epsilon) ).
- use the ODE to solve for dy/dx directly with the boundary condition. This step is the same as what you did in Problem set 2.