Answer all parts of all three questions. The value of the question is given in parentheses next to the question number.

Question 1) (45 points) $S$ is an index of environmental quality and $x$ is the flow of pollution. The evolution of $S$ is given by $\dot{S} = x - g(S; \alpha)$, where $g$ is increasing and concave in $S$, and $\alpha$ is a parameter of the function $g$. The instantaneous (flow) payoff is $x - \frac{x^2}{2} - \frac{S^2}{2}$ and the instantaneous discount rate is $r$. You want to maximize the present discounted integral of the flow of welfare from time 0 to infinity. The initial condition $S_0$ is given.

(i) Write the current value Hamiltonian and necessary conditions.

(ii) Sketch the phase portrait in $(S, x)$, space. The portrait should include the 0-isoclines, the directional arrows, and the stable saddle path (the optimal control rule).

(iii) Write the equations that determine the optimal steady state.

(iv) Explain how to find the comparative statics of the steady state with respect to $\alpha$. You should set up the comparative statics expression and explain how you use the stability of the steady state in finding the comparative statics. (It is not necessary to carry out all of the calculations – just explain clearly how you would proceed.)

(v) Find the differential equation that the optimal control rule, $x^*(S)$, must satisfy. (Write the expression for $\frac{dx}{dS}$) Explain one method of numerically solving this ODE.

Question 2) (25 points) Specialize problem number (1) by letting $g(x; \alpha) \equiv \alpha x$.

(i) Write down the dynamic programming equation.

(ii) Explain how you would solve the DPE (i.e., find the optimal control rule and the optimal value function). It is not necessary to complete the calculations – just explain how you would proceed.

Question 3) (30 points) The stock of greenhouse gases, $S$, evolves according to

$$S_{t+1} = \Delta S_t + x_t$$
where $x_t$ is the flow of emissions associated with economic activity. Society’s benefits from this economic activity in period $t$ is $B(x_t)$, and the discount factor is $\beta$. If the stock ever exceeds an unknown threshold, $\bar{S}$, an irreversible change occurs. The magnitude of stock-related damages depend on whether the stock has ever exceeded this unknown threshold. These damages are:

$$\bar{D}(S_t) = \begin{cases} 0 & \text{if } S_\tau \leq \bar{S} \text{ for all } \tau \leq t \\ D(S_t) & \text{if } S_\tau > \bar{S} \text{ for any } \tau \leq t \end{cases},$$

where $D(S_t)$ is a known function.

If the stock ever exceeds the unknown threshold, society begins to incur damages $D(S_t)$. At that time, society knows that the irreversible change has occurred. At the initial time, the stock is $S_0$ and damages are 0. Thus, at the initial time society knows that the irreversible change has not yet occurred. Society begins with a subjective cumulative distribution function for the unknown threshold ($\bar{S}$) of $F(S)$. The support of this distribution is $(\underline{S}_0, \bar{S})$, where $\bar{S}$ is a known parameter.

a) Society wants to maximize the present discounted value of the expectation of the flow of welfare, $B(x_t) - \bar{D}(S_t)$. Write down society’s maximization problem. (If you decide to invent any new notation, define it carefully.)

b) Write down the Dynamic Programming Equation (DPE) for this problem.

c) Describe an algorithm that you could use to solve this DPE.

d) Very briefly, explain how the solution to the problem would change if the equation of motion were stochastic, i.e. if $S_t$ evolves according to

$$S_{t+1} = \Delta S_t + x_t + \epsilon_t$$

where $\epsilon_t$ is an iid random variable with a known distribution.