IX. Linear Control Problems

1) One-state variable linear control problem.
2) Necessary and sufficient condition for optimality.
3) Most Rapid Approach Path (MRAP), and singular arc.
4) Fishing example.
5) Two-state variable problem, a durable non-renewable resource with decay.

K & S Section I.16 and II.13, Clark Ch 2.7

General 1 state variable problem

\[
\max_{x_0} \int_{t_0}^{t_1} \left[ G(t,x) + H(t,x)\dot{x} \right] dt
\]

s.t. \( A(t,x) \leq \dot{x} \leq B(t,x) \), \( x_0, x_{t_1} \) given

Euler equation

\[
F_x = \frac{d(F_x)}{dt}
\]

\[
G_x + H_x \dot{x} = \frac{d}{dt}H = H_t + H_x \dot{x}
\]

(2) \( G_x = H_t \)

Any \( x^* \) that satisfies (2) is singular solution (arc)

Thrm. Suppose \( \frac{\partial G}{\partial x} > \frac{\partial H}{\partial t} \) whenever \( x > x^*(t) \).

The optimal solution is

(i) If \( x_0 > x^*(t_0) \), set \( \dot{x}(t) = A( ) \) until \( x^* \) hit
    If \( x_0 < x^*(t_0) \), set \( \dot{x}(t) = B( ) \) until \( x^* \) hit
Define $t_a$ as time you hit $x^*$ using (i)

(ii) Analogous description at RHS, with $t_b$ as time you leave singular path.

over $t_0$, $t_a$, set $\dot{x} = B$, over $t_a$, $t_b$, set $\dot{x} = A$. (MRAP) over $t_a$, $t_b$, solution follows singular path.

Example from fisheries (Clark). This problem is autonomous.

\begin{equation}
\max \int_0^\infty e^{-rt}(p - c(x))h \, dt
\end{equation}

s.t.

\begin{equation}
\dot{x} = F(x) - h \quad x_0 \text{ given} \quad 0 \leq h \leq h_{\max}
\end{equation}

Exercise. Eliminate $h$ to write (3) and (4) so it looks like (1). Check to make sure that assumptions of Thrm are satisfied.

Solve (3) and (4) using maximum principle.
\[ H = (p - c(x))h + \lambda(F(x) - h) \]

F.O.C.

\[
\frac{\partial H}{\partial h} = p - c(x) - \lambda \begin{cases} 
< 0 \Rightarrow h = 0 \\
= 0 \Rightarrow h = h^* \\
> 0 \Rightarrow h = h_{\text{max}}
\end{cases}
\]

\[ \dot{\lambda} = r\lambda + c'(x)h - \lambda F'(x) \quad (6) \]

Suppose we have an interior sol’n: \( h = h^* \), then differentiate mid equation (5), use (6) and (4) and (5)

\[
0 = -c'(x)x - \dot{\lambda} = -c'(x)(F(x) - h) - \lambda (r - F'(x)) - c'(x)h \\
= -c'(x)(F(x) - h) - (p - c(x))(r - F'(x)) - c'(x)h
\]

\[ (7) \]

\[ [p - c(x)]F'(x) - c'(x)F(x) = r(p - c(x)) \]

I could have gotten (7) by recognizing that "singular arc" is a point, so has to be a steady state. Set (4) and (6) = 0, use (5) to obtain (7).

Interpret (7). Define \( \rho = [p - c(x)]F(x) \equiv \) "sustained economic rent"

Rewrite (7) as

\[ \frac{1}{r} \left[ \frac{d\rho(x)}{dx} \right] = p - c(x) \quad (8) \]

An extra unit of harvest yields current revenue of \( p - c(x) \), but flow of future loss is \( \frac{d\rho}{dx} \).

Discount this to obtain LHS of (8)
\[
\frac{d^2p}{dx^2} = -2c'F' + (p - c(x))F'' - c''F
\]

(Even with concave \( F \), can't guarantee unique soln of (8). I've assumed it.)

\( x_0 \) corresponds to rent maximizing stock \((r = 0)\)

\( x_\infty \) corresponds to competitive equilibrium \((r = \infty)\)

A two state variable problem durable, nonrenewable resource (e.g. iron, not oil). (Notes follow Karp, Monopoly Power can be Disadvantageous in the Extraction of a Durable Nonrenewable Resource.) The point of this exercise is to show you how to find and interpret the singular arc in a two state variable autonomous control problem. Here the arc is 1-dimensional. In the one-state variable problem, the arc is 0-dimensional, i.e. a point.

(1) \[ \dot{S} = -m, \quad S_0 \text{ given} \quad \text{(resources stock)} \]

(2) \[ \dot{Q} = m - \delta Q, \quad Q_0 \text{ given} \quad \text{(durable good stock)} \]

\[ U(Q) = \text{utility of stock of good} \]

\[ c(S)m = \text{extraction cost.} \]

Social planner’s problem
(3) \[ \max_{0}^{\infty} e^{-t}[U(Q) - c(S)m]dt \]

s.t. (1) and (2)

\[ H = U(Q) - c(S)m - \lambda m + \eta (m - \delta Q) = U - \eta \delta Q + (\eta - \lambda - c(S))m \]

\[ \lambda = \text{S.V. of } S, \quad \eta = \text{S.V. of } Q \]

(4) \[ \frac{\partial H}{\partial m} = -c(s) - \lambda + \eta = 0 \quad \text{(on singular arc)} \]

(\( \eta - c - \lambda \) is called the "switching function")

(5) \[ \dot{\lambda} = r\lambda + c'(s)m \]

(6) \[ \eta = (r + \delta)\eta - U'(Q) \]

[Exercise: integrate equation 6 (use integrating factor) to obtain the expression \( \eta_t = \int e^{(r+\delta)t}U'(Q_t)dt \), where limits of integration are \( t, \infty \). Interpret this expression.]

Find singular arc.

Define \( U'(Q) \equiv F(Q) = \text{rental of a unit of the durable good} \). \( \eta \) is the price that buyers of the
good would be willing to pay. (6) says capital gains + dividends = opportunity cost of purchase + depreciation.

Differentiate (4)

\[ \eta = c'(S)\dot{S} + \dot{\lambda} \]
\[ = -c(S)m + r\dot{\lambda} + c'(S)m \]

(7) \[ r(\eta = c(S)) \] (The Hotelling rule!)

Equating (7) and (6)

\[ (r + \delta)\eta - F = r(\eta - c(S)) \]

(8) \[ \eta = \frac{F - rc(S)}{\delta} \]

(8) gives equil. goods price on singular arc.

Define singular arc as \( Q = G(S) \). Differentiate wrt time using (1) and (2)

(9) \[ m = \frac{\delta G}{1 + G'} \]

Differentiate (8), use (6), (8) and (9) to get

\[ rF(G(S)) - r\dot{r} + \delta c(S) = \]

(10) \[ -F'(G(S))\delta G(s) + \left[ \frac{rc'(S) + F'(G(S))}{1 + G'(S)} \right] \delta G(S) \]

This is an ODE in \( S \), solve to find singular arc (need B.C.)

e.g. if

9:6
\[ F = a - bQ, \quad C(S) = k_0 - kS, \] with \[ \frac{a}{r - \delta} < k_0 \] then \( G(s) = gS \), where \( g \) is positive solution to a quadratic.