You have three hours to work on this exam – closed book, closed notes. Please return the exam to me or to my mailbox in 207 Giannini by 5 PM on Friday.

Answer all questions. You cannot avoid using mathematics to answer these questions. However, DO NOT GET BOGGED DOWN IN ALGEBRA. It is sufficient to provide a clear statement of the algebraic manipulations that you would perform, and the nature of the result that you would obtain.

I hope that there are no typos or ambiguities in the following questions. If you find a problem of this sort, just be sure to let me know what assumptions you have made.

Question 1. Consider the following two-sector model, with total stock of labor = 1. The amount of labor in the manufacturing sector at time $t$ is equal to $L(t)$. The value of output in the manufacturing sector is $L f(L)$ with $f'(<1) > 0$. The increasing returns to scale in the sector are external to the firm, so each worker receives the wage $w = f(L)$. The value of output in the agricultural sector is $(1 - L)a$; the agricultural wage is the constant $a$. Assume that $f(0) < a < f(1)$. The flow of labor into the manufacturing sector is $m$. The social cost of migration is $\gamma m^2/2$, and each worker who changes sectors pays the migration cost $\gamma |m|$. The constant discount rate is $r$.

a) (Economic interpretation) (i) How would the model change if the increasing returns to scale were internal to the firm? (ii) What is the implication of the assumption that $f(0) < a < f(1)$?

b) Discuss the qualitative solution to the optimization problem of the social planner who is able to directly choose migration in order to maximize the present discounted value of output minus adjustment costs.

c) Discuss the qualitative solution to the equilibrium problem of (non-strategic) workers with rational point expectations, each of whom decides (at each point in time) whether to migrate. A worker wants to maximize...
the present discounted value of wages minus adjustment costs. (Your answer should explain what you mean by “rational point expectations”.)

d) Can you design a tax policy that causes the competitive equilibrium to produce the socially optimal program?

e) Describe an algorithm (or algorithms) for solving the optimization problem and the equilibrium problem. What (if any) differences between these two problems do you need to take into account?

Sketch of answers:

a) i) A "standard" competitive equilibrium does not exist, because if firms have internal IRTS and take the wage as given they want to increase their size without bound. ii) The assumption means that a corner solution is a stable steady state.

b) I wanted you to set up the problem and recognize that this is a "Skiba problem", and discuss how to solve it.

c) This is just Krugman’s model, with a more general wage function (\( f \) instead of a linear function).

d) Might be possible, but not obvious in the case where there are multiple rational expectations equilibria.

e) For the optimization problem (when there are two candidate trajectories) you could proceed in a couple of ways. Either approximate the value functions under the two candidate decision rules and compare them, or approximate the costate variable under the two candidate decision rules and use the Skiba technique. For the equilibrium problem you need to solve the rational expectations equilibrium that takes the economy to either steady state, and find the domains of these equilibria. Remember that the condition under which there are two candidates in the optimization problem, or two rational expectations equilibria, are different.

Question 2 Consider the following two problems:

Problem A: You have a three-stage (two decision periods) cake eating model. In each period the utility from eating \( c \) units is \( U(c) \). The total size of the cake is 1, so \( c_1 + c_2 + c_3 \leq 1 \). The one-step ahead discount rate is \( r_1 \) and the two-step ahead discount rate is \( r_2 \). (These are known numbers.) The corresponding one-step and two-step discount factors are \( \beta_i \). That is, the present value at time 1 of a unit of cake at time 3 is \( \beta_1 \beta_2 \); the present
value at time 1 of a unit of cake at time 2 is $\beta_1$; the present value at time 2 of a unit of cake at time 3 is $\beta_1$ (NOT $\beta_2$). The decision-maker at time 1 wants to maximize the present discounted value of the stream of utility.

Problem B: You have a three-stage (two decision periods) cake eating model. In each period the utility from eating $c$ units is $U(c)$. The total size of the cake is 1, so $c_1 + c_2 + c_3 \leq 1$. The discount rate in period $i$ is

$$r_i = r_{i-1} (1 + \varepsilon_i)$$

where $\varepsilon_i$ are independently and identically distributed random variables with mean 0 and support $(-1, 1)$. In period $i$, when choosing how much cake to eat, the decisionmaker knows $r_{1-1}$. The present value at time 1 of a unit of cake at time 2 is $\beta_1$; the present value at time 1 of a unit of cake at time 3 is $\beta_1 \beta_2$; the present value at time 2 of a unit of cake at time 3 is $\beta_2$ (NOT $\beta_1$). The decision-maker at time 1 wants to maximize the expected present discounted value of the stream of utility.

a) Discuss the differences in assumptions between these two models, and explain the economic significance of these differences. (I see two important differences in assumptions. I want you to tell me how these differences reflect differences in the underlying story that the models are intended to describe.)

b) Explain how you would find the decision rules, and the first period decision, in these two models. (Tell me what problems you need to solve; I'm not asking for an algorithm that produces a numerical solution.)

Sketch of answers

a) In problem A the discount rate for a period depends on the amount of time until the period arrives. This model can be used to describe the temptation to procrastinate. In this case, the trajectory that is optimal for the decisionmaker in the first period is time inconsistent. The decisionmaker does not learn anything as time goes on; but the amount of cake that the agent in period 1 wants to have eaten is not the same as the amount of cake that the agent in period 2 wants to eat. In problem B the discount rate is stochastic, but the discount rate for a period is not a function of the amount of time until that period. In this case, there is no time inconsistency, but the decision maker should recognize that she will get more information in the second period.

b) For both problems you need to work backwards. In problem A, the
second period problem gives
\[ c_2^* (c_1) = \text{arg max} \ U (c_2) + \beta_1 U (1 - c_1 - c_2) \]
and the first period problem solves
\[ \max_{c_1} U (c_1) + \beta_1 (U (c_2^*) + \beta_2 U (1 - c_1 - c_2^*)) . \]

For problem B the second period problem gives
\[ c_2^{**} (c_1, r_1) = \text{arg max} \ U (c_2) + E_{r_2|r_1} \left( \frac{1}{1 + r_2} U (1 - c_1 - c_2) \right) \]
and
\[ V (c_1, r_1) = U (c_2^{**}) + E_{r_2|r_1} \left( \frac{1}{1 + r_2} U (1 - c_1 - c_2^{**}) \right) . \]
The first period problem is
\[ \max_{c_1} E_{r_1} \left\{ U (c_1) + \frac{1}{1 + r_1} V (c_1, r_1) \right\} . \]

Question 3. Two fishers harvest from a common property resource. Agent i’s harvest in period t is \( h_i(t) \). The stock of the resource in period t is \( x_t \). The stock changes according to
\[ x_{t+1} = (x_t - h_1(t) - h_2(t))^\alpha . \]
Agent i’s single period utility in period t is \( \ln(h_i(t)) \), and both agents have the constant discount factor \( \beta \). They each want to maximize the present discounted infinite stream of utility.

a) How would you find the cooperative equilibrium (the one that maximizes joint welfare)? What, if any, restriction do you need to impose on \( \alpha \) for your proposed solution to be correct?
b) How would you find the non-cooperative Nash equilibrium?

a) This is the same maximization problem that was in one of the problem sets. I wanted you to set up the DPE and recognize that the value function
equals a constant plus another constant time ln x. Equate coefficients to find the unknown constants. See the key to PS 8 for details.

b) We never discussed this problem in detail, but I talked about it in passing on a couple of occasions. Write down "our firm’s" optimization problem when it takes as given "the other firm’s" decision rule \( H(x) \). Let our firm’s decision rule be \( h(x) \). Use the necessary conditions to find an ODE for this function, of the form

\[
\frac{dh}{dx} = f (x, H(x), h(x)).
\]

You can write down the function \( f \) explicitly, but of course you do not know the solution to this ODE. Assume that firms are symmetric, so \( H(x) = h(x) \). The problem is that you do not have a boundary condition for the resulting ODE

\[
\frac{dh}{dx} = f (x, h(x), h(x)),
\]

so the best that you can do is to find a range of values of \( x \) that are asymptotically stable.