Question 1) (40 points) $S$ is an index of environmental quality and $x$ is the flow of pollution. The evolution of $S$ is given by \[ \dot{S} = x - g(S; \alpha), \] where $g$ is increasing and concave in $S$, and $\alpha$ is a parameter of the function $g$. The instantaneous (flow) payoff is $x - \frac{x^2}{2} - \frac{S^2}{2}$ and the instantaneous discount rate is $r$. You want to maximize the present discounted integral of the flow of welfare from time 0 to infinity. The initial condition $S_0$ is given.

(i) Write the current value Hamiltonian and necessary conditions.

(ii) Sketch the phase portrait in $(S, x)$ space. The portrait should include the 0-isoclines, the directional arrows, and the stable saddle path (the optimal control rule).

(iii) Write the equations that determine the optimal steady state.

(iv) Explain how to find the comparative statics of the steady state with respect to $\alpha$. You should set up the comparative statics expression and explain how you use the stability of the steady state in finding the comparative statics. (It is not necessary to carry out all of the calculations – just explain clearly how you would proceed.)

(v) Find the differential equation that the optimal control rule, $x^*(S)$, must satisfy. (Write the expression for $\frac{dx^*}{dS}$) Explain one method of numerically solving this ODE.

Question 2) (40 points) Let $K_t$ be the stock of capital in a period, and let $\pi(K_t)$ be the restricted profit function. The cost of investment in period $t$ is an increasing convex function of investment, $c(I_t)$. The firm has a discount factor of $\beta$ and it wants to maximize

\[ \sum_{t=0}^{\infty} \beta^t (\pi(K_t) - c(I_t)) \]

subject to $K_{t+1} = I_t - \delta K_t$, $K_1$ given.

(i) Write the Dynamic Programming equation for this problem.
(ii) Describe an algorithm for solving the DPE.
(iii) Derive the Euler Equation associated with this problem.
(iv) Interpret the Euler Equation.
(v) Find an expression that the steady state stock of capital must satisfy.
(vi) Interpret the condition that determines the steady state stock of capital.
(vii) How does a decrease in the depreciation rate (an increase in $\delta$) affect the steady state stock of capital?

Problem 3 (20 points) Consider the following utility maximization problem:

$$\text{Max} \sum_{t=0}^{T} \beta^t \ln (C_t)$$

subject to

$$K_{t+1} = (K_t - C_t)^\alpha, \quad \alpha > 1.$$ 

$K_0$ given; $K_t \geq 0, C_t \geq 0$, for all $t$. Here $C_t$ is the consumption at time $t$, and $K_t$ is the stock of money at time $t$. Derive the optimal consumption rule and show how it changes as the “time to go”, $T - t$ (where $t$ is calendar time) increases.