Problem 1 just requires that you run the program you are given.

2. The DPE is

\[ J(i) = \max \left[ g(1) - R + \beta \sum_k p_{1k} J(k), g(i) + \beta \sum_k p_{ik} J(k) \right] \]

The first term is the value of replacing the machine, and the second term is the value of not replacing the machine. A policy is optimal if it calls for replacing the machine in those states \( i \) where

\[ g(1) - R + \beta \sum_k p_{1k} J(k) > g(i) + \beta \sum_k p_{ik} J(k). \]

3

a) The DPE is

\[ J(S) = \max_h (U(h) + \beta J(f(S - h))) \] (1)

and the first order condition is

\[ U'(h) = \beta J'(f(S - h)) f'(S - h). \] (2)

The necessary condition states that the marginal utility of consumption in the current period equals the discounted shadow value of the state \( (J') \times \) time the marginal productivity of the state \( (f') \).

b) Differentiate equation 1 using the envelope theorem to obtain and expression for \( J'(S) \):

\[ J'(S) = \beta J'(f(S - h)) f'(S - h). \] (3)

Equations 2 and 3 imply

\[ U'(h) = J'(S) \] (4)

Equation 4 must hold at every calendar time; in particular it holds in the next period. Re-introducing the time subscripts for clarity, and using

\[ J'(f(S - h)) = J'(S_{t+1}) = U'(h_{t+1}) \]
we have (using equation 2) the Euler equation:

\[ U'(h_t) = \beta U'(h_{t+1}) f'(S_t - h_t). \]

The interpretation is that along the optimal trajectory, any perturbation has a zero first order effect. A particular perturbation increases current harvest by one unit (leading to additional utility of \( U'(h_t) \)) and decreases harvest in the next period by \( f'(S_t - h_t) \) so that in the subsequent period the stock returns to the optimal trajectory. The present value of the utility cost of the reduction in the next period is \( \beta U'(h_{t+1}) f'(S_t - h_t) \). On the optimal trajectory, the benefit must just equal the cost.

c) In the steady state the harvest and the stock are constant, leading to the algebraic conditions

\[
\begin{align*}
S &= f(S - h) \\
U'(h) &= \beta U'(h) f'(S - h) \implies \beta f'(S - h) = 1.
\end{align*}
\]

4.

a) The DPE is

\[ J(S_t,t) = \max_{h_t} \log(h_t) + \beta J(S_{t+1},t) \]  

and the boundary condition is that \( h_T = S_T \) which implies that \( J(S_T) = \log(S_T) \). (In the last period there is no reason to save any of the stock.) Note that \( J(S_t,t) \) is the current value at calendar time \( t \) given stock \( S \).

b and c) Recall that \( t \) is calendar time and \( \tau = T - t \) is "time to go". An inductive proof involves three steps:

1. • Show that the guessed form is true for \( \tau = 0 \)
   • Assume that the guessed form is true for \( \tau = s \)
   • Show that the guessed form is true for \( \tau = s + 1 \)

The guess is that \( J(S_\tau, \tau) = B_\tau + A_\tau \log(S_\tau) \). Note that \( \tau = 0 \) when \( t = T \). Consequently, \( J(S,T) = \log(S_T) \). When \( \tau = 0 \) the value function has the guessed form with \( B_0 = 0 \) and \( A_0 = 1 \).

Now assume that the guessed form is true for \( \tau = s \). This implies that \( J(S,s) = B_s + A_s \log(S) \). Finally we need to show that the guessed form is
true for \( \tau = s + 1 \). Write the DPE going forward but in terms of \( s \) and \( s + 1 \) (note that in terms of \( t \), \( s + 1 < s \)). This gives

\[
J(S, s + 1) = \max_h [\log(h) + \beta J((S - h)^{\alpha}, s)]
\] (6)

Substitute

\[
J((S - h)^{\alpha}, s) = B_s + A_s \log((S - h)^{\alpha})
\]

into the RHS of equation 6 and perform the maximization to obtain and then derive the first order condition. This gives

\[
\frac{1}{h} - \frac{\alpha \beta A_s}{S-h} = 0
\] (7)

Simplify this expression to write

\[
h = \frac{S}{1 + \alpha \beta A_s}.
\]

Substitute this expression into the DPE, equation 6, and remove the max operator to obtain the maximized DPE:

\[
J(S, s + 1) = \log(S) - \log(1 + \alpha \beta A_s) + \beta B_s + \\
\beta \alpha A_s [\log(S) + \log(\alpha \beta A_s) - \log(1 + \alpha \beta A_s)]
\] (8)

Using our guess, the left hand side is

\[
J(S, s + 1) = B_s + A_s \ln S
\]

This guess satisfies the maximized DPE, equation 8, if and only if

\[
B_{s+1} = \beta B_s + \beta \alpha A_s \log(\alpha \beta A_s) - (1 + \alpha \beta A_s) \log(1 + \alpha \beta A_s)
\] (9)

\[
A_{s+1} = 1 + \alpha \beta A_s.
\] (10)

You can solve the difference equation 10, or more simply you can use an inductive proof to show that the sequence in increasing in "time to go". The last two equations are the difference equations for \( B_{\tau} \) and \( A_{\tau} \). The control rule is given by

\[
h_{\tau+1} = \frac{S_{\tau+1}}{1 + \alpha \beta A_{\tau}}
\] (11)
The fraction of stock harvested is a decreasing function of $A_\tau$ and from the above comment, $A_\tau$ is an increasing function of $\tau$. Therefore, the fraction harvested decreases as the horizon becomes more distant; equivalently, the fraction harvested increases as the final period approaches.

d) Equation 10 has a unique steady state, $A_\infty = \frac{1}{1-\alpha\beta}$. Using the graphical argument discussed in class and the facts that $\alpha > 0$ and $\beta > 0$, we know that this steady state is stable iff $\alpha\beta < 1$, which is ensured by our assumptions above. (Note that the assumption $\alpha < 1$ is really stronger than we need. We only require $\alpha < \frac{1}{\beta}$.) The stationary control rule for the infinite horizon problem is

$$h = \frac{S}{1 + \frac{\alpha\beta}{1-\alpha\beta}} = S (1 - \alpha\beta).$$

This is the same as the rule we would have obtained if we had started with an infinite horizon problem, rather than taking the limit of the finite horizon problem.

The equation of motion, evaluated at the optimal control, is

$$S' = (S - S (1 - \alpha\beta))^{\alpha} = (S\alpha\beta)^{\alpha}.$$

Evaluate this equation at the steady state, where $S' = S$ to obtain

$$S_\infty = (\alpha\beta)^{\frac{\alpha}{1-\alpha}}.$$